Noncommutative Connectives

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Abstract: The paper starts from discussing the noncommutativity for MV algebras, and continues with the noncommutative connectives in the context of fuzzy logic, respectively. The aim of the paper is to consider the case of a fuzzy logic with 4 values (the minimum number of values to admit noncommutativity), and to represent the corresponding algebra as linear transformations of the plane, entering this way the realm of the theory of general action.

Keywords: fuzzy logic, MV algebras, partially ordered semigroups, representation of semigroups, actions

1.The Algebra PL_4

The general idea of a connective (logical connective), not necessarily commutative, finds its natural cradle in the theory of partially ordered semigroups (see, for the theory of partially ordered semigroups Fuchs, [3]). Let us recall the general definitions:

Let S be a **partial ordered set** (the order being denoted by "$\leq$") and suppose $T : S \times S \to S$ be given such that:

i. T is associative $T(T(x, y),z) = T(x, T(y, z)) \quad \forall\, x,y,z \in S$

ii. T is increasing in each variable.

$(x \leq y \Rightarrow T(x, z) \leq T(y, z)$ and $T(z,x) \leq T(z, y) \qquad \forall\, z \in S$

Such an S is called a **partially ordered semigroup**.

In a setting a partially ordered semigroup is a set of "degrees of truth" (or a "graded world") which is ordered by "$\leq$ " together with a logical connective T (for example T may represent "conjunction" or "disjunction"). We consider the connective to be associative but not necessarily commutative. This can be the source of long discussions, but we do not enter into more detail here. Anyway, it seems that, for giving good reasons for noncommutativity, the idea of time has to be tacitly assumed. This is not so restrictive if one wants to apply the theory to "actions"; after all repeated actions are in general not commutative actions.

The condition ii) is a natural condition of compatibility between T and $\leq$.

Integral partially ordered semigroups will be next discussed: T has a neutral element 1 which is also the greatest element of S.

So we have

iii. $T(x,1)=T(1,x)=x$ for all $x\in S$ and $x \leq 1$ for all $x\in S$.

In fact the very simple case of $PL_4= \{0,a,b,1\}$ has been considered, the order being total $0 < a < b < 1$ and T being defined by

T	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	a	b	b
1	0	a	b	1

It is easy to prove the associativity of T; we see that PL_4 has not only an unity but also a 'zero" 0 which is the least element of PL_4.

The intuition behind T is "noncommutative conjunction" so a noncommutative "and" such that) is the "false", 1 is the "absolute truth" and a, b are "degrees" of truth.

The noncommutativity is T(a b) $= 0 \neq$ T(b, a) = a. As usual there are two implications "$\rightarrow$" and "$\rightsquigarrow$" connected with T (in fact the corresponding residuations). We obtain:

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	a	a	1	1
1	0	a	b	1

$$x \rightarrow y = \max \{z;\ T(z, x) \leq y\}$$

$\rightsquigarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	0	a	1	1
1	0	a	b	1

$$x \rightarrow y = \max \{z;\ T(x, z) \leq y\}$$

PL_4 will not be discussed into more detail. The next Section represents PL_4 as linear maps of $\Re^2$.

2. A Representation of PL_4

Let End($\Re^2$) be the algebra of the linear endomorphisms of $\Re^2$. By 0 the null map is defined and by 1 the identity map is defined. Let $\varepsilon : \Re^2 \to \Re^2$ be the "twisted" projection $\varepsilon(x,y)=(0,x)$ and $q : \Re^2 \to \Re^2$ defined by $q(x,y)=(0,y)$.

We have

$$(\varepsilon \circ q)(x,y) = \varepsilon(o,y) = (0,0) \qquad \text{so} \qquad \varepsilon \circ q = 0$$

$$(q \circ \varepsilon)(x,y) = q(0,x) = (0,x) \qquad \text{so} \qquad q \circ \varepsilon = \varepsilon$$

and clearly $\varepsilon^2 = 0$ and $q^2 = q$.

The map $PL_4 \to \text{End}(\Re^2)$ given by $0 \to 0$, $1 \to 1$, $a \to \varepsilon$, $b \to q$ is considered.

It is easy to see that a representation of the semigroup PL_4 in End($\Re^2$) is obtained.

We can think of ε, q as actions in the place and the logic of PL_4 as the logic of repeated actions of type 0, 1, ε, q. The order relation in PL_4 can be transferred to 0, 1, ε, q and considered as some kind of "preference".

The study of finite noncommutative sets of degrees of truth may, in our opinion, be of interest as approximating the noncommutative fuzzy logic in [0,1] (for interesting noncommutative conjunctions in [0,1] see [2].

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