On Global Optimization and Fuzzy Mathematical Programming

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Abstract: A subset of $\mathbf{R}^n$ defined by restrictions can be represented by a projection of the maximum level set of a fuzzy set. Some consequences of this representation are presented in the context of global optimization with restrictions and mixed variables in $\mathbf{R}^n$.

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1. Indicator Fuzzy Sets

Let $\mathbf{R}$ be the set of real numbers, $\mathbf{Z}$ be the set of integer numbers and $K \subseteq \mathbf{R}^n$ be a set defined by restrictions

(1.1) $f_i(x) = 0 \quad (i = 1, \ldots, m)$

(1.2) $g_i(x) \le 0 \quad (i = 1, \ldots, p)$

(1.3) $x = (x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$

where

$(f_i : \mathbf{R}^n \to \mathbf{R}^n)_{i=1,\ldots,m}$, $(g_i : \mathbf{R}^n \to \mathbf{R})_{i=1,\ldots,p}$ are functions and for all $j = 1, \ldots, n$ the set X_j satisfies one and only one of the following conditions:

$X_j = \mathbf{R}$

or

$X_j = \mathbf{Z}$.

A fuzzy set

$\varphi_K : \mathbf{R}^n \times \mathbf{R}^p \to [0, 1]$

will be called an *indicator fuzzy set* of K if the following condition holds:

$x \in K$ *iff* $(\exists\, y \in \mathbf{R}^p)\ \varphi_K(x, y) = 1$.

For each set $K \subseteq \mathbf{R}^n$ defined by restrictions there exists an *indicator fuzzy set* φ_K of K. A concrete expression of φ_K will follow. Define m fuzzy sets as follows:

$\alpha_i : \mathbf{R}^n \to (0, 1]$

$\alpha_i(x) = \cos[\arctan[f_i(x)]]$

for all $x \in \mathbf{R}^n$ and $i=1, \ldots, m$.

Define p fuzzy sets as follows:

$\beta_i : \mathbf{R}^n \times \mathbf{R} \to (0, 1]$

$\beta_i(x, y_i) = \cos[\arctan[g_i(x) + y_i^2]]$

for all $x \in \mathbf{R}^n$, $y_i \in \mathbf{R}$ and $i = 1, \ldots, p$.

Let $J \subseteq \{1, \ldots, n\}$ be the set defined by

$j \in J$ iff $X_j = \mathbf{Z}$.

If $j \in J$ define a fuzzy set

$\gamma_j : \mathbf{R} \to [0, 1]$

$\gamma_j(x_j) = [\cos(\pi \cdot x_j)]^2$

for all $x_j \in \mathbf{R}$. Suppose that $n_J > 0$ is the number of elements of J.

Then an *indicator fuzzy set* of K can be defined by the following relation:

$$\varphi_K(x, y) = \frac{1}{3} \cdot \left[\alpha(x) + \beta(x, y) + \gamma(x)\right],$$

where

$$\alpha(x) = \frac{1}{m} \cdot \sum_{i=1}^{m} \alpha_i(x);$$

$$\beta(x, y) = \frac{1}{p} \cdot \sum_{i=1}^{p} \beta_i(x, y_i);$$

$$\gamma(x) = \frac{1}{n_J} \cdot \sum_{j \in J} \gamma_j(x_j),$$

for all $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $y = (y_1, \ldots, y_p) \in \mathbf{R}^p$.

Thus, every subset K of $\Re^n$ defined by restrictions can be represented by the projection on $\mathbf{R}^n$ of the maximum level set of an indicator fuzzy set φ_K.

2.Global Optimization Problems

Let $K \subseteq \mathbf{R}^n$ be a compact set and $\theta : \mathbf{R}^n \to \mathbf{R}$ be a continuous function. The problem consists in finding a global maximizer $x^0 \in K$ of θ on K, i.e.

$$(\forall x \in K)\ \theta(x) \le \theta(x^0).$$

Suppose that K is non-empty and K is defined by restrictions (1.1)-(1.3). This problem can then be reduced to the following standard form:

(2.1) $\max \theta(x)$

such that

(2.2) $\varphi_K(x, y) = 1$

where

(2.3) $(x, y) \in \mathbf{R}^n \times \mathbf{R}^p$

and $\varphi_K : \mathbf{R}^n \times \mathbf{R}^p \to [0, 1]$ is an *indicator fuzzy set* of K, i.e. $K = \pi_n[\varphi_K^{-1}(1)]$.

3. Discussion

Global optimization problems should be considered once standard optimization techniques failed because of the existence of non-global local optima.

The standard form (2.1)-(2.3) can be associated with a large class of global optimization problems such that the indicator fuzzy set φ_K is a Lipschitzian function as well. For example, consider the following problem:

$\max \theta(x)$

such that

$f_i(x) = 0 \quad (i = 1, \ldots, m)$

$x = (x_1, \ldots, x_n) \in X \subseteq \mathbf{R}^n$

where $f_i : X \to \mathbf{R}$ is a Lipschitzian function on X with the Lipschitz constant L_i ($i = 1, \ldots, m$). Let $K \subseteq \mathbf{R}^n$ be the set defined by the precedent restrictions, i.e.

$$K = \{ x \in X / f_i(x) = 0, \text{ for all } i \in \{1, \ldots, m\} \}.$$

Let $\varphi_K : X \to (0, 1]$ be the fuzzy set defined by

$$\varphi_K(x) = \frac{1}{m} \cdot \sum_{i=1}^{m} \cos[\arctan[f_i(x)]], \text{ for all } x \in X.$$

Then φ_K is an *indicator fuzzy set* of $K \subseteq X$, i.e.

$x \in K$ iff $x \in X$ and $\varphi_K(x) = 1$.

In addition, φ_K is a Lipschitzian function on X with the Lipschitz constant $L = \frac{1}{m} \cdot \sum_{i=1}^{m} L_i$.

Indeed the function $\rho: \mathbf{R} \to (0, 1]$ being defined by

$$\rho(u) = \cos[\arctan[u]], \ \forall u \in \Re$$

satisfies

$$|\rho'(u)| = |\sin[\arctan[u]]| \cdot \frac{1}{1+u^2} \le 1, \text{ for all } u \in \mathbf{R},$$

which implies

$$|\rho(u) - \rho(v)| \le |u - v|, \ \forall u, v \in \mathbf{R}$$

Therefore

$$|\varphi_K(x) - \varphi_K(y)| \le \left(\frac{1}{m} \cdot \sum_{i=1}^{m} L_i \right) \cdot \|x - y\|,$$

for all $x, y \in X$.

It follows that the following problem in standard form:

(3.1) $\max \theta(x)$

(3.2) $\varphi_K(x) = 1$

(3.3) $x \in X \subseteq \mathbf{R}^n$

can be solved using specific algorithms for the determination of global maximizers on sets defined by finitely many Lipschitzian inequalities [3].

A simple but useful optimality criterion for the problem in the standard form (3.1)-(3.3) is expressed by the following proposition.

Proposition

If $\lambda^0 \geq 0$ *is a positive real number such that* $x^0 \in X$ *is a global maximizer of the Lagrangean function*

$L(x, \lambda^0) = \theta(x) + \lambda^0 \cdot [\varphi_K(x) - 1]$

on the set X *then* x^0 *is a solution of the problem*:

$\max \theta(x)$

$\varphi_K(x) \geq \varphi_K(x^0)$

$x \in X \subseteq \mathbf{R}^n$.

The proposition follows from the fact that if $x^0 \in X$ is a global maximizer of $L(\cdot, \lambda^0)$ on X then the couple (x^0, λ^0) is a saddle point of the function

$$L^0(x, \lambda) = \theta(x) + \lambda \cdot [\varphi_K(x) - \varphi_K(x^0)],$$

$\forall (x, \lambda) \in X \times \mathbf{R}_+$, i.e. the inequalities

$$L^0(x, \lambda^0) \leq L^0(x^0, \lambda^0) \leq L^0(x^0, \lambda)$$

hold for all $x \in X$ and $\lambda \in \mathbf{R}_+$.

Also a penalty method can be used, based on the minimization of a sequence of functions having the form

$$P(x, r_k) = -\theta(x) + \frac{1}{r_k} \cdot [1 - \varphi_K(x)]^2,$$

where $(r_k)_{k \in N}$ is a decreasing sequence of positive real numbers such that $r_k \to 0$. Thus satisfactory solutions are reachable if only using local minimization techniques [1], but the use of branch-and-bound methods is the most recommended [3].

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