Some Algebraic Properties of Non-commutative Fuzzy Structures

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Abstract: Pseudo-BL algebras[2, 5] are non-commutative generalizations of BL-algebras, algebraic structures for Basic Logic, the fuzzy logic introduced by Hájek [7]. In this paper we study some algebraic properties concerning co-annihila-tors and minimal prime filters on pseudo-BL algebras, extending part of the results obtained in [1] to distributive lattices.

Keywords: Non-commutative fuzzy logic, pseudo-BL algebra, co-annihilators, minimal prime filters

1 Introduction

BL-algebras are the algebraic structures for Hájek's Basic Logic [7]. The main example of a BL-algebra is the interval $[0,1]$ endowed with the structure induced by a t-norm. MV-algebras, the Gödel algebras and product algebras are the most known classes of BL-algebras. Recent investigations have been concerned with non-commutative generalizations for these structures.

In [2, 5], pseudo-BL algebras were defined as non-commutative generalizations of BL-algebras. In [4], there was introduced a notion of pseudo-t-norm, in order to recapture some of the properties of pseudo-BL algebras. For the interval $[0,1]$, this notion induces more general algebras named weak pseudo-BL algebras.

Davey [1] studied the interrelation between minimal prime ideals conditions and annihilators conditions on distributive lattices.

In this paper we extend some of Davey's results to pseudo-BL algebras. The same results hold in BL-algebras too, since they are a particular case.

2 Definitions and First Properties

A *pseudo-BL algebra* ([2, 5]) is an algebra $\mathbf{A} = (A, \wedge, \vee, \odot, \rightsquigarrow, \rightarrow, 0, 1)$ with five binary operations $\wedge, \vee, \odot, \rightsquigarrow, \rightarrow$ and two constants $0, 1$ such that:

(A1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;
(A2) $(A, \odot, 1)$ is a monoid;
(A3) $a \odot b \leq c$ iff $a \leq b \rightsquigarrow c$ iff $b \leq a \rightarrow c$;
(A4) $a \wedge b = (a \rightsquigarrow b) \odot a = a \odot (a \rightarrow b)$;
(A5) $(a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$;
$(a \rightarrow b) \vee (b \rightarrow a) = 1$.

In the sequel, we shall agree that the operations $\wedge, \vee, \odot$ have priority to the operations $\rightsquigarrow, \rightarrow$. Sometimes, for the sake of clearness, we shall put parantheses even if superfluous.

It is proved in [2] that commutative pseudo-BL algebras are BL-algebras. For details on BL-algebras see [7, 8].

A pseudo-BL algebra $\mathbf{A}$ is non-trivial iff

$0 \neq 1$. For any pseudo-BL algebra $\mathbf{A}$, the reduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. A *pseudo-BL chain* is a linear pseudo-BL algebra, i.e. a pseudo-BL algebra such that its lattice order is total.
For any $a \in A$, we define $a^{\sim} = a \rightsquigarrow 0$ and $a^{-} = a \rightarrow 0$. We shall write $a^{\approx}$ instead of $(a^{\sim})^{\sim}$ and $a^{=}$ instead of $(a^{-})^{-}$.
The set of natural numbers is denoted by ω. We define $a^0 = 1$ and $a^n = a^{n-1} \odot a$ for $n \in \omega - \{0\}$. The *order* of $a \in A$, in symbols $ord(a)$, is the smallest $n \in \omega$ such that $a^n = 0$. If no such n exists, then $ord(a) = \infty$.
The following properties hold in any pseudo-BL algebra $\mathbf{A}$ and will be used in the sequel. See [2] for details.
(1) $a \odot b \leq a, b$;
(2) $a \leq b$ implies $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$;
(3) $a \odot b \leq a \wedge b$;
(4) $(a \vee b) \odot (a \vee c) \leq a \vee (b \odot c)$
Let $\mathbf{A}$ be a pseudo-BL algebra. According to [2], a *filter* of $\mathbf{A}$ is a non-empty subset F of A such that for all $a, b \in A$,
(i) if $a, b \in F$, then $a \odot b \in F$;
(ii) if $a \in F$ and $a \leq b$, then $b \in F$.
By (3), it is obvious that any filter of $\mathbf{A}$ is also a filter of the lattice $L(A)$.
A filter F of $\mathbf{A}$ is *proper* if $F \neq A$. A proper filter P of $\mathbf{A}$ is *prime* if for all $a, b \in A$, $a \vee b \in P$ implies $a \in P$ or $b \in P$. $Spec(A)$ will denote the set of prime filters of the pseudo-BL algebra $\mathbf{A}$.
A proper filter U of $\mathbf{A}$ is an *ultrafilter* (or a *maximal filter*) if no other proper filter contains it. We shall denote by $\mathcal{M}(A)$ the intersection of all ultrafilters of $\mathbf{A}$. Obviously, $\mathcal{M}(A)$ is a proper filter of $\mathbf{A}$.
Some properties of filters, to be used in the sequel, are reminded.

Proposition 2.1 ([2], Theorem 3.25)
Let F be a filter of the pseudo-BL algebra $\mathbf{A}$ and let S be a $\vee$-closed subset of A (i.e. if $a, b \in S$, then $a \vee b \in S$) such that $F \cap S = \emptyset$. Then there exists a prime filter P of $\mathbf{A}$ such that $F \subseteq P$ and $P \cap S = \emptyset$.

Proposition 2.2 Any proper filter of $\mathbf{A}$ can be extended to a prime filter.

Proof: By [2], Corollary 3.26. □

Proposition 2.3 ([2], Corollary 3.32)
Any ultrafilter of $\mathbf{A}$ is a prime filter of $\mathbf{A}$.

Proposition 2.4 ([2], Remark 3.33)
Any proper filter of $\mathbf{A}$ can be extended to an ultrafilter.

Let $X \subseteq A$. The filter of $\mathbf{A}$ generated by X will be denoted by $< X >$. Given $< \emptyset >= \{1\}$ and $< X >= \{a \in A \mid x_1 \odot \cdots \odot x_n \leq a$ for some $n \in \omega - \{0\}$ and some $x_1, \cdots, x_n \in X\}$ if $\emptyset \neq X \subseteq A$. For any $a \in A$, $< a >$ denotes the principal filter of $\mathbf{A}$ generated by $\{a\}$. It follows that $< a >= \{b \in A \mid a^n \leq b$ for some $n \in \omega - \{0\}\}$.

Proposition 2.5 ([2], Lemma 3.11)
For any $a, b \in A$,
$< a \vee b >=< a > \cap < b >$.

Let us denote by $\mathcal{F}(\mathcal{A})$ the set of all filters of $\mathbf{A}$. Then

Proposition 2.6 ([2], Proposition 3.8)
$(\mathcal{F}(\mathcal{A}), \subseteq)$ is a complete lattice. For every family $\{F_i\}_{i \in I}$ of filters of A, we have that $\bigwedge_{i \in I} F_i = \cap_{i \in I} F_i$ and $\bigvee_{i \in I} F_i =< \cup_{i \in I} F_i >$.

3 Co-annihilators and Co-annihilator Filters

Let $\mathbf{A}$ be a pseudo-BL algebra, F be a filter of $\mathbf{A}$ and $a \in A$. The *co-annihilator of a relative to F* is the set $(F, a) = \{x \in A \mid x \vee a \in F\}$. To indicate the relevant pseudo-BL algebra, sometimes (F, a) is written as $(F, a)_A$. The co-annihilator $(< b >, a)$ is abbreviated to (b, a).

Proposition 3.1 Let F be a filter of $\mathbf{A}$ and $a \in A$. Then (F, a) is a filter of $\mathbf{A}$.

Proof: Given $a \vee 1 = 1 \in F$, hence $1 \in (F, a)$. If $x \leq y$ and $x \in (F, a)$, then $x \vee a \in F$ and $x \vee a \leq y \vee a$, so $y \vee a \in F$, that is $y \in (F, a)$. Suppose now that $x, y \in (F, a)$, i.e. $x \vee a, y \vee a \in F$. It follows that $(x \vee a) \odot (y \vee a) \in F$. But, by (4), it follows that $(x \vee a) \odot (y \vee a) \leq (x \odot y) \vee a$. Hence, $(x \odot y) \vee a \in F$, so $x \vee y \in (F, a)$. □

Proposition 3.2 Let F, G be filters of $\mathbf{A}$ and $a, b \in A$. Then
(i) $F \subseteq (F, a)$;
(ii) $a \leq b$ implies $(F, a) \subseteq (F, b)$;
(iii) $F \subseteq G$ implies $(F, a) \subseteq (G, a)$;
(iv) $(F, a) = A$ iff $a \in F$;
(v) $(F, a \wedge b) = (F, a) \cap (F, b)$;
(vi) $(F \cap G, a) = (F, a) \cap (G, a)$;
(vii) $(b, a) = (b, a \wedge b) = (a \vee b, a) = (b, a \odot b)$.

Proof: (i) Let $x \in F$. Then $x \vee a \geq x \in F$, hence $x \vee a \in F$. That is, $x \in (F, a)$.
(ii) Let $x \in (F, a)$. Then $x \vee a \in F$ and $x \vee a \leq x \vee b$, since $a \leq b$. It follows that $x \vee b \in F$, that is $x \in (F, b)$.
(iii) Let $x \in (F, a)$. Then $x \vee a \in F \subseteq G$, hence $x \in (G, a)$.
(iv) If $(F, a) = A$, then $0 \in (F, a)$, hence $a = a \vee 0 \in F$. Conversely, if $a \in F$, then for any $x \in A$, $a \leq x \vee a$, so $x \vee a \in F$. That is, for any $x \in A$, $x \in (F, a)$.
(v) Since $a \wedge b \leq a, b$, by (ii), it follows that $(F, a \wedge b) \subseteq (F, a) \cap (F, b)$. Conversely, let $x \in (F, a) \cap (F, b)$, i.e. $x \vee a \in F$ and $x \vee b \in F$. Since F is also a filter of the distributive lattice $L(A)$, it results that $x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b) \in F$. That is, $x \in (F, a \wedge b)$.
(vi) Applying (iii) and the fact that $F \cap G \subseteq F, G$, it follows that $(F \cap G, a) \subseteq (F, a) \cap (G, a)$. The converse inclusion is obvious.
(vii) Since $a \odot b \leq a \wedge b \leq a$, by (ii), it comes that $(b, a \odot b) \subseteq (b, a \wedge b) \subseteq (b, a)$. We shall prove now that $(b, a) \subseteq (b, a \odot b)$. Let $x \in (b, a)$, so $x \vee a \in < b >$. Then, there is $n \in \omega - \{0\}$ such that $b^n \leq x \vee a$. Twice applying (2) and (4), it follows that $b^{n+1} \leq (x \vee a) \odot b \leq (x \vee a) \odot (x \vee b) \leq x \vee (a \odot b)$. Hence, $x \vee (a \odot b) \in < b >$, that is $x \in (b, a \odot b)$. Thus, we have proved that $(b, a) = (b, a \wedge b) = (b, a \odot b)$. Applying Proposition 2.5, (v) and (iv), it comes that $(a \vee b, a) = (< a \vee b >, a) = (< a > \cap < b >, a) = (< a >, a) \cap (< b >, a) = A \cap (b, a) = (b, a)$. □

If X is a non-empty subset of A, then $^{\perp}X = \{a \in A \mid x \vee a = 1 \text{ for any } x \in X\}$ is a filter of $\mathbf{A}$ called the *co-annihilator filter of* X (see [2]). It is easy to see that for any $a \in A$, $^{\perp}a = (\{1\}, a) = \{x \in A \mid x \vee a = 1\}$.

Proposition 3.3 ([2], Proposition 3.37) Let $\emptyset \neq X, Y \subseteq A$. Then,
(i) If $X \subseteq Y$, then $^{\perp}Y \subseteq {}^{\perp}X$;
(ii) $X \subseteq {}^{\perp\perp}X$;
(iii) $^{\perp}X = {}^{\perp\perp\perp}X$;
(iv) $^{\perp}X = {}^{\perp} < X >$;
(v) $< X > \cap {}^{\perp}X = \{1\}$.

Now, let us recall some facts from the lattice theory (see [6]). Let $(L, \vee, \wedge, 0)$ be a lattice with 0. An element $a^* \in L$ is a *pseudocomplement* of $a \in L$ iff $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \leq a^*$. The lattice L is called *pseudocomplemented* iff every element has a pseudocomplement.

Proposition 3.4 Let $\mathbf{A}$ be a pseudo-BL algebra. Then the lattice $\mathcal{F}(A)$ is pseudocomplemented. For any filter F, its pseudocomplement is $^{\perp}F$.

Proof: By Proposition 3.3(v), we have that $F \cap {}^{\perp}F = \{1\}$. Let G be a filter of $\mathbf{A}$ such that $F \cap G = \{1\}$. We shall prove that $G \subseteq {}^{\perp}F$. Let $a \in G$. For any $x \in F$, we have that $x \vee a \in F \cap G = \{1\}$, since $x \vee a \geq x \in F$ and $a \vee x \geq a \in G$. Hence, $x \vee a = 1$ for any $x \in F$, so $a \in {}^{\perp}F$. It follows that $^{\perp}F$ is the pseudocomplement of F. □

Let $Co - An(A) = \{^{\perp}X \mid X \subseteq A\}$ be the set of co-annihilator filters of $\mathbf{A}$. Applying Proposition 3.3(iv), we get that $Co - An(A) = \{^{\perp}F \mid F \in \mathcal{F}(A)\}$. Hence, $Co - An(A)$ is the set of pseudocomplements of the pseudocomplemented lattice $\mathcal{F}(A)$. Applying known results from the lattice theory, the following proposition follows.

Proposition 3.5 Let **a** be a pseudo-BL algebra and F, G filters of $\mathbf{A}$. Then
(i) $^{\perp}1 = A$ and $^{\perp}A = \{1\}$;
(ii) $\{1\}, A \in Co-An(A)$;
(iii) $F \in Co-An(A)$ iff $^{\perp\perp}F = F$;
(iv) if $F, G \in Co-An(A)$, then $F \cap G \in Co-An(A)$;
(v) if $F, G \in Co-An(A)$, then
$F \vee_{Co-An(A)} G = {}^{\perp}({}^{\perp}F \cap {}^{\perp}G)$;
(vi) $(Co-An(A), \cap, \vee_{Co-An(A)}, \perp, \{1\}, A)$ is a Boolean algebra;
(vii) $^{\perp\perp}(F \cap G) = {}^{\perp\perp}F \cap {}^{\perp\perp}G$.

Proof: See [6], Theorem 6.4, p. 58 and Theorem 15.1, p. 166 □

Proposition 3.6 Let $a, b \in A$. Then
(i) $a \leq b$ implies $^{\perp}a \subseteq {}^{\perp}b$ and $^{\perp\perp}b \subseteq {}^{\perp\perp}a$;
(ii)$^{\perp\perp}a = \{x \in A \mid x \vee y = 1$ for any $y \in A$ such that $y \vee a = 1\}$;
(iii) $a \in {}^{\perp\perp}a$;
(iv) $^{\perp\perp}a \cap {}^{\perp\perp}b = {}^{\perp\perp}(a \vee b)$.

Proof: (i) $a \leq b$ implies $< b > \subseteq < a >$. Hence, applying Proposition 3.3(v), (ii), we have that $^{\perp}a = {}^{\perp} < a > \subseteq {}^{\perp} < b > = {}^{\perp}b$. Applying again Proposition 3.3(ii), we get $^{\perp\perp}b \subseteq {}^{\perp\perp}a$.
(ii) By definition.
(iii) It follows from (ii).
(iv) Applying Proposition 3.3(iv), Proposition 3.5(vii) and Proposition 2.5 we get that $^{\perp\perp}a \cap {}^{\perp\perp}b = {}^{\perp\perp} < a > \cap {}^{\perp\perp} < b > = {}^{\perp\perp}(< a > \cap < b >) = {}^{\perp\perp} < a \vee b > = {}^{\perp\perp}(a \vee b)$.
□

4 Minimal Prime Filters Belonging to A Filter F

A prime filter M which is minimal in the poset of prime filters containing a filter F is called a *minimal prime filter belonging to* F. A minimal prime filter belonging to $\{1\}$ is simply called a *minimal prime filter*. Hence, a minimal prime filter of $\mathbf{A}$ is a minimal element of the poset $(Spec(A), \subseteq)$.

In the sequel, we shall present some results concerning minimal prime filters belonging to a filter. All these propositions are inspired by [1].

Proposition 4.1 If S is $\vee$-closed and F is a proper filter of $\mathbf{A}$, then there is a minimal prime filter M belonging to F and disjoint from S.

Proof: If F is a proper filter, then the set $\{P \in Spec(A) \mid F \subseteq P$ and $F \cap S = \emptyset\}$ is non-empty, by Proposition 2.1. Apply Zorn's Lemma to get a minimal element of this set. □

Proposition 4.2 Let F be a filter of $\mathbf{A}$ and M be a prime filter including F. The following are equivalent:
(i) M is a minimal prime filter belonging to F;
(ii) for all $a \in M$, there is $b \notin M$ such that $a \vee b \in F$.

Proof: (i)⇒(ii) Let $a \in M$ and let $S = \{a \vee b \mid b \in A - M\}$. If $a \vee b_1, a \vee b_2 \in S$, then $(a \vee b_1) \vee (a \vee b_2) = a \vee (b_1 \vee b_2) \in S$, since $A - M$ is $\vee$-closed, so $b_1, b_2 \in A - M$ implies $b_1 \vee b_2 \in A - M$. Hence, S is $\vee$-closed. Let us suppose that $F \cap S = \emptyset$. Applying Proposition 2.1, there exists a prime filter P such that $F \subseteq P$ and $P \cap S = \emptyset$. Since $a = a \vee 0$ and $0 \in A - M$, it comes that $a \in S$, so $a \notin P$. It follows that $M \neq P$, since $a \in M$ and $a \notin P$. If there is $x \in P$ such that $x \notin M$, then $x \in A - M$, so $a \vee x \in S$. But $x \leq a \vee x$ and $x \in P$, hence $a \vee x \in P$. We have got that $a \vee x \in P \cap S = \emptyset$, that is a contradiction. Hence, $P \subseteq M$ and $P \neq M$, which contradicts the fact that M is a minimal prime filter belonging to F. It follows that $F \cap S \neq \emptyset$. Hence, there is $b \in A - M$ such that $a \vee b \in F$.
(ii)⇒(i) Let P be a prime filter of $\mathbf{A}$ such that $F \subseteq P \subseteq M$. We shall prove that $M \subseteq P$ too. Let $a \in M$. Then there is $b \notin M$ such that $a \vee b \in F \subseteq P$. Since P is a prime filter of $\mathbf{A}$, from $a \vee b \in P$ it follows that $a \in P$ or $b \in P$. But $b \notin M$ and $P \subseteq M$,

so $b \notin P$. We get that $a \in P$. □

A simple induction gives the following corollary for $n \in \omega - \{0\}$.

Corollary 4.3 Let F be a filter of $\mathbf{A}$ and $M_0, \cdots M_n$ be $n+1$ distinct minimal prime filters belonging to F. Then, there are $a_0, \cdots a_n \in A$ such that $a_i \vee a_j \in F$ $(i \neq j)$ and $a_i \notin M_i$ $(i = 0, \cdots, n)$.

Proof: If $n = 1$, let $x_0 \in M_1 - M_0$ and $x_1 \in M_0 - M_1$. From the above proposition, it follows that there is $y_1 \notin M_0$ such that $x_1 \vee y_1 \in F$. Since M_0 is prime, from $x_0 \notin M_0$ and $y_1 \notin M_0$, we get that $x_0 \vee y_1 \notin M_0$. It follows that $a_0 = x_0 \vee y_1$ and $a_1 = x_1$ are the required elements. Assume the result is true for $n = k$ and let us prove it for $n = k+1$. Let $M_0, \cdots M_{k+1}$ be $k+2$ distinct minimal prime filters belonging to F. Let x_i $(i = 0, \cdots, k)$ satisfy $x_i \vee x_j \in F$ $(i \neq j)$ and $x_i \notin M_i$ $(i = 0, \cdots, k)$. For any $i = 0, \cdots k$, there is $y_i \in M_{k+1}$ such that $y_i \notin M_i$. If we take $y = y_1 \wedge \cdots \wedge y_k$, then $y \in M_{k+1} - \cup_{i=0}^{k} M_i$. Applying Proposition 4.2, we get $z \notin M_{k+1}$ such that $y \vee z \in F$. It follows that $a_i = x_i \vee y$ $(i = 0, \cdots k)$ and $a_{k+1} = z$ establish the result. □

¿From now on all indexed joins and meets range from 0 to n, where $n \in \omega - \{0\}$, and all joins of filters are taken in the lattice $\mathcal{F}(A)$ of filters of $\mathbf{A}$. A family of filters is *comaximal* if its join is A.

Proposition 4.4 Let F be a filter of $\mathbf{A}$. Then for $n \in \omega - \{0\}$ the following are equivalent:

(i) any $n+1$ distinct minimal prime filters belonging to F are comaximal;

(ii) any prime filter containing F contains at most n distinct minimal prime filters belonging to F;

(iii) if $a_0, \cdots, a_n \in A$ with $a_i \vee a_j \in F$ $(i \neq j)$, then $\bigvee_i (F, a_i) = A$;

(iv) $(F, \vee_j a_j) = \bigvee_i (F, \vee_{j \neq i} a_j)$ holds identically in $\mathbf{A}$.

Proof: (i)⇔(ii) Trivial.

(ii)⇒(iii) Let $a_0, \cdots, a_n \in A$ such that $a_i \vee a_j \in F$ $(i \neq j)$ and $\bigvee_i (F, a_i) \neq A$. It follows that $(F, a_i) \neq A$ for all $i = 0, \cdots, n$ and, by Proposition 3.2(iv), we get that $a_i \notin F$ for all i. Since $\bigvee_i (F, a_i)$ is a proper filter of $\mathbf{A}$, applying Proposition 2.2 we obtain a prime filter P of $\mathbf{A}$ such that $\bigvee_i (F, a_i) \subseteq P$. For $i = 0, \cdots, n$, let $S_i = \{x \vee y \mid x \leq a_i, y \notin P\}$. If $x_1 \vee y_1, x_2 \vee y_2 \in S_i$, then $x_1 \leq a_i, x_2 \leq a_i$ and $y_1, y_2 \notin P$. Hence $x_1 \vee x_2 \leq a_i$ and $y_1 \vee y_2 \notin P$, since P is prime. It follows that $(x_1 \vee y_1) \vee (x_2 \vee y_2) = (x_1 \vee x_2) \vee (y_1 \vee y_2) \in S_i$. Hence, S_i is $\vee$-closed. If $F \cap S_i \neq \emptyset$, then there are $x \leq a_i$ and $y \notin P$ such that $x \vee y \in F$. It follows that $a_i \vee y \in F$, hence $y \in (F, a_i)$. But $y \notin P$, so $(F, a_i) \nsubseteq P$, which is a contradiction. Hence $F \cap S_i = \emptyset$ for all i. Applying Proposition 4.1, for each i there is M_i a minimal prime filter belonging to F such that $M_i \cap S_i = \emptyset$. Suppose that $M_i \nsubseteq P$, so there is $x \in M_i$ such that $x \notin P$. Then $x = x \vee 0 \in S_i$, hence $x \in M_i \cap S_i = \emptyset$, that is a contradiction. Hence $M_i \subseteq P$ for all i, so $\bigvee_i M_i \subseteq P \neq A$. Since $a_i \notin P$, we get that $a_i \notin M_i$ for any i. But, $a_i \vee a_j \in F \subseteq M_i$ $(i \neq j)$ and M_i is prime, hence $a_j \in M_i$ $(i \neq j)$. It follows that $M_i \neq M_j$ $(i \neq j)$. Thus, we have obtained $n+1$ minimal prime filters belonging to F such that their join is not A, that is a contradiction with (ii).

(iii)⇒(iv) Applying Proposition 3.2(ii) it follows that $\bigvee_i (F, \vee_{j \neq i} a_j) \subseteq (F, \vee_j a_j)$. Thus, it is still to prove that $(F, \vee_j a_j) \subseteq \bigvee_i (F, \vee_{j \neq i} a_j)$. Let $x \in (F, \vee_j a_j)$, i.e. $x \vee \vee_j a_j \in F$. For any i, let $b_i = x \vee \vee_{j \neq i} a_j$. Then, $b_i \vee b_j = x \vee \vee_j a_j \in F$ for all $j \neq i$, hence, by (iii), we get that $\bigvee_i (F, b_i) = A$. From $x \in \bigvee_i (F, b_i)$, we get $k \in \omega - \{0\}$ and $y_1^0, \cdots, y_k^0 \in (F, b_0)$, $\cdots$, $y_1^n, \cdots, y_k^n \in (F, b_n)$ such that $y_1^0 \odot \cdots \odot y_1^n \odot \cdots \odot y_k^0 \odot \cdots \odot y_k^n \leq x$. Letting $t_p^i = x \vee y_p^i$ $(i = 0, \cdots, n$, and $p = 1, \cdots, k)$, we have that $x \leq t_p^i$ and $t_p^i \in (F, b_i)$, since $y_p^i \leq t_p^i$ and $y_p^i \in (F, b_i)$. Since $x \leq t_p^i$, we have that $t_p^i \vee \vee_{j \neq i} a_j = t_p^i \vee x \vee \vee_{j \neq i} a_j = t_p^i \vee b_i \in F$, because $t_p^i \in (F, b_i)$. Hence, $t_p^i \in (F, \vee_{j \neq i} a_j)$ $(i = 0, \cdots, n, p = 1, \cdots, k)$. Applying (4), we get that $(t_1^0 \odot \cdots \odot t_1^n \odot \cdots \odot t_k^0 \odot \cdots \odot t_k^n) = (x \vee y_1^0) \odot \cdots \odot (x \vee y_1^n) \odot \cdots \odot (x \vee y_k^0) \odot \cdots \odot (x \vee$

$y_k^n) \leq x \vee (y_1^0 \odot \cdots \odot y_1^n \odot \cdots \odot y_k^0 \odot \cdots \odot y_k^n) = x$. Thus, we have got that $x \in \bigvee_i (F, \vee_{j \neq i} a_j)$

(iv) $\Rightarrow$(i) Let $M_0, \cdots M_n$ be $n+1$ distinct prime filters belonging to F. Then, by Corrolary 4.3 there are $a_0, \cdots a_n \in A$ such that $a_i \vee a_j \in F$ $(i \neq j)$ and $a_i \notin M_i$ $(i = 0, \cdots, n)$. We have that for $n \in \omega - \{0\}$,

$\vee_j (\wedge_{k \neq j} a_k) = \wedge_{j<k} (a_j \vee a_k)$ is an identity in the class of distributive lattices, hence in the class of pseudo-BL algebras. We shall denote this identity by (I). Applying (I), it follows that $\vee_{j \neq i} (\wedge_{k \neq j} a_k) = \vee_{j \neq i} (a_i \wedge (\wedge (a_k \mid k \notin \{i, j\}))) =$

$a_i \wedge (\vee_{j \neq i} (\wedge (a_k \mid k \notin \{i, j\}))) = a_i \wedge (\wedge (a_j \vee a_k \mid j < k, i \notin \{j, k\}))$. But $\wedge (a_j \vee a_k \mid j < k, i \notin \{j, k\}) \in F$, hence, by Proposition 3.2(iv) we get that $(F, \wedge (a_j \vee a_k \mid j < k, i \notin \{j, k\})) = A$. Applying Proposition 3.2(v), it follows that $(F, \vee_{j \neq i} (\wedge_{k \neq j} a_k)) = (F, a_i \wedge (\wedge (a_j \vee a_k \mid j < k, i \notin \{j, k\}))) = (F, a_i) \cap (F, \wedge (a_j \vee a_k \mid j < k, i \notin \{j, k\})) = (F, a_i) \cap A = (F, a_i)$. If $x \in (F, a_i)$, then $x \vee a_i \in F \subseteq M_i$. Since M_i is a prime filter and $a_i \notin M_i$, we get that $x \in M_i$. Hence, $M_i \supseteq (F, a_i)$ for all $i = 0, \cdots, n$. It follows that $\bigvee_i M_i \supseteq \bigvee_i (F, a_i) =$

$\bigvee_i (F, \vee_{j \neq i} (\wedge_{k \neq j} a_k)) = (F, \vee_j (\wedge_{k \neq j} a_k))$, by (iv). But, applying (I), we obtain that $\vee_j (\wedge_{k \neq j} a_k) = \wedge_{j<k} (a_j \vee a_k) \in F$, so, by Proposition 3.2(iv), $(F, \vee_j (\wedge_{k \neq j} a_k)) = A$. It follows that $\bigvee_i M_i \supseteq A$, that is $\bigvee_i M_i = A$, as required. □

Corollary 4.5 Let $n \in \omega - \{0\}$. The following are equivalent:

(i) any $n+1$ distinct minimal prime filters are comaximal;

(ii) any prime filter contains at most n distinct minimal prime filters;

(iii) if $a_0, \cdots, a_n \in A$ such that $a_i \vee a_j = 1$ $(i \neq j)$, then $\bigvee_i {}^{\perp} a_i = A$;

(iv) ${}^{\perp}(\vee_j a_j) = \bigvee_i {}^{\perp}(\vee_{j \neq i} a_j)$ holds identically in **A**.

Proof: Take $F = \{1\}$ in Proposition 4.4. □

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