Control Of An Inverted Pendulum With the Help Of A Piece-wise Affine Fuzzy Controller

Jean-Yves Dieulot[1] and Pierre Borne[2]

LAIL- Laboratoire d'Automatique et Informatique Industrielle de Lille
[1]IAAL - Institut Agro-Alimentaire de Lille
Bâtiment C6, Cité Scientifique
59655 Villeneuve d'Ascq Cedex
[2]Ecole Centrale de Lille, Cité Scientifique
59651 Villeneuve d'Ascq Cedex
FRANCE
E-mails: dieulot@univ-lille1.fr; p.borne@ec-lille.fr

Abstract : A fuzzy controller, a conventional variable structure controller, and a region-wise affine fuzzy controller are compared for the control of an inverted pendulum. The region-wise affine controller is designed via a conventional fuzzy controller with a linear defuzzification algorithm; the input and output membership functions parameters are chosen in order to ensure the stability of the linearised system.

Keywords : fuzzy control, stability, fuzzy controller design

Jean- Yves Dieulot graduated from Institut Industriel du Nord (French Grande Ecole d'Ingénieurs) in 1990 and received the Ph. D degree in Control Engineering from the University of Lille in 1993. Since 1995 Dr. Dieulot has been an Associate Professor with IAAL (Food Processing Institute of the University of Lille). His research interests include fuzzy controllers and food engineering modelling and control.

Professor **Pierre Borne** received the MSc. degree in Physics in 1967 and the MSc. degree in Electronics, Mechanics and Applied Mathematics in 1968. He graduated from Institut Industriel du Nord (French Grande Ecole d'Ingénieurs) in 1968 and received Ph.D and DSc. in Control Engineering from the University of Lille (France) in 1970 and 1976, respectively. He is the author and co-author of more than 200 journal articles, book chapters and communications at conferences. He is the author of 6 books in automatic control, the co-editor of the Concise Encyclopedia of Modelling and Simulation (Pergamon, NY), and he organised 7 international conferences and symposia. He received an IEEE Outstanding Award in 1992 and the Kuhlmann Prize in 1994. Professor Borne was IMACS Vice-President from 1988 to 1994. He has been Fellow of the IEEE since 1996. Since 1999 he has been IEEE System, Man and Cybernetics President. He is currently Professor at the Ecole Centrale de Lille, France where he is Director of the Research and Head of the Department of Automatic Control. His activities concern automatic control and robust control, including the implementation of fuzzy controllers and neural networks.

1. Introduction

Experimental robustness properties of fuzzy controllers remain theoretically difficult to prove and their synthesis is still an open problem. The non-linear structure of the final controller is derived from all controllers at the different stages of fuzzy control [1], particularly from common defuzzification methods (such as Centre of Area). In general, fuzzy controllers have a region-wise structure given the partition of its input space by the fuzzification stage. Local controls designed in these regions are then combined into sets to make up the final global control. A partition of the state space can be found for which the controller has region-wise constant parameters [3], [4]. Moreover, each fuzzy controller tuning parameter (i.e. the shapes and the values of input or output variables membership functions) influences the values of parameters in several regions at the same time. In the particular case of a switching line separating the phase plane into one region where the control is positive whereas in the other it is negative, the fuzzy controller may be seen as a variable structure controller [5] [6] [7] [8]. This kind of a fuzzy controller can be assimilated to a variable structure controller with boundary layer such as in [9], for which stability theorems exist, but with a non-linear switching surface [7] [8].

With the use of trapezoidal input membership functions and appropriate composition and inference methods, it will be shown that it is possible to obtain rule membership functions which are region-wise affine functions of the controller input variable. We propose a linear defuzzification algorithm that keeps this region-wise affine structure and yields a piece-wise affine controller . A particular and systematic parameter tuning method will be given which allows to turn this controller into a variable structure-like controller. We will compare this region-wise affine controller with a Fuzzy and Variable Structure Controller through the application to an inverted pendulum control [10].

In the first Section, the equations of the inverted pendulum system will be recalled. In the second Section, the classical fuzzy and variable structure controllers, to be used in the system control, will be presented. A fuzzy controller of which output is an affine function of its input variables will then be designed, and a condition on the fuzzy controller parameters values will be set so that the resulting control should be piece-wise linear and stable when used to control linear systems. In the third Section, simulations will be referred for the three controllers and a short discussion on the piece-wise affine controller behaviour will be undertaken.

2. Inverted Pendulum System

The system to be controlled is the classical inverted pendulum system described in Figure 1 [10].

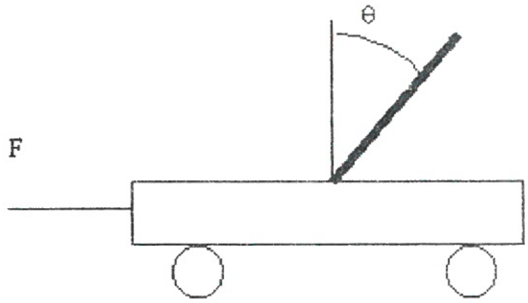

Figure 1. Inverted Pendulum

A force F is applied to the cart. The goal is to stabilise the rod to a position $\theta = 0$ from an initial position θ_0, following a reference trajectory

$$\theta_d = 0.5\theta_0 \left(1 + \cos(\pi t)\right). \qquad (1)$$

The length of the rod is $2l$, with $l = 1$ m, the mass of the rod is $m = 0.1$ kg, the mass of the cart is $m_c = 1$ kg.

The angle follows the equation:

$$\ddot{\theta} = f(\vartheta) + g(\vartheta)F \,, \qquad (2)$$

where F is the control, and

$$f(\theta) = \left(g \sin(\theta) - \frac{ml}{m + m_c} \cos(\theta) \sin(\theta) \dot{\theta}^2 \right) \Big/ \left(l \left(\frac{4}{3} - \frac{m}{m + m_c} \cos^2(\theta) \right) \right), \qquad (3)$$

$$g(\theta) = -\cos(\theta) \Big/ \left(l \left(\frac{4}{3} - \frac{m}{m + m_C} \cos^2(\theta) \right) \right). \qquad (4)$$

This model can be linearised around the equilibrium point using the model

$$\ddot{\theta} = \tilde{f}(\vartheta) + \tilde{g}(\vartheta)F = \left(g\theta - \frac{m}{m + m_C} F \right) \Big/ \left(l \left(\frac{4}{3} - \frac{m}{m + m_C} \right) \right) \qquad (5)$$

which can take the form :

$$\dot{x} = Ax + BF, \qquad (6)$$

We will define the control error as

$\varepsilon = \vartheta - \theta_d$ and the vector x as $x = \begin{pmatrix} \dot{\varepsilon} \\ \varepsilon \end{pmatrix}$.

3. Fuzzy and Variable Structure Controllers

A. Fuzzy Control of the Inverted Pendulum

The control error ε and its derivative $\dot{\varepsilon}$ are the controller inputs.

The proposed fuzzy controller [1] follows the linguistic rules $R_{j,k}$ such as:

$if\ \varepsilon\ is\ A_j \quad AND\ \ \dot{\varepsilon}\ is\ B_k \quad THEN\ u\ is\ U_{j,k}$,

where A_j and B_k are linguistic predicates. For the sake of simplicity, $U_{j,k}$ will be a scalar number.

A membership function $\mu_j(\varepsilon)$ to the predicate "$\varepsilon\ is\ A_j$" and $\mu_k(\dot{\varepsilon})$ of the predicate "$\dot{\varepsilon}\ is\ B_k$" can be defined. The value of the membership functions to the predicate $if\ \varepsilon\ is\ A_j \quad AND\ \ \dot{\varepsilon}\ is\ B_k$ is defined as

$$\mu_{j,k}(\varepsilon, \dot{\varepsilon}) = \min(\mu_j(\varepsilon), \mu_k(\dot{\varepsilon})). \qquad (7)$$

The output F is computed with the centroid method:

$$F(\varepsilon,\dot{\varepsilon})=\frac{\sum \mu_{j,k}U_{j,k}}{\sum \mu_{j,k}}. \qquad (8)$$

Practically, 3 predicates (Negative (N), Zero (Z), Positive (P)) will be considered; the composition function AND will be the function min(.) as stated in formula (7). The trapezoidal membership functions are defined in Figure 2.

ξ is a linear control which ensures for a known system the condition :

$$\dot{s}=-ps\,,$$

where p is a chosen real positive number.

Calling $\Delta f=f-\tilde{f}$ and $\Delta g=g-\tilde{g}$, it is

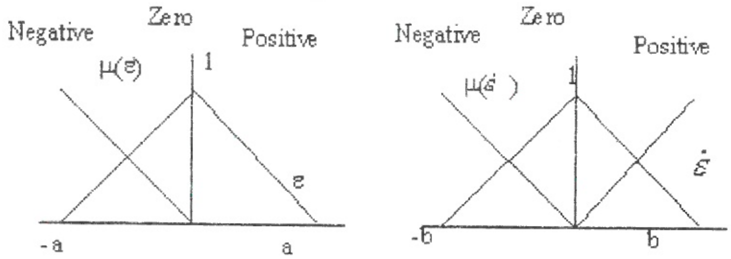

Figure 2. Input Membership Functions

The look-up table is defined in Table 1.

Table 1. Lookup Table and Rule Number (in parentheses) for the Fuzzy Controller Output

ε / $\dot{\varepsilon}$	Negative	Zero	Positive
Positive	$F_{1,1}$ ($R_{1,1}$)	$F_{1,2}$ ($R_{1,2}$)	$F_{1,3}$ ($R_{1,3}$)
Zero	$F_{2,1}$ ($R_{2,1}$)	$F_{2,2}$ ($R_{2,2}$)	$F_{2,3}$ ($R_{2,3}$)
Negative	$F_{3,1}$ ($R_{3,1}$)	$F_{2,2}$ ($R_{2,2}$)	$F_{2,3}$ ($R_{2,3}$)

After some tuning trials, the following parameters are obtained:

a = 0.005, b = 0.02, $F_{1,1}$ = -90, $F_{1,2}$ = 6, $F_{1,3}$ = 90, $F_{2,1}$ = -30, $F_{2,3}$ = 30, $F_{3,1}$=-90, $F_{3,2}$ = -6, $F_{3,3}$ = 90, $F_{2,2}$ = 0.

B. Sliding Mode Control of the Inverted Pendulum

The idea of the sliding mode control is to use a discontinuous control through a variable structure system. A hypersurface

$$s(\varepsilon,\ \dot{\varepsilon})=C^T x \qquad (9)$$

defines the discontinuity manifold [11]. If we consider the approximation of the system (2) using Eq (5), the control is

$$F=\frac{1}{\tilde{g}}\left(-pS-\tilde{f}+\ddot{y}_d-\xi\right)-\frac{K}{\tilde{g}}sign(s(t))\,. \qquad (10)$$

shown that the condition $\dot{s}s<0$, which ensures the convergence of s to zero, is satisfied if the gain K is chosen big enough:

$$K\geq\left|\frac{\Delta f+\frac{\Delta g}{\tilde{g}}\left(-pS-\tilde{f}+\ddot{y}_d-\xi\right)}{1+\frac{\Delta g}{\tilde{g}}}\right|. \qquad (11)$$

To avoid high frequency oscillations close to the sliding surface $s(t)$, it is proposed in [11] to replace the function sign(.) in Eq (10) by the function saturation defined by:

$$\left\{sat(x)=\frac{x}{\alpha},\left|\frac{x}{\alpha}\right|<1, \qquad sat(x)=sign\left(\frac{x}{\alpha}\right),\left|\frac{x}{\alpha}\right|>1.\right. \qquad (12)$$

The value of α is so chosen as to realise a trade-off between the robustness of the control and the damping of the chattering. In the simulations, the value of the parameters is

$s(\varepsilon,\ \dot{\varepsilon})=5\varepsilon+\dot{\varepsilon}$, p = 5, α = 0.01, K= 4.

4. Piece-wise Affine Fuzzy Control

The membership functions $\mu_j(\varepsilon)$ to the predicate *if* ε *is* A_j of the fuzzy controller defined in Section 2.A, are piece-wise affine functions of ε. It becomes obvious that the membership function $\mu_{j,k}(\varepsilon,\dot{\varepsilon})$ =min($\mu_j(\varepsilon)$,$\mu_k(\dot{\varepsilon})$) to the predicate *if* ε *is* A_j *AND* $\dot{\varepsilon}$ *is* B_k is a piece-wise linear function of vector x. Thus there exists a number M of regions $\Re_p$ with p

$=1,2,...,M$, in the plane $\{\varepsilon, \dot{\varepsilon}\}$ where these membership functions are affine functions of x.

Yet, the centroid method defined in (8) yields a non-linear controller with constant parameters in each region $\Re_p$.

Let us suppose that N rules are active in each region $\Re_p$ considered, and rule $R_{j,k}$ reads:

$R_{j,k}$: *if* ε *is* A_j *AND* $\dot{\varepsilon}$ *is* B_k then F is $F_{j,k}$.

In order to remove the non-linearities due to the defuzzification stage, the following defuzzification operator is introduced:

$$F = \frac{1}{N}\sum_{j,k}\left(1 + \mu_{j,k} - \mu_{h(j,k)}\right)F_{j,k}. \qquad (13)$$

where $\mu_{j,k}$ corresponds to the membership function for the rule $R_{j,k}$ and the function h is defined as follows:

h: $\{1..N\} \rightarrow \{1..N\}$

$h(l) \neq l$, and h is a bijection from $\{1..N\}$ into $\{1..N\}$.

The belief in rule $R_{j,k}$ is thus altered by the belief in another rule $R_{h(j,k)}$: in the inverted pendulum example, rules $R_{1,1}$, $R_{2,2}$, $R_{2,1}$ and $R_{1,2}$ are active at the same time in the quadrant $\varepsilon < 0$ and $\dot{\varepsilon} > 0$. For all the regions $\Re_p$ included in this quadrant, we compensate the membership function of the cell of the look-up table which is far from the origin (that is to say rule $R_{1,1}$) with the nearest (rule $R_{2,2}$) and the other two active rules by each other (rule $R_{2,1}$ and rule $R_{1,2}$) . The operation is repeated for the other quadrants as well.

In every region $\Re_p$ where the membership functions are piece-wise linear, the resulting fuzzy control also becomes piece-wise linear and can thus be written:

$$F(\varepsilon, \dot{\varepsilon}) = \alpha_p x + V_p,$$

for every region $\Re_p$, $p = 1 \ldots M$. (14)

N is the number of active rules in the region $\Re_p$. Since the $\mu_{j,k}$ are piece-wise affine functions, V_p and α_p are linear functions of the output membership functions $F_{j,k}$. As a result, there exist some parameters $\beta_{j,k,p}$ such that:

$$V_p = \sum_{j,k}\beta_{j,k,p}F_{j,k}. \qquad (15)$$

The control defined above has a variable structure because the parameter value is region-wise. A simple way to obtain a stable controller is to tune the parameters ($F_{j,k}$) so that a switching line separates the phase plane into one region where the control is positive, whereas in the other it is negative. In this case, it is possible to assimilate the controller to a Variable Structure Controller with Boundary Layer [9] as described in [7] [8]. The difference with the last scheme is that we may obtain a linear switching surface using the formalism of fuzzy control. We may thus choose the membership functions such that there exists a common frontier to some of the regions $\Re_p$, $s(x) = C^T x$ where C is a constant matrix; in the regions $\Re_p$ where the surface $C^T x$ is always positive (respectively negative), a constant strictly negative (respectively positive) term will be introduced using the term V_p in equation (7); if $s(x)$ is both negative and positive in some region $\Re_p$, then the term V_p in Eq (7) will be cancelled by tuning the control parameters $F_{j,k}$, the control being then linear in the regions considered. Considering the control of the linearised system (6), the algorithm will thus be:

$$\dot{x} = Ax + B\left(\alpha_p^T x + V_p\right), \qquad (16)$$

where we choose $V_p > 0$ *if* $C^T x \leq 0$ for every $x \in \Re_p$, $V_p < 0$ *if* $C^T x \geq 0$ for every $x \in \Re_p$, else $V_p = 0$.

The last condition can be rewritten, using Eq (7),

$$V_p = \sum_{j,k}\beta_{j,k,p}F_{j,k} = 0 \qquad (17)$$

The control $F = \alpha_p^T x + V_p$ may be discontinuous with respect to the surface $s(x) = C^T x = 0$. Eq (16) can be rewritten as:

$$\dot{x} = \left(A + B\alpha_p^T\right)x - B\left|V_p\right| sign\left(C^T x\right). \qquad (18)$$

Let us suppose that $B^T C > 0$; a sufficient condition for the stability of the system (19) is that, for every region $\Re_p$, the matrix

$$Q_p = \left(\left(A + B\alpha_p^{\ T} \right)^T CC^T + CC^T \left(A + B\alpha_p^{\ T} \right) \right) < 0. \qquad (19)$$

The proof is straightforward using the candidate Lyapunov function $V = s^T s$. Derivating V, we obtain:

$$\dot V = \dot x^T CC^T x + x^T CC^T \dot x\,,$$

$$\dot V = x^T \left(A + B\alpha_p^{\ T} \right)^T CC^T x - B^T \left| V_p \right| sign\left(C^T x \right) CC^T x + x^T CC^T \left(A + B\alpha_p^{\ T} \right) x - x^T CC^T B \left| V_p \right| sign\left(C^T x \right),$$

$$\dot V = x^T \left(\left(A + B\alpha_p^{\ T} \right)^T CC^T + CC^T \left(A + B\alpha_p^{\ T} \right) \right) x - 2\left(B^T C \right) \left| V_p \right| sign\left(C^T x \right)\left(C^T x \right)$$

Obviously, $\dot V < 0$ if condition (19) is met.

C. Application to the Inverted Pendulum

Let us consider the quadrant called Q_1 where $-a < \varepsilon < 0$ and $0 < \dot\varepsilon < b$; only rules $R_{1,1}$, $R_{2,1}$, $R_{1,2}$ and $R_{2,2}$ are active. The membership functions for these rules are respectively,

$$\mu_{R_{1,1}} = \min\left(-\frac{\varepsilon}{a}, \frac{\dot\varepsilon}{b} \right), \quad \mu_{R_{2,1}} = \min\left(1 + \frac{\varepsilon}{a}, \frac{\dot\varepsilon}{b} \right),$$

$$\mu_{R_{1,2}} = \min\left(1 + \frac{\varepsilon}{b}, 1 - \frac{\dot\varepsilon}{b} \right), \quad \mu_{R_{2,2}} = \min\left(-\frac{\varepsilon}{a}, 1 - \frac{\dot\varepsilon}{b} \right)$$

Thus we have 4 regions $\Re_1$, $\Re_2$, $\Re_3$, $\Re_4$, for which the membership functions are all linear with respect to ε and $\dot\varepsilon$ and have constant parameters (Figure 3).

$$\Re_1:\ -\frac{\varepsilon}{a} > \frac{\dot\varepsilon}{b}\ and\ 1 + \frac{\varepsilon}{a} < \frac{\dot\varepsilon}{b}, \quad \Re_2:\ -\frac{\varepsilon}{a} < \frac{\dot\varepsilon}{b}\ and\ 1 + \frac{\varepsilon}{a} < \frac{x_2}{b},$$

$$\Re_3:\ -\frac{\varepsilon}{a} < \frac{\dot\varepsilon}{b}\ and\ 1 + \frac{\varepsilon}{a} > \frac{\dot\varepsilon}{b} \quad \Re_4:\ -\frac{\varepsilon}{a} > \frac{\dot\varepsilon}{b}\ and\ 1 + \frac{\varepsilon}{a} > \frac{\dot\varepsilon}{b}.$$

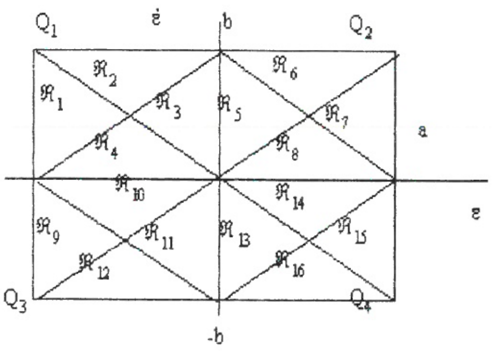

Figure 3. Regions with Constant Parameters Membership Functions

For example, for region $\Re_1$, we have: $\mu_{R_{1,1}} = \frac{\dot\varepsilon}{b}$, $\mu_{R_{1,2}} = 1 + \frac{\varepsilon}{a}$, $\mu_{R_{2,2}} = 1 + \frac{\varepsilon}{a}$, $\mu_{R_{2,1}} = 1 - \frac{\dot\varepsilon}{b}$.
This operation can be repeated for the other three quadrants $0 < \varepsilon < a$ and $-b < \dot\varepsilon < 0$, $-a < \varepsilon < 0$ and $0 < \dot\varepsilon < b$, $0 < \varepsilon < a$ and $0 < \dot\varepsilon < b$ which finally makes 16 regions where the $\mu_{R_{i,j}}$ has a constant expression (see Figure 3).

The resulting control in the quadrant Q_1 ($-a < \varepsilon < 0$ and $0 < \dot\varepsilon < b$) is then:

$$F = \frac{1}{2}\left(1 + \mu_{R_{1,1}} - \mu_{R_{2,2}} \right) F_{1,1} + \frac{1}{2}\left(1 + \mu_{R_{1,2}} - \mu_{R_{2,1}} \right)$$
$$F_{1,2} + \frac{1}{2}\left(1 + \mu_{R_{2,1}} - \mu_{R_{1,2}} \right) F_{2,1} + \frac{1}{2}\left(1 + \mu_{R_{2,2}} - \mu_{R_{1,1}} \right) F_{2,2}.$$

Replacing the expression of the $\mu_{R_{i,j}}$ in the region $\Re_1$, we have

$$\Re_1 : F = \frac{1}{2}\begin{pmatrix} \left(\frac{\dot\varepsilon}{b} - \frac{\varepsilon}{a} \right) F_{1,1} + \left(1 + \frac{\dot\varepsilon}{b} + \frac{\varepsilon}{a} \right) F_{1,2} + \left(1 - \frac{\varepsilon}{a} - \frac{\dot\varepsilon}{b} \right) \\ F_{2,1} + \left(2 + \frac{\varepsilon}{a} - \frac{\dot\varepsilon}{b} \right) F_{2,2} \end{pmatrix}. \qquad (20)$$

Due to the symmetry of the operator and of the membership function, the expression for control F is the same in regions $\Re_2$, $\Re_3$, and $\Re_4$, so that the above formula holds for the whole quadrant $-a < \varepsilon < 0$ and $0 < \dot\varepsilon < b$.

For the other quadrants, using the same method, we get:

Q_2: $0 < \varepsilon < a$ and $0 < \dot\varepsilon < b$,

$$F = \frac{1}{2}\begin{pmatrix} \left(\frac{\dot\varepsilon}{b} + \frac{\varepsilon}{a} \right) F_{1,3} + \left(1 - \frac{\dot\varepsilon}{b} + \frac{\varepsilon}{a} \right) F_{2,3} + \\ \left(1 - \frac{\varepsilon}{a} + \frac{\dot\varepsilon}{b} \right) F_{1,2} + \left(2 - \frac{\varepsilon}{a} - \frac{\dot\varepsilon}{b} \right) F_{2,2} \end{pmatrix},$$

Q_3: $-a < \varepsilon < 0$ and $-b < \dot\varepsilon < 0$,

$$F = \frac{1}{2}\begin{pmatrix} \left(-\frac{\dot\varepsilon}{b} - \frac{\varepsilon}{a} \right) F_{3,1} + \left(1 - \frac{\dot\varepsilon}{b} + \frac{\varepsilon}{a} \right) F_{2,1} + \\ \left(1 + \frac{\varepsilon}{a} - \frac{\dot\varepsilon}{b} \right) F_{3,2} + \left(2 + \frac{\varepsilon}{a} + \frac{\dot\varepsilon}{b} \right) F_{2,2} \end{pmatrix},$$

Q_4: $0 < \varepsilon < a$ and $-b < \dot{\varepsilon} < 0$,

$$F = \frac{1}{2}\left(\begin{array}{l}\left(-\frac{\dot{\varepsilon}}{b}+\frac{\varepsilon}{a}\right)F_{3,3}+\left(1+\frac{\dot{\varepsilon}}{b}+\frac{\varepsilon}{a}\right)F_{2,3}+\\ \left(1-\frac{\varepsilon}{a}-\frac{\dot{\varepsilon}}{b}\right)F_{3,2}+\left(2-\frac{\varepsilon}{a}+\frac{\dot{\varepsilon}}{b}\right)F_{2,2}\end{array}\right).$$

The controller is region-wise linear in the four quadrants, the symmetry of the membership functions reduces the number of the regions to be considered.

D. Stability Analysis and Synthesis

The control (20) is thus piece-wise affine with constant parameters in each of the four quadrants Q_1, Q_2, Q_3 and Q_4. A common frontier to some of the regions $\mathfrak{R}_j$ is the surface:

$$s(x)=\frac{\dot{\varepsilon}}{b}+\frac{\varepsilon}{a}=0\,, \tag{21}$$

The control will not be discontinuous along this surface since it belongs to quadrants Q_1 and Q_4 where the control is continuous. However, we can choose the controls to be discontinuous around these quadrants, that is to say in Q_2 and Q_3. We can thus choose the control to be linear in quadrants Q_1 and Q_4 (V_i =0) and have a discontinuous term in Q_2 and Q_3 so that, using Eq (16), $V_p \neq 0$ and $V_p.s(x) \geq 0$. The application of the method yields, as an application of formula (17):

$$\begin{array}{l}Q_1 : F_{1,2} + F_{2,1} + 2F_{2,2} = 0,\\ Q_4 : F_{3,2} + F_{2,3} + 2F_{2,2} = 0\end{array} \tag{22}$$

and in the discontinuity parts:

$$\begin{array}{l}Q_2 : V_2 = F_{1,2} + F_{2,3} + 2F_{2,2} < 0,\\ Q_3 : V_3 = F_{3,2} + F_{2,1} + 2F_{2,2} > 0\end{array} \tag{23}$$

Substituting Eq (22) in Eq (23) yields:

$$\begin{array}{l}Q_2 : V_2 = F_{2,3} - F_{2,1} < 0,\\ Q_3 : V_3 = F_{2,1} - F_{2,3} > 0\end{array} \tag{24}$$

To satisfy equation (24), it suffices to take $F_{2,1} \neq F_{2,3}$. The expression of the controller is this time:

$$\begin{cases} Q_1 : u = \frac{1}{2}\left(\left(\frac{\dot{\varepsilon}}{b}-\frac{\varepsilon}{a}\right)F_{1,1}+F_{2,1}+\left(-\frac{3\dot{\varepsilon}}{b}-\frac{\varepsilon}{a}\right)F_{2,2}\right) \\ Q_2 : u = \frac{1}{2}\left(\begin{array}{l}\left(\frac{\dot{\varepsilon}}{b}+\frac{\varepsilon}{a}\right)F_{1,3}+\left(-\frac{\dot{\varepsilon}}{b}+\frac{\varepsilon}{a}\right)F_4+\\ \left(\frac{\varepsilon}{a}-\frac{\dot{\varepsilon}}{b}\right)F_{2,3}+F_{2,3}-F_{2,1}\end{array}\right) \\ Q_3 : u = \frac{1}{2}\left(\begin{array}{l}\left(-\frac{\dot{\varepsilon}}{b}-\frac{\varepsilon}{a}\right)F_{3,1}+\left(\frac{\dot{\varepsilon}}{b}-\frac{\varepsilon}{a}\right)F_{2,1}+\\ \left(-\frac{\varepsilon}{a}+\frac{\dot{\varepsilon}}{b}\right)F_{2,3}+F_{2,1}-F_{2,3}\end{array}\right) \\ Q_4 : u = \frac{1}{2}\left(\begin{array}{l}\left(-\frac{\dot{\varepsilon}}{b}+\frac{\varepsilon}{a}\right)F_{3,3}+\left(\frac{2\dot{\varepsilon}}{b}+\frac{2\varepsilon}{a}\right)F_{2,3}+\\ \left(\frac{3\dot{\varepsilon}}{b}+\frac{\varepsilon}{a}\right)F_{2,2}\end{array}\right) \end{cases} \tag{25}$$

For the inverted pendulum stabilisation, we take $a = b = 0.02$. The surface $s(x)$ will thus be $C^T x = \dot{\varepsilon} + \varepsilon = 0$.

With the choice of $F_{1,1}$ = 1, $F_{1,3}$ = 18, $F_{2,1}$ = -0.55, $F_{2,3}$ = 0.85, $F_{3,1}$=-10, $F_{3,3}$ = 0.1, $F_{2,2}$ = -0.1 we obtain the following matrices $Q_p = \left(\left(A + B\alpha_p^{\,T}\right)^T CC^T + CC^T\left(A + B\alpha_p^{\,T}\right)\right)$ corresponding to the quadrants Q_p:

$$Q_1 = \begin{pmatrix}-1.85 & -0.96\\ -0.96 & -0.07\end{pmatrix},\quad Q_2 = \begin{pmatrix}-318 & -306\\ -306 & -295\end{pmatrix},$$

$$Q_3 = \begin{pmatrix}-185 & -168\\ -168 & -152\end{pmatrix},\quad Q_4 = \begin{pmatrix}-9.56 & -4.8\\ -4.8 & -0.07\end{pmatrix}.$$

It can be checked that $Q_p < 0$ for every j and, following condition (19), the overall system is stable.

To avoid chattering due to the discontinuity term, we use the *sat(.)* function defined in (12) with α=0.03.

As for the VSC controller defined in Section 3. A, it is to hope that the modelling error will be small enough so that the convergence of the overall system should not be affected.

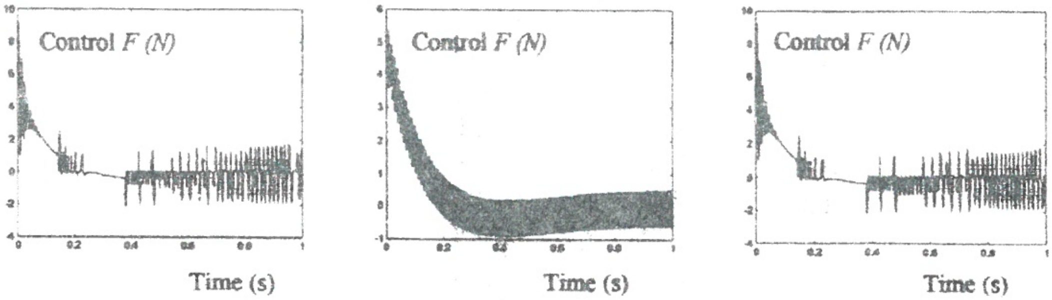

Figure 4. (a) Fuzzy Control, (b) Variable Structure Control, (c) Piece-wise Affine Fuzzy Control

5. Numerical Results

The structure of the piece-wise affine controller is now clear: a fast tracking is allowed by the "discontinuity" terms, which are region-wise, and the regulation is ensured nearby the surface by linear terms. The Sliding Mode Controller described in [9] has a better performance but it might rather difficult to design as the equations of the system should be known. The controller also resembles the one which was described in [7] or [8] which can be seen as an extension of the previously cited controller, but it has the advantage of being piece-wise linear and displaying a linear switching surface which is a great simplification. A major advantage over the classical fuzzy controller lies in that Eqs (17) and (19) provide a framework for a systematic controller design. These equations provide some inequalities which restrict the admissible parameter domain and give stability or performance conditions. Parameter tuning can thus be immediately compared to the trial-and-error method used for fuzzy control

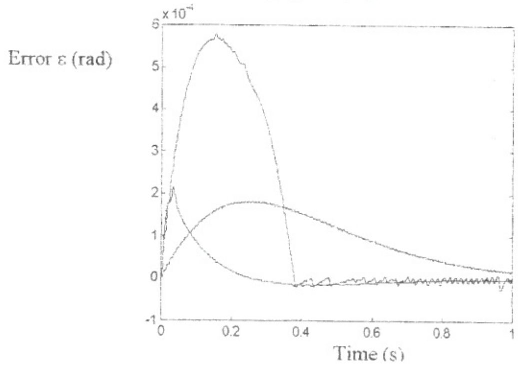

Figure 5. Control Errors Fuzzy Control, Variable Structure Control, Piece-wise Affine Fuzzy Control

6. Conclusion

A linear defuzzification method has allowed to turn a fuzzy controller into a region-wise affine state space controller; the control output has a region-wise constant term and a region-wise term which is a linear function of the controller inputs. A systematic parameter tuning ensures the stability of the controller, and can be of a real help for control synthesis. It was shown that this controller could be seen as a particular case of variable structure control with boundary layer.

The performance and interests of the control was shown through the control of an inverted pendulum and compared to variable structure control and fuzzy control. A simple defuzzification configuration was tested, and future work was suggested for other strategies.

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