Piecewise functioning systems: bi-sampled controllers

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Abstract: This article introduces a new formalism describing a class of systems with double time scale discrete functioning associated to two input spaces. These systems have specific properties enabling design of new control architectures particularly well adapted to tracking problems in discrete time.

Keywords: Discrete system, multiple time scale, tracking.

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1. Introduction

Discrete control of continuous or sampled-data plants has been studied in various works dealing with different criteria defining: stability, precision, robustness etc. corresponding to control context [1, 2]. In this area several researchers have contributed with original approaches. Among them, Kabamba [3] has proposed to generalize the sample data hold function in order to control linear plants. The main idea was to generate a control from a periodical matrix and the plant sampled output. Other robustness properties of sampled-data systems with generalized hold function have been investigated by Kabamba. Urikura and Nagata [4] have proposed a discrete control with reduction of intersample ripples. Yamamoto [5] used a concept of piecewise defined function for which the state is observed at sampling times and during the sampling periods. Krishan, Nagpal and Khargonekar [6] have studied $H\infty$ control and filtering problems for sampled-data systems. They have taken a state-space approach to give solutions to a number of $H\infty$ filtering and control problems. They have introduced the notion of a linear system with finite discrete jumps and have also stated that both standard continuous-time system and discrete-time system are special cases of linear system with jumps.

More recently, we have defined a class of systems with finite discrete jumps defined in the discrete space S containing commutation moments t_k where $S=\{t_k, k=0,1,2,...\}$ called also "commutation space" [7, 8, 9]. Between two commutations, a plant is controlled from the input space U^r. At the moments of commutation, a plant is controlled from the second input space V^r.

In our previous works, the control approach based on piecewise continuous systems [7, 8] admitted that the system is continuous between two commutations. In this article we proposed the study of the case where the system is defined as sample-data system between two commutations. In the first time a control application based on our approach to piecewise functioning systems has been developed in order to realize the tracking of a state trajectory. In the second time, we have developed a ripple free control tracking strategy based on the optimal control method between commutation moments. We have considered in both cases that the plant state is available entirely and that a noise is not present.

The organization of the paper is as follows: in the next section, we will introduce some elementary properties of bi sampled-data systems with jumps (piecewise functioning system). These may be regarded as a general framework for dealing with hybrid systems using multiple sampled-data approach. We will present results of tracking problems in Section 3. The intersample ripple reduction algorithm, based on the optimal control theory, will be exposed in the Section 4 and finally the conclusion will be given in the section 5. The organization of the paper should optimize the readability.

We end this introduction with some remarks on the notation most of which is standard: R denotes the set of real numbers, R^n denotes the n dimensional Euclidian space (identified with n x 1 vectors of real numbers), and R^{nxm} denotes the set of all n x m real matrices. We will use A^T to denote the transpose matrix of A, $Ker(A)$ the kernel of A, $Im(A)$ the image of A and $dim(A)$ the dimensions of A.

2. Definitions and preliminary results

In this section, we will discuss the notion of linear bi sampled-data piecewise functioning system with finite discrete jumps. It turns out that the solutions of the tracking problem treated in this paper have this structure.

2.1 Two sampled-data time scale

Two sampled-data time scale used in our approach to tracking problem is illustrated in the Figure 1.

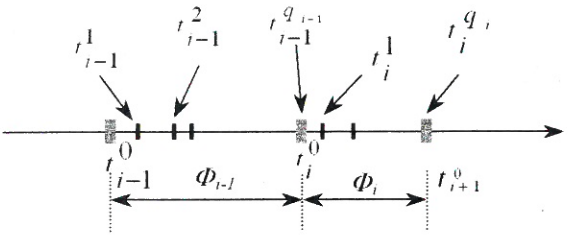


Figure 1: Two sampled-data time scale

The discrete moments are noted t_i^k where i indicates the time scale relating to commutations and k the time scale relating to system evolution between two commutations. Therefore two successive commutations t_{i-1}^0 and t_i^0 delimit the piece noted Φ_{i-1}.

Inside the piece Φ_{i-1} the subscript does not change while the superscript increases from 0 (initial moment) to q_{i-1}.

Generally the time length of a piece is not constant nor the sampling period and the number of sampling periods inside one piece. Finally, the last sampling moment of the piece Φ_{i-1} is superposed with the first sampling moment of the piece Φ_i.

$$t_{i-1}^{q_{i-1}} = t_i^0$$

2.2 Piecewise functioning sampled-data system

A sampled-data system with piecewise functioning is a system whose properties are completely defined over all pieces Φ_i by the following state-space model:

$$x_i^{k+1} = A_i.x_i^k + B_i^c.u_i^k \text{ , } k = 0,\ldots,q_i - 1 \text{ .[2.1a]}$$

$$y_i^k = C_i.x_i^k \text{ .[2.1b]}$$

$$x_i^0 = B_i^d.v_i^0 \text{ .[2.1c]}$$

$x_i^k \in R^n$ denotes the state, $y_i^k \in R^m$ the system output at t_i^k. Two inputs spaces are then defined as

$u_i^k \in R^r$ that is a control inside a piece and

$v_i^0 \in R^\sigma$ that is a control defining the initial state x_i^0 for considered piece.

Generally $x_i^0 \neq x_{i-1}^{q_{i-1}}$, implies the discontinuity at two pieces transition. (Figure 2.).

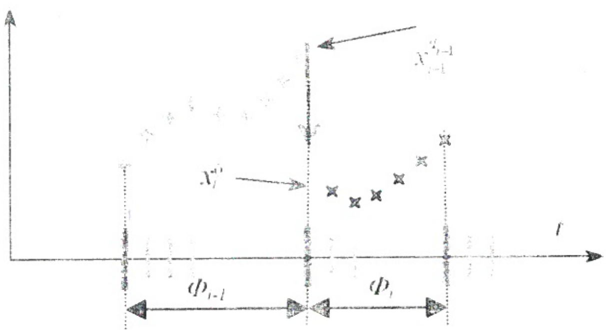


Figure 2: Discontinuity at commutation moments

The matrices A_i, B_i^c and C_i are commonly used state matrices with appropriate dimensions. The matrix B_i^d defines the relation $v_i^0 \rightarrow x_i^0$.

REMARK. The previous system definition can be applied on standard sampled-data system without any discontinuities or limited jumps. In this case the following definitions are sufficient in the relation [2.1c] :

$B_i^d = I_n$ and $v_i^0 = x_{i-1}^{q_{i-1}}$.

These guaranty the continuity at commutation moments:

$x_i^0 = x_{i-1}^{q_{i-1}}$.

3. Tracking problem – bi-sampled controller

In this section we will define the two time scales as follows:

The pieces of constant time length are noted T_e,

Inside each piece, the system evolution is defined by a sampling period of constant length t_e with $T_e = q.t_e$, and q positive integer.

3.1 Controller definition

We will use previously defined piecewise functioning system in order to define a linear sampled-data time-invariant plant controller. The control objective is to track a predefined trajectory with one sampling period T_e delay.

All over the piece Φ_i, a plant is defined as follows:

$x_i^{k+1} = f.x_i^k + h.u_i^k \quad k = 0,\dots,\ q-1$,[3.1a]

$y_i^k = C.x_i^k$,[3.1b]

$x_i^0 = x_{i-1}^q$.[3.1c]

In the same manner, during Φ_i, the controller is defined by expressions:

$\lambda_i^{k+1} = \alpha.\lambda_i^k \quad k = 0,\dots,q-1$,[3.2a]

$a_i^k = \gamma.\lambda_i^k$,[3.2b]

$\lambda_i^0 = \beta^d.v_i^0$,[3.2c]

λ_i^k and a_i^k are respectively the state vectors and the output vectors at the moment t_i^k. The controller output is used to control a plant, therefore: $u_i^k = a_i^k\ ,\ \forall i,k$. If we use the formalism defined by [2.1] then we may take $A_i = \alpha$, $B_i^c = 0$, $C_i = \gamma$ and $B_i^d = \beta^d$.

Dimensions have to be defined as follows:

dim(λ) = dim(x) = n and dim(a) = dim(u) = r,
dim(α) = dim(f) = $n\times n$, dim(h) = $n\times r$ and dim(γ) = $r\times n$.

It is important to note that a controller is free ($B_i^c = 0$). We will detail in the next sections how to exploit all possibilities of this type of controller ($B_i^c \neq 0$).

Finally, if $c(t)$ denotes the state trajectory to track, the tracking problem can be presented by the expression: $x((i+1)T_e) = c(iT_e),\ \forall\ i$.

It is equivalent to $x_{i+1}^0 = c_i^0$ or following [3.1c]:

$x_i^q = c_i^0$.[3.3]

3.2 Solution

By combining the systems of equations [3.1] and [3.2] during the piece Φ_i, one obtains:

$$x_i^q = f^q.x_i^0 + \left[f^{q-1}.h,\ f^{q-2}.h,\cdots f^0.h\right]\begin{bmatrix}\gamma.\alpha^0\\ \gamma.\alpha^1\\ .\\ .\\ \gamma.\alpha^{q-1}\end{bmatrix}.\lambda_i^0 \quad .[3.4]$$

In the matrix form, if we note:

$$M = \left[f^{q-1}.h,\ f^{q-2}.h,\cdots f^0.h\right]\begin{bmatrix}\gamma.\alpha^0\\ \gamma.\alpha^1\\ .\\ .\\ \gamma.\alpha^{q-1}\end{bmatrix} \quad ,[3.5]$$

$$x_i^q = f^q . x_i^0 + M . \lambda_i^0 \quad .[3.6]$$

In this case and regarding [3.2c], the solution of [3.3] implies:

$$\beta^d v_i^0 = M^{-1} . \left[c_i^0 - f^q . x_i^0 \right].$$

If M is non-singular matrix then M -1 exists, the following results are obtained:

$\beta^d = M^{-1}$, that identifies completely the controller and

$v_i^0 = c_i^0 - f^q x_i^0$, that identifies the state feedback.

REMARK. The initial state $\lambda_i^0 = \beta^d . v_i^0$ can now be computed. This initial state can also be obtained as the resolution of linear system [3.6]. The proposed control scheme guarantees that the trajectory tracking condition [3.3] is fulfilled, nevertheless the global closed-loop system stability is not guaranteed.

3.3 Conditions of existence of non-singular *M* (existence of M^{-1})

It is possible to note $M = K . \Omega$ with:

$$K = \left[f^{q-1} .h, f^{q-2} .h, \cdots f^0 .h \right] \text{ and } \Omega^T = [(\alpha^T)^0 . \gamma^T, (\alpha^T)^1 . \gamma^T, \cdots, (\alpha^T)^{q-1} . \gamma^T] .$$

M is a *n x n* square matrix. It is invertible if and only if:

Ker(*K*.Ω)={0} which is equivalent to:
Ker(Ω)={0} and *Ker*(*K*) ∩ *Im*(Ω) = {0}.

REMARK. If $q = n$, the number of sampling periods between two commutations is equal to a plant order *n*, then *K* and Ω^T are respectively a plant controlability matrix and a controller observability matrix. Generally the matrix M^{-1} does not exist if $q<n$.

3.4 Architecture

The Figure 3 shows the control architecture. The feedforward part of the control configuration is sampled with t_e sampling period. The feedback part is sampled with T_e period in order to generate λ_i^0 . The controller initialisation procedure at commutation time is marked with a point.

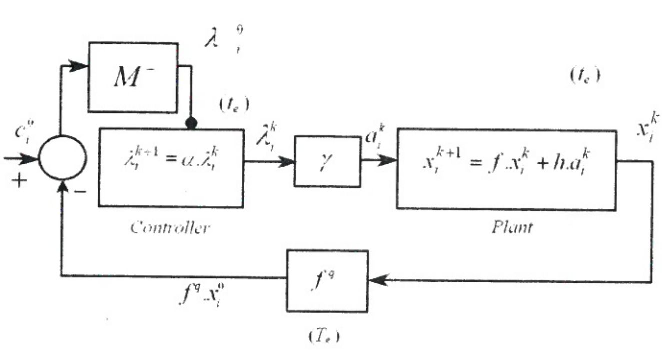


Figure 3. Sampled-data plant tracking architecture

3.5 Numerical example

3.5.1 Introduction

A plant is defined as follows:

$x' = A.x + B.u$, with :

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

This plant is the same as one presented in [8]. The plant sampling with t_e period leads to

$$f = e^{A.t_e} \text{ and } h = \int_0^{t_e} e^{A(t_e - \tau)}.B.d\tau .$$

Moreover let define T_e= 1s.

The piecewise continuous bi-sampled controller is then defined by:

$$\alpha = \begin{bmatrix} -84.139 & 83.005 \\ -94.863 & 58.156 \end{bmatrix}, \gamma = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

The state trajectory to track is defined as follows:

$$c(t) = \begin{bmatrix} -0.17 * t^3 + 2.5 * t^2 - 10 * t + 10 \\ -0.51 * t^2 + 5 * t - 10 \end{bmatrix}.$$

The matrices M^{-1}, f and h are obtained by the symbolic math. calculus and by assigning then to couple $\{t_e;q\}$ different values satisfying $q.t_e$=T_e=1s. The numerical couples that have been tested are: {2;0.5} , {4;0.25} and {20;0.05}.

3.5.2 Performances

The numerical simulation results are given in Figures 5a, b and c. In order to simplify the presentation and the comparison we have plotted on the same graph, the system response and the desired state trajectory T_e delayed, $w(t)=c(t- T_e)$. The values at commutation moments are plotted as inversed triangles. It is evident that the tracking is realized at commutation moments. On the other side, between two successive commutations the global control system is functioning in open loop configuration and its dynamics depends on α, γ and q. The consequence of this open-loop control is the intersample ripple between commutation moments. In our numerical example the intersample ripple does not exist for q = 2. Nevertheless, for q = 4 and 20, very important intersample ripples are observed. As expected, the control signal shows jumps at commutation moments and the magnitude of the control increases with the increase of q.

In the next section, we will introduce the control that enables the ripple reduction between commutations. This control is based on the same principle as piecewise functioning systems and exploits their properties.

4. Ripple reduction

In this section the previous tracking problem is completed with the additional constraint in order to minimize the error state-desired state trajectory between commutations. The optimal control based on the principle of Pontryagin [10] is used to achieve the new objective. Therefore, the bi-sampled controller using two control spaces defined in section 2. will be designed.

4.1 Controller design

4.1.1 Cost criterion

The cost criterion including the additional constraint for the piece Φ_i is defined as follows:

$$r = \frac{1}{2} * \sum_{k=0}^{q-1} \left[\left(c_{i-1}^{k} - x_i^k \right) E . \left(c_{i-1}^{k} - x_i^k \right)^T + a_i^{k^T} . G . a_i^k \right] . [4.1]$$

In this expression the matrices E and G are symmetric positive definite with appropriate dimensions. The cost criterion r minimization assures the reduction of intersample ripple and the control magnitude moderation. Following the optimal control theory the corresponding Hamiltonian is denoted as [11]:

$$H = -\frac{1}{2} * \sum_{k=0}^{q-1} \left[\left(c_{i-1}^{k} - x_i^k \right) E . \left(c_{i-1}^{k} - x_i^k \right)^T + a_i^{k^T} . G . a_i^k \right] + \sum_{k=0}^{q-1} \lambda_i^{k+1^T} . \left[f . x_i^k + h . a_i^k \right].$$

In this expression λ_k is the $n \times 1$ Lagrange multiplication vector. The principle of Pontryagin leads to:

$$\lambda_i^k = -E.(c_{i-1}^k - x_i^k) + f^T \lambda_i^{k+1} \quad \text{and}$$

$$a_i^k = G^{-1} h^T . \lambda_i^{k+1} , [4.2]$$

therefore:

$$\lambda_i^{k+1} = (f^T)^{-1} \lambda_i^k + (f^T)^{-1} . E . (c_{i-1}^k - x_i^k) \; . [4.3]$$

4.1.2 Comprehension

The equations [4.3] and [4.2] can be interpreted respectively as the state equation and the output equation of bi-sampled system for the piece Φ_i. The system input is $(c_{i-1}^k - x_i^k)$ that defines a unit gain feedback.

The value of λ_i^0 has to be determined in order to satisfy the first tracking objective ($x_i^q = c_i^0$ equation [3.3]).

4.2 Computation of λ_i^0

To compute λ_i^0 , the augmented system has to be designed using equations [3.1a], [4.2] and [4.3]:

$$\begin{bmatrix} x_i^{k+1} \\ \\ \lambda_i^{k+1} \end{bmatrix} = \mathrm{H} . \begin{bmatrix} x_i^k \\ \\ \lambda_i^k \end{bmatrix} + \mathrm{K} . c_{i-1}^k , [4.4]$$

with:

$$\mathrm{H} = \begin{bmatrix} [f - h.G^{-1}.h^T.(f^T)^{-1}.E] & [hG^{-1}.h^T.(f^T)^{-1}] \\ [-(f^T)^{-1}.E] & -(f^T)^{-1} \end{bmatrix},$$

$$K = \begin{bmatrix} h.G^{-1}.h^T.(f^T)^{-1}.E \\ (f^T)^{-1}.E \end{bmatrix},$$

The resolution gives the following solution:

$$\begin{bmatrix} x_i^q \\ \lambda_i^q \end{bmatrix} = H^q . \begin{bmatrix} x_i^0 \\ \lambda_i^0 \end{bmatrix} + \begin{bmatrix} H^{q-1}.K \cdots H^{q-2}.K \cdots H^0.K \end{bmatrix} \begin{bmatrix} c_{i-1}^0 \\ c_{i-1}^1 \\ . \\ . \\ c_{i-1}^{q-1} \end{bmatrix} .[4.5]$$

Finally let note:

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = H^q \text{ and } \begin{bmatrix} I_h \\ I_h \end{bmatrix} = \begin{bmatrix} H^{q-1}.K \cdots H^{q-2}.K \cdots H^0.K \end{bmatrix} \begin{bmatrix} c_{i-1}^0 \\ c_{i-1}^1 \\ . \\ . \\ c_{i-1}^{q-1} \end{bmatrix} .[4.6]$$

These lead, for x_i^q, to following result:

$x_i^q = \Theta_{11}.x_i^0 + \Theta_{12}.\lambda_i^0 + I_h$ and we want to achieve $x_i^q = c_i^0$.

Thus, the initial condition for bi sampled controller is:

$$\lambda_i^0 = \Theta_{12}^{-1}.\left[c_i^0 - \Theta_{11}.x_i^0 - I_h\right].[4.7]$$

4.3 Outline

If Θ_{12} non singular, the bi sampled controller is completely defined by:

$$\lambda_i^{k+1} = \alpha.\lambda_i^k + \beta^c.(c_{i-1}^k - x_i^k),$$

$$a_i^k = \gamma.\lambda_i^{k+1},$$

$$\lambda_i^0 = \beta^d.v_i^0,$$

with

$\alpha = (f^T)^{-1}$ and $\beta^c = (f^T)^{-1}.E$, following the equation 4.3,

$\gamma = G^{-1}h^T$, following the equation 4.2 and

$\beta^d = \Theta_{12}^{-1}$, following 4.7.

Therefore, the expression $v_i^0 = \left[c_i^0 - \Theta_{11}.x_i^0 - I_h\right]$ contained in 4.7 expresses the feedback at commutation moments.

4.4 Conditions of existence of non-singular Θ_{12}

In the case if q tends toward infinite value and if the product $q.t_e$ remains finite equal to T_e it is possible to express the geometrical condition of existence of the non-singular matrix Θ_{12} [6]. In the case when q is finite value there is only numerical condition of non-singular Θ_{12} existence i.e. $\det(\Theta_{12}) \neq 0$.

4.5 Tracking architecture with intersample ripple reduction

The architecture of the tracking system with intersample ripple is given in the Figure 4.

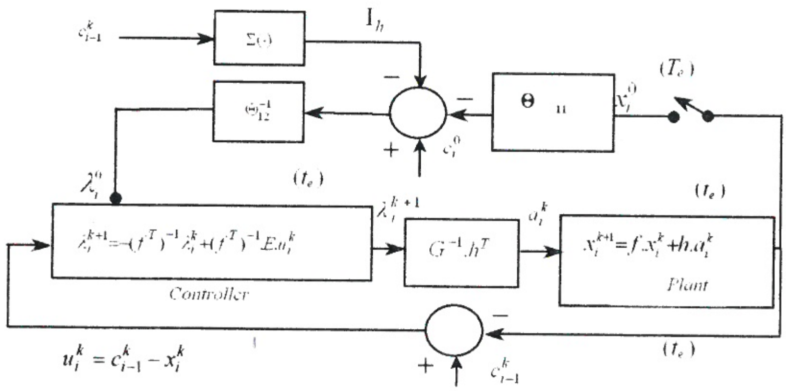


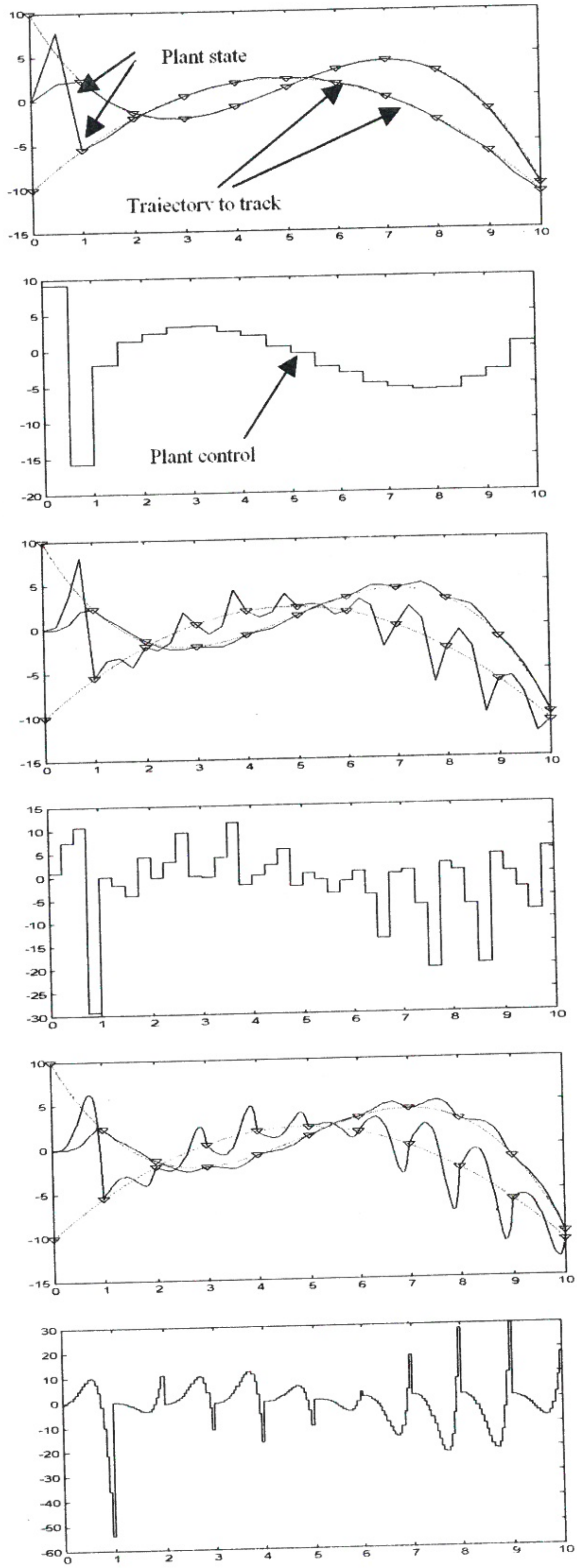
Figure 4: Tracking architecture with intersample ripple reduction

The block $\Sigma(\cdot)$ is defined following the equation 4.6. It is important to note that the controller has two inputs λ_i^0 and u_i^k fitting with two time scales that are defined using T_e and t_e. The procedure of controller initialisation at commutation is marked by a point.

4.5 Numerical example

We have used a plant defined in previous numerical example with the same trajectory to be tracked and the same couples $\{q, t_e\}$ in order to compare performances of two different tracking strategies.

We have noticed that for the couple {2;0.5}, there is no difference between two tracking strategies (dead beat control in two cases). On the other hand, for the couples {4;0.25} et {20 ;0.05}, the tracking with intersample ripple reduction reduces strongly the oscillations between the moments of commutation. In addition the second tracking strategy makes it possible to obtain a control signal of lower magnitude without strong discontinuities at commutation moments.



a

b

c

Figures 5-a, b, c. Discrete time tracking
5-a: q=2, t_e=0.5 s., T_e=1 s.
5-b: q=4, t_e=0.25 s., T_e=1 s.
5-c: q=20, t_e=0.05 s., T_e=1 s.

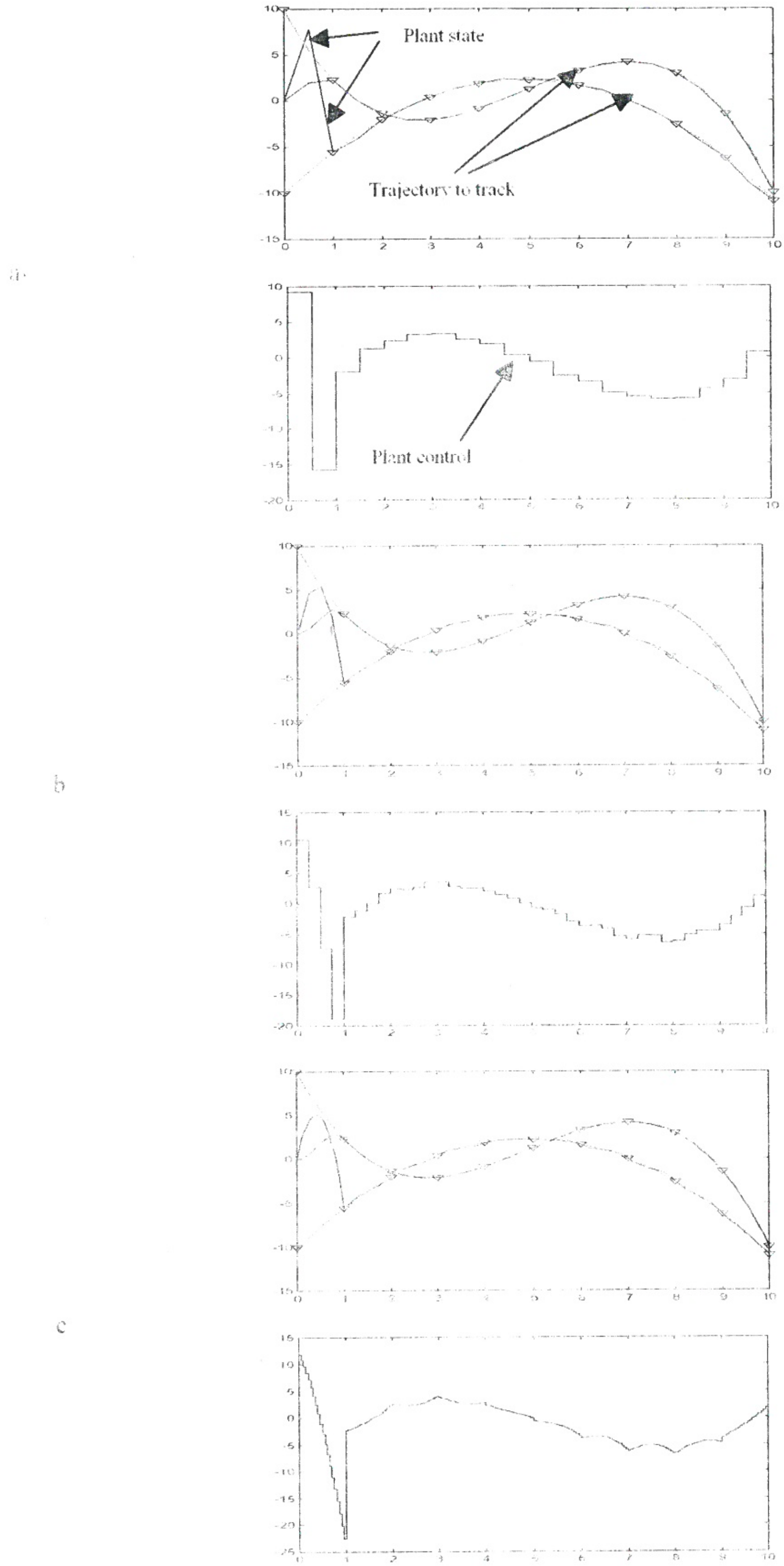


Figure 6-a, b, c. Discrete time tracking, ripple reduction

6-a: q=2, t_e=0.5 s., T_e=1 s.

6-b: q=4, t_e=0.25 s., T_e=1 s.

6-c: q=20, t_e=0.05 s., T_e=1 s.

For very large values of q satisfying the equality $q.te=Te=1$ $s.$ we have the piecewise continuous system defined in [6] for both tracking (Figure 7.) and ripple free tracking (Figure 8.)

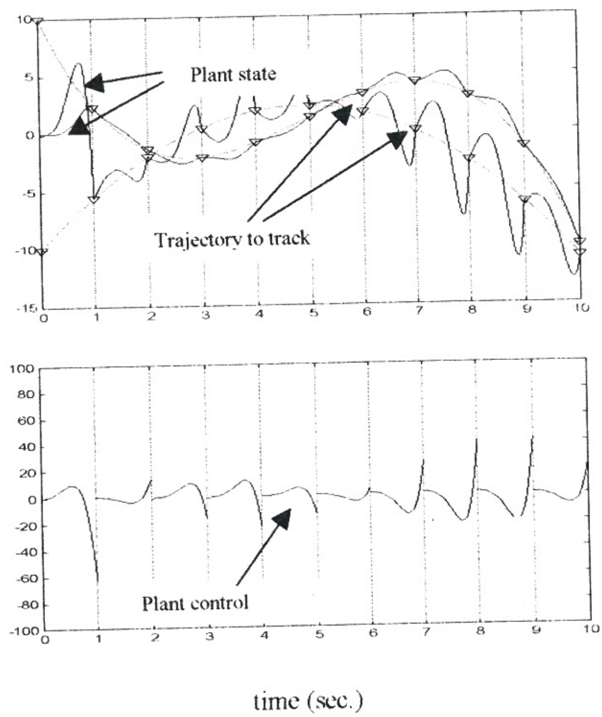


Figure 7: Continuous time tracking (T_e=1 s. [8])

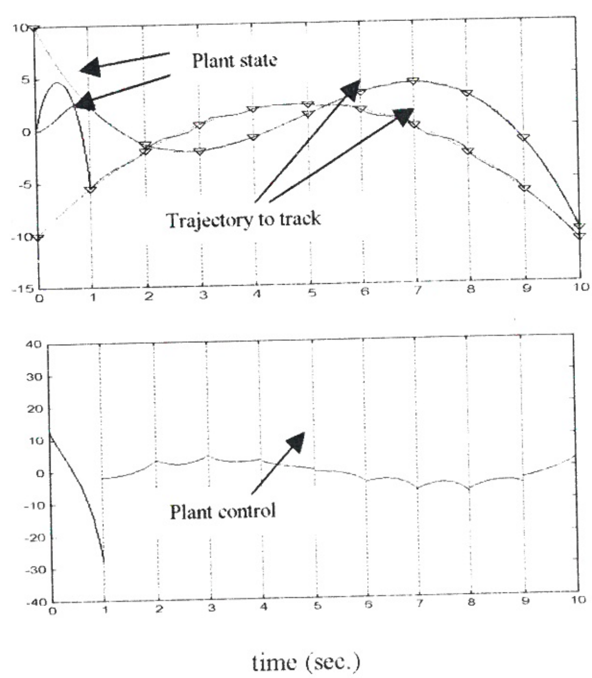


Figure 8: Ripple free continuous time tracking (T_e=1 s. [6])

Conclusion and future investigations

The study given in this article is the continuation of our works on the piecewise functioning systems presented in [8]. Two time scale introduction for bi-sampled controllers, based on piecewise functioning systems, leads to a specific formalism enabling the definition of q parameter where $q=T_e/t_e$. Thus, interesting perspective appears concerning the possibility to control q and commutation moments in the same time. This possibility of q control can be interpreted as a new degree of freedom in the system control. For instance, we can plan to control q following exponential decreasing low (until $q=1$) in order to reduce the tracking time delay to t_e. Another advantage: the bi-sampled formalism can be easily implemented in real time. Finally this tracking strategy can be applied also to non linear plants.

On the other side a state observer design based on our piecewise functioning systems in the case when only a part of the state vector can be measured is one of our future objectives including also a study of the disturbed plant state case.

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