# On Many-valued Equivalence and Distance Functions

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**Abstract:** The variety of biresiduated algebras including the structures of D-algebra and MV-algebra was introduced together with a corresponding logical system. A many-valued space over a biresiduated algebra is a set equipped with an equivalence function and a distance function such that these functions are complementary. A cartesian closed category of many-valued spaces is presented.

**Keywords:** fuzzy set, MV-algebra, Heyting algebra, Brouwer algebra, D-algebra, residuated lattice, biresiduated algebra, equivalence function, distance function, category.

## Introduction

The notion of *fuzzy set* was introduced by L. A. Zadeh in 1965 as "*a class of objects with a continuum of grades of membership*" [12]. A fuzzy set $A$ is characterized by a mapping $f_A$ from X to [0, 1], called *membership function* on X, where $[0, 1] \subset \mathbb{R}$ is the complete bounded chain of positive real numbers. In this paper the acceptance of a fuzzy set is that of a couple (X, f), where X is a set and $f : X \to [0, 1]$ is a function.

The notion of L-*set* including the notion of fuzzy set was introduced by J. A. Goguen as a couple (X, f), where L is a lattice and $f : X \to L$ is a function. Goguen considers that the *algebra of inexact concepts* is a *residuated lattice* [6]. Adjoint couples and residuated lattices are often used in the fuzzy set theory [11].

The theory of *MV-algebras* is a mathematical development arising from algebraic foundations of many-valued reasoning [3, 4].

An MV-algebra has both a structure of residuated lattice and a structure of dual residuated lattice.

In order to identify a standard logical system which includes features common to some basic many-valued logical systems, a variety of *biresiduated algebras* was introduced in [10]. This class of biresiduated algebras includes *Heyting algebras* [1, 2], *Brouwer algebras and MV-algebras*. A *D-algebra* [9, 10] is a structure isomorphic to a subdirect product between a Heyting algebra and a Brouwer algebra. Thus, every D-algebra is also a biresiduated algebra.

A general description of the connection between some basic algebraic structures from the category of biresiduated algebras, is given.

The notions of *equivalence and distance functions on a set* over a biresiduated algebra are introduced together with the notion of *many-valued space*. Different examples of these notions are given. The purpose of this paper is to present a *cartesian closed category of many-valued spaces over a complete D-algebra*. This category can be considered as a starting point leading to a new suitable mathematical development of the fuzzy set theory.

## 1. Basic algebraic structures

### 1.1 Biresiduated algebras

Let $\underline{K}$ be the class of algebras

$$\mathbf{A} = (A, \wedge, \vee, \otimes, \to, \oplus, -, \neg, 0, 1)$$

of type (2, 2, 2, 2, 2, 2, 1, 0, 0).

A *biresiduated algebra* is an algebra **A** of $\underline{K}$ with seven operations $\wedge$ (*meet*), $\vee$ (*join*), $\otimes$ (*multiplication*), $\to$ (*residuation*), $\oplus$ (*addition*), $-$ (*dual residuation*), $\neg$ (*negation*) and two constants $0, 1 \in A$ such that:

(BR1) (A, $\wedge$, $\vee$, 0, 1) is a *bounded distributive lattice with the minimum element* 0 *and the maximum element* 1.

(BR2) (A, $\otimes$) and (A, $\oplus$) are *commutative semigroups*.

(BR3) The following equations hold:

(i) $x \vee (x \otimes y) = x$
(i°) $x \wedge (x \oplus y) = x$
(ii) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$
(ii°) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$

(iii) $x \otimes (y \wedge z) = [(x \otimes y) \wedge (x \otimes z)] \otimes 1$
(iii°) $x \oplus (y \vee z) = [(x \oplus y) \vee (x \oplus z)] \oplus 0$

(iv) $(x \otimes y) \otimes 1 = x \otimes y$
(iv°) $(x \oplus y) \oplus 0 = x \oplus y$

(v) $(x \otimes y) \oplus 0 = (x \oplus 0) \otimes (y \oplus 0)$
(v°) $(x \oplus y) \otimes 1 = (x \otimes 1) \oplus (y \otimes 1)$

(vi) $(x \to y) \otimes 1 = x \to y$;
(vi°) $(x - y) \oplus 0 = x - y$;

(vii) $(x \to y) \oplus 0 = \neg(x - y)$
(vii°) $(x - y) \otimes 1 = \neg(x \to y)$

(viii) $\neg x = x \to 0$
(viii°) $\neg x = 1 - x$

(ix) $x \to (y \to z) = (x \otimes y) \to z$
(ix°) $(x - y) - z = x - (y \oplus z)$

(x) $x \otimes (x \to y) = (x \wedge y) \otimes 1$
(x°) $(x - y) \oplus y = (x \vee y) \oplus 0$

(xi) $(x \wedge y) \to x = 1$
(xi°) $x - (x \vee y) = 0$

(xii) $(x \otimes 1) \vee \neg x = x \vee \neg x$
(xii°) $(x \oplus 0) \wedge \neg x = x \wedge \neg x$

Let BR be the class of biresiduated algebras.

## 1.2 Residuated algebras

A biresiduated algebra will be called *residuated algebra* if verifying:

$$(R1)\ x \otimes 1 = x.$$

The following condition holds in every residuated algebra:

$$z \le x \to y \text{ iff } z \otimes x \le y.$$

Let R be the class of residuated algebras.

## 1.3 Heyting algebras

A *Heyting algebra* is a system

$$(A, \wedge, \vee, \to, 0, 1)$$

such that $(A, \wedge, \vee, 0)$ is a relatively pseudo-complemented lattice with the minimum element 0, the binary operation of relative pseudocomplementation $\to$ and $0, 1 \in A$ satisfying $1 = 0 \to 0$ and for every $x, y, z \in A$:

$$z \le x \to y \text{ iff } z \wedge x \le y.$$

Let H be the class of Heyting algebras and $H^*$ be the class of biresiduated algebras verifying the following equation:

$$(R2)\ x \otimes y = x \wedge y.$$

The next result expresses the relation between the classes of algebras $H^*$, R, BR and H.

## 1.4 Proposition

(i) $H^* \subset R \subset BR$.
(ii) The algebraic categories associated with the classes H and $H^*$ are isomorphic.

***Proof***

(i) Relation 1.3(R2) implies 1.2(R1).
(ii) A biresiduated algebra of $H^*$ can be associated with every Heyting algebra

$$(A, \wedge, \vee, \to, 0, 1) \in H$$

such that it verifies 1.3(R2) together with the following equations:

$$\neg x = x \to 0;$$
$$x - y = \neg(x \to y);$$
$$x \oplus y = \neg\neg(x \vee y).$$

Using this correspondence an isomorphism between the algebraic categories associated with H and $H^*$ is reached. □

## 1.2° Dual residuated algebras

A biresiduated algebra will be called *dual residuated algebra* if verifying:

$$(R1°)\ x \oplus 0 = x.$$

The following condition holds in every dual residuated algebra:

$$x - y \le z \text{ iff } x \le y \oplus z.$$

Let $R^o$ be the class of dual residuated algebras.

## 1.3° Brouwer algebras

A *Brouwer algebra* is a system

$$(A, \wedge, \vee, -, 0, 1)$$

such that $(A, \wedge, \vee, 1)$ is a dual Heyting lattice with a binary operation $-$ (relative pseudo-subtraction) and $0, 1 \in A$ satisfying $1 = 0 - 0$ and for every $x, y, z \in A$:

$$x - y \le z \text{ iff } x \le y \vee z.$$

Let Br be the class of Brouwer algebras and $Br^*$ be the class of biresiduated algebras verifying the following equation:

(R2°) $x \oplus y = x \vee y$.

The next result expresses the relation between the classes of algebras $\underline{Br}^*$, $\underline{R}^\circ$, $\underline{BR}$ and $\underline{Br}$.

## 1.4° Proposition

(i) $\underline{Br}^* \subset \underline{R}^\circ \subset \underline{BR}$.

(ii) The algebraic categories associated with the classes $\underline{Br}$ and $\underline{Br}^*$ are isomorphic.

### *Proof*

(i) Relation 1.3°(R2°) implies 1.2°(R1°).

(ii) A biresiduated algebra of $\underline{Br}^*$ can be associated with every Brouwer algebra

$$(A, \wedge, \vee, -, 0, 1) \in \underline{Br}$$

such that it verifies 1.3°(R2°) together with the following equations:

$$\neg x = 1 - x;$$
$$x \rightarrow y = \neg(x - y);$$
$$x \otimes y = \neg\neg(x \wedge y).$$

Using this correspondence an isomorphism between the algebraic categories associated with $\underline{Br}$ and $\underline{Br}^*$ is obtained. □

## 1.5 MV-algebras

An MV-algebra [4] is a system $(A, \oplus, \neg, 0)$ of type (2, 1, 0) such that the following equations hold:

MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
MV2) $x \oplus y = y \oplus x$;
MV3) $x \oplus 0 = x$;
MV4) $\neg\neg x = x$;
MV5) $x \oplus \neg 0 = \neg 0$;
MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

From MV1)-MV3) it follows that $(A, \oplus, 0)$ is an *abelian monoid*. We define a constant 1 and the operations $\otimes$, $\rightarrow$ and $-$ together with a binary relation $\leq$ on A as follows, for any two elements x, y of A:

(1) $1 = \neg 0$;
(2) $x \otimes y = \neg(\neg x \oplus \neg y)$;
(3) $x \rightarrow y = \neg x \oplus y$;
(4) $x - y = x \otimes \neg y$;
(5) $x \leq y$ iff $\neg x \oplus y = 1$.

Then $\leq$ is an order relation which determines on A a structure of *distributive lattice with the smallest element 0 and the greatest element* 1, $(A, \wedge, \vee, 0, 1)$, such that:

(6) $x \vee y = (x \otimes \neg y) \oplus y$;
(7) $x \wedge y = \neg(\neg x \vee \neg y)$.

Let $\underline{MV}$ be the class of MV-algebras and $\underline{MV}^*$ be the class of algebras

$$\mathbf{A}^* = (A, \wedge, \vee, \otimes, \rightarrow, \oplus, -, \neg, 0, 1)$$

associated with an MV-algebra

$$\mathbf{A} = (A, \oplus, \neg, 0),$$

where the operations $\wedge$, $\vee$, $\otimes$, $\rightarrow$, $\neg$ and 1 are defined as above. The next result expresses the relation between the classes $\underline{MV}$, $\underline{MV}^*$ and $\underline{BR}$.

## 1.6 Proposition

(i) $\underline{MV}^* = \underline{R} \cap \underline{R}^\circ \subset \underline{BR}$.

(ii) The algebraic categories associated with the classes $\underline{MV}$ and $\underline{MV}^*$ are isomorphic.

### *Proof*

(i) Every algebra $\mathbf{A}^* \in \underline{MV}^*$ associated with an MV-algebra $\mathbf{A} \in \underline{MV}$ and defined as in 1.5 is a biresiduated algebra (see Definition 1.1) satisfying equations 1.2(R1) and 1.2°(R1°). The following specific relations hold:

$$x \wedge y = x \otimes (x \rightarrow y);$$
$$x \vee y = (x - y) \oplus y;$$

(ii) Using the precedent correspondence an isomorphism between the algebraic categories associated with $\underline{MV}$ and $\underline{MV}^*$ can be obtained.□

The result below was established in [10].

## 1.7 Theorem

The class $\underline{BR}$ of biresiduated algebras is the variety of algebras of $\underline{K}$ generated by $\underline{R} \cup \underline{R}^\circ$.

## 1.8 Boolean algebras

A Boolean algebra [8] is a system

$$(A, \wedge, \vee, \neg, 0, 1)$$

of type (2, 2, 1, 0, 0) such that $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and it satisfies the equations $x \wedge \neg x = 0$ and $x \vee \neg x = 1$.

Let $\underline{B}$ be the class of Boolean algebras and $\underline{B}^*$ be the class of algebras $\mathbf{A} \in \underline{K}$ associated with Boolean algebras $(A, \wedge, \vee, \neg, 0, 1)$ such that:

$$x \otimes y = x \wedge y;$$
$$x \rightarrow y = \neg x \vee y;$$
$$x \oplus y = x \vee y;$$
$$x - y = x \wedge \neg y.$$

## 1.9 D-algebras

A D-algebra [9] is a system

$$\mathbf{A} = (A, \wedge, \vee, \rightarrow, -, 0, 1)$$

of type (2, 2, 2, 2, 0, 0) such that it is isomorphic to a subdirect product of two structures

$$\mathbf{H} = (H, \wedge, \vee, \to, -, 0, 1)$$

and

$$\mathbf{Br} = (Br, \wedge, \vee, \to, -, 0, 1),$$

where $(H, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra such that $x - y = (x \to y) \to 0$, for all $x, y \in H$ and $(Br, \wedge, \vee, -, 0, 1)$ is a Brouwer algebra such that $x \to y = 1 - (x - y)$, for all $x, y \in Br$.

Let $\underline{D}$ be the class of D-algebras and $\underline{D}^*$ be the class of biresiduated algebras $\mathbf{A} \in \underline{BR}$ such that the following equations hold:

(D1) $x \otimes y = (x \wedge y) \otimes 1$;
(D2) $x \oplus y = (x \vee y) \oplus 0$.

Then the algebraic categories associated with $\underline{D}$ and $\underline{D}^*$ are isomorphic.

The structures of Boolean algebra, Heyting algebra and Brouwer algebra are related to the structure of D-algebra as follows:

### 1.10 Theorem

( i) $\underline{B}^* = \underline{H}^* \cap \underline{Br}^* \subset \underline{D}^* \subset \underline{BR}$.
(ii) $\underline{D}^*$ is a variety of biresiduated algebras generated by $\underline{H}^* \cup \underline{Br}^*$.

## 2. Many-valued spaces over a biresiduated algebra

Let X be a set and **A** be a biresiduated algebra.

In this Section the notions of *equivalence and distance functions* on X and the term of *many-valued space over* **A** will be introduced.

### 2.1 Definition

An *equivalence function* on X over **A** is a mapping $e : X \times X \to A \otimes 1$ such that the following conditions hold for all $x, y, z \in X$:

( i) $e(x, x) = 1$;
( ii) $e(x, y) = e(y, x)$;
(iii) $e(x, y) \otimes e(y, z) \le e(x, z)$.

### 2.2 Definition

A *distance function* on X over **A** is a mapping

$$d : X \times X \to A \oplus 0$$

such that the following conditions hold for all $x, y, z \in X$:

( i) $d(x, x) = 0$;
( ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \le d(x, y) \oplus d(y, z)$.

### 2.3 Definition

A *many-valued space* over **A** with the carrier set X (called more simply an **A**-valued space) is a system $\mathcal{X} = (X, e, d)$ such that

$$e : X \times X \to A \otimes 1$$

is an equivalence function on X over **A**,

$$d : X \times X \to A \oplus 0$$

is a distance function on X over **A** and the following conditions hold for all $x, y \in X$:

( i) $e(x, y) \otimes d(x, y) = 0$;
(ii) $e(x, y) \oplus d(x, y) = 1$.

### 2.4 Example

Let $R \subseteq X \times X$ be an equivalence relation on X,

$$e_R = \alpha_R : X \times X \to \{0, 1\}$$

be the Boolean characteristic function of R, i.e.

$$e_R(x, y) = \begin{cases} 1, & \text{if } (x, y) \in R \\ 0, & \text{if } (x, y) \notin R \end{cases},$$

for all $x, y \in X$ and

$$d_R = \neg\alpha_R : X \times X \to \{0, 1\}$$

be the Boolean complement of $\alpha_R$, i.e.

$$d_R(x, y) = \begin{cases} 0, & \text{if } (x, y) \in R \\ 1, & \text{if } (x, y) \notin R \end{cases}$$

for all $x, y \in X$.

Then *the standard* **A**-*valued space associated with* R is the triplet

$$\mathcal{X}[R] = (X, e_R, d_R).$$

Thus, the standard **A**-valued space associated with the identity relation R on X,

$$R = \Delta = \{(x, x) / x \in X\},$$

is the system

$$\mathcal{X}[\Delta] = (X, e_\Delta, d_\Delta)$$

where for all $x, y \in X$:

$$e_\Delta(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \ne y \end{cases}$$

and

$$d_\Delta(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \ne y \end{cases}.$$

## 2.5 Example

Let

$$e_A : A \times A \to A \otimes 1$$

and

$$d_A : A \times A \to A \oplus 0$$

be the mappings defined for all x, y ∈ A by

$$e_A(x, y) = (x \to y) \otimes (y \to x);$$
$$d_A(x, y) = (x - y) \oplus (y - x).$$

Then *the standard* **A**-*valued space associated with* **A** is the triplet

$$\mathcal{A} = (A, e_A, d_A).$$

The mapping $e_A$ is called *the equivalence function on* **A** and the mapping $d_A$ is called *the distance function on* **A**.

## 2.6 Example

If **A** ∈ $\underline{D}^*$ is associated with a *D-algebra*

$$(A, \wedge, \vee, \to, -, 0, 1)$$

then the equivalence and distance functions on **A** are defined by:

$$e_A(x, y) = [(x \to y) \wedge (y \to x)] \otimes 1;$$
$$d_A(x, y) = [(x - y) \vee (y - x)] \oplus 0,$$

for all x, y ∈ A.

If **A** ∈ $\underline{H}^* \subset \underline{D}^*$ is associated with a *Heyting algebra* (A, ∧, ∨, →, 0, 1) then for all x, y ∈ A:

$$e_A(x, y) = (x \to y) \wedge (y \to x);$$
$$d_A(x, y) = \neg\neg[\neg(x \to y) \vee \neg(y \to x)],$$

where ¬u = u → 0, ∀ u ∈ A. Then concrete expressions of equivalence and distance functions on **A** can be obtained, if **A** is associated with the complete Heyting algebra of open subsets of a topological space.

If **A** ∈ $\underline{Br}^* \subset \underline{D}^*$ is associated with a *Brouwer algebra* (A, ∧, ∨, –, 0, 1) then for all x, y ∈ A:

$$e_A(x, y) = \neg\neg[\neg(x - y) \wedge \neg(y - x)];$$
$$d_A(x, y) = (x - y) \vee (y - x),$$

where ¬u = 1 – u, ∀ u ∈ A. Then concrete expressions of equivalence and distance functions on **A** can be also obtained, if **A** is associated with the complete Brouwer algebra of closed subsets of a topological space.

If **A** ∈ $\underline{B}^* \subset \underline{D}^*$ is associated with a *Boolean algebra* (A, ∧, ∨, ¬, 0, 1) then for all x, y ∈ A:

$$e_A(x, y) = (\neg x \vee y) \wedge (\neg y \vee x);$$
$$d_A(x, y) = (x \wedge \neg y) \vee (y \wedge \neg x).$$

## 2.7 Example

Let **A** be the *Lukasiewicz structure* on the real unit interval [0, 1] i.e. for all x, y ∈ [0, 1]:

$$x \wedge y = \min(x, y);$$
$$x \vee y = \max(x, y);$$
$$x \otimes y = \max(0, x + y - 1);$$
$$x \to y = \min(1, 1 - x + y);$$
$$x \oplus y = \min(1, x + y);$$
$$x - y = \max(0, x - y);$$
$$\neg x = 1 - x,$$

where in the second member of the precedent relations + and - are the usual operations of addition and subtraction of real numbers. Then the following relations hold:

$$e_A(x, y) = 1 - |x - y|;$$
$$d_A(x, y) = |x - y|.$$

Therefore, the distance function $d_A$ is the *usual distance* on [0, 1] ⊂ **R** and the equivalence function $e_A$ is the negation of $d_A$.

## 2.8 Example

Let **A** be a structure of biresiduated algebra on the real unit interval [0, 1] defined by the following relations for all x, y ∈ [0, 1]:

$$x \wedge y = \min(x, y);$$

$$x \vee y = \max(x, y);$$

$$x \otimes y = x \cdot y;$$

$$x \to y = \begin{cases} 1, & \text{if } x \le y \\ \frac{y}{x}, & \text{if } x > y \end{cases};$$

$$x \oplus y = \begin{cases} 1, & \text{if } x \vee y \ne 0 \\ 0, & \text{if } x \vee y = 0 \end{cases};$$

$$x - y = \begin{cases} 0, & \text{if } x \le y \text{ or } 0 < y < x \\ 1, & \text{if } 0 = y < x \end{cases};$$

$$\neg x = x \to 0 = 1 - x = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \ne 0 \end{cases},$$

where in the second member of the precedent relations x · y is the multiplication of x by y and $\frac{y}{x}$ is the division of y by x ≠ 0 in **R**. Then the following relations hold:

$$e_A(x, y) = \begin{cases} 1, & \text{if } x = y = 0 \\ \dfrac{x \wedge y}{x \vee y}, & \text{if } x \neq 0 \text{ or } y \neq 0 \end{cases};$$

$$d_A(x, y) = \begin{cases} 0, & \text{if } (x = y = 0) \text{ or } (x \cdot y \neq 0) \\ 1, & \text{if } (x = 0 < y) \text{ or } (y = 0 < x) \end{cases}.$$

### 2.9 Example

Let **A** be a structure of biresiduated algebra on the real unit interval [0, 1] defined by the following relations for all $x, y \in [0, 1]$:

$$x \wedge y = \min(x, y);$$

$$x \vee y = \max(x, y);$$

$$x \otimes y = \begin{cases} 1, & \text{if } x \wedge y = 1 \\ 0, & \text{if } x \wedge y \neq 1 \end{cases};$$

$$x \to y = \begin{cases} 1, & \text{if } x \leq y \text{ or } y < x < 1 \\ 0, & \text{if } y < 1 = x \end{cases};$$

$$x \oplus y = x + y - x \cdot y;$$

$$x - y = \begin{cases} 0, & \text{if } x \leq y \\ \dfrac{x - y}{1 - y}, & \text{if } x > y \end{cases};$$

$$\neg x = 1 - x = x \to 0 = \begin{cases} 0, & \text{if } x = 1 \\ 1, & \text{if } x \neq 1 \end{cases}.$$

The following relations will then hold:

$$e_A(x, y) = \begin{cases} 1, & \text{if } (x = y = 1) \text{ or } x \vee y \neq 1 \\ 0, & \text{if } (x < 1 = y) \text{ or } (y < 1 = x) \end{cases};$$

$$d_A(x, y) = \begin{cases} 0, & \text{if } x = y = 1 \\ \dfrac{|x - y|}{1 - (x \wedge y)}, & \text{if } x \wedge y \neq 1 \end{cases}.$$

### 2.10 Example

Let **A** be the structure of biresiduated algebra on the real unit interval [0, 1] defined as in Example 2.8. Suppose that $p \in \mathbb{R}$ and $p > 1$. Define the mapping

$$e_p : \mathbb{R} \times \mathbb{R} \to [0, 1]$$

for all $x, y \in \mathbb{R}$ by

$$e_p(x, y) = p^{-|x - y|}.$$

It follows that for all $x, y, z \in \mathbb{R}$

$$e_p(x, y) \cdot e_p(y, z) \leq e_p(x, z).$$

Thus $e_p$ is an equivalence function on $\mathbb{R}$ over **A**.

Let **A** be the structure of biresiduated algebra on the real unit interval [0, 1] defined as in Example 2.9. Define the mapping

$$d_p : \mathbb{R} \times \mathbb{R} \to [0, 1]$$

for all $x, y \in \mathbb{R}$ by

$$d_p(x, y) = 1 - e_p(x, y) = 1 - p^{-|x - y|}.$$

Then $d_p$ is a distance function on $\mathbb{R}$ over **A**. □

The next section is the final part of this paper and it presents a notion of morphism and a corresponding category of **A**-valued spaces.

## 3. The category $\underline{\mathcal{S}}$[A]

Let **A** be a biresiduated algebra.

If X, Y, Z are sets and $f : X \to Y$, $g : Y \to Z$ are functions then $g \cdot f : X \to Z$ is the function defined by $(g \cdot f)(x) = g(f(x))$, for all $x \in X$. For all sets X and Y, let [X, Y] be the set of all functions from X to Y. A *specific terminology used in fuzzy set theory* [7] will be adopted in this section. A function $\mu \in [X, A]$ is called an **A**-*subset* of X. For all $f \in [X, Y]$, if $v \in [Y, A]$ is an **A**-subset of Y then $v \cdot f \in [X, A]$ is called *the inverse image of v under f*.

The next definition introduces a notion of morphism which makes the class $\underline{\mathcal{S}}$[**A**] of all **A**-valued spaces be a category.

### 3.1 Definition

Let $\mathcal{X} = (X, e_X, d_X)$ and $\mathcal{Y} = (Y, e_Y, d_Y)$ be two **A**-valued spaces. A *morphism from* $\mathcal{X}$ *to* $\mathcal{Y}$ is a function

$$f : X \to Y$$

such that the following conditions hold for all $x_1, x_2 \in X$ and $y \in Y$:

(i) $e_X(x_1, x_2) \otimes e_Y(f(x_1), y) \leq e_Y(f(x_2), y)$;

(ii) $d_Y(f(x_2), y) \leq d_Y(f(x_1), y) \oplus d_X(x_1, x_2)$.

The set of all morphisms from $\mathcal{X}$ *to* $\mathcal{Y}$ will be denoted by *Hom*$(\mathcal{X}, \mathcal{Y})$.

### 3.2 Definition

Let $f \in$ *Hom*$(\mathcal{X}, \mathcal{Y})$ and $g \in$ *Hom*$(\mathcal{Y}, \mathcal{Z})$ be morphisms, where $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \underline{\mathcal{S}}$[**A**]. Then

$$g \cdot f \in Hom(\mathcal{X}, \mathcal{Z})$$

and it is called the product of the couple (g, f). The class $\underline{\mathcal{S}}$[**A**] and the class of all morphisms together with this product of morphisms is a category denoted by $\mathcal{S}[\mathbf{A}]$.

The next definition introduces the notion of monotone **A**-subset of an **A**-valued space and it will be used to characterize morphisms.

### 3.3 Definition

Let $\mathcal{X} = (X, e, d) \in \mathcal{S}[\mathbf{A}]$. *A monotone* **A***-subset of* $\mathcal{X}$ is a function

$$\mu : X \to A$$

such that the following conditions hold for all $x, y \in X$:

(i) $\mu(x) \otimes e(x, y) \leq \mu(y)$;

(ii) $\mu(y) \leq \mu(x) \oplus d(x, y)$.

The set of all monotone **A**-subsets of $\mathcal{X}$ will be denoted by $\mathcal{M}[\mathcal{X}]$.

### 3.4 Lemma

Let $\mathcal{X} = (X, e, d) \in \mathcal{S}[\mathbf{A}]$ and $x \in X$. Define two functions

$$e[x] : X \to A \text{ and } d[x] : X \to A$$

such that for all $y \in X$:

$$e[x](y) = e(x, y);$$
$$d[x](y) = d(x, y).$$

Then $e[x], d[x] \in \mathcal{M}[\mathcal{X}]$.

### *Proof*

Now we show that $e[x] \in \mathcal{M}[\mathcal{X}]$. The functions e and d satisfy all the conditions 2.1(i)-(iii), 2.2(i)-(iii), 2.3(i) and 2.3(ii). Let $x', y \in X$. From Definition 2.1 (ii) it follows that

(1) $e[x](x') \otimes e(x', y) = e(x, x') \otimes e(x', y)$
$\leq e(x, y)$
$= e[x](y)$.

From Definition 2.2 (ii) and the relation

$$(\forall a \in A)\ a \oplus \neg a = 1$$

it follows that

$$1 = d(x, x') \oplus \neg d(x, x')$$
$$\leq d(x, y) \oplus d(x', y) \oplus \neg d(x, x').$$

This implies that

$$\neg d(x, y) \leq \neg d(x, x') \oplus d(x', y),$$

but the conditions 2.3(i) and 2.3(ii) imply $(\forall x, y \in X)\ \neg d(x, y) = e(x, y) \oplus 0$, therefore

$$= \neg d(x, y)$$
$$\leq (e(x, x') \oplus 0) \oplus d(x', y)$$
$$= e[x](x') \oplus d(x', y).$$

The relations (1) and (2) imply $e[x] \in \mathcal{M}[\mathcal{X}]$. The other condition $d[x] \in \mathcal{M}[\mathcal{X}]$ follows using similar arguments. □

The next result characterizes morphisms as functions preserving the property of monotony by making inverse images.

### 3.5 Proposition

Let $\mathcal{X} = (X, e_X, d_X)$ and $\mathcal{Y} = (Y, e_Y, d_Y)$ be two objects of $\mathcal{S}[\mathbf{A}]$ and $f : X \to Y$ be a function. Then the following conditions are equivalent:

(i) $f \in Hom(\mathcal{X}, \mathcal{Y})$;

(ii) $\nu \cdot f \in \mathcal{M}[\mathcal{X}]$, for all $\nu \in \mathcal{M}[\mathcal{Y}]$.

### *Proof*

(i) ⇒ (ii). Suppose that 3.5 (i) holds. Using Definitions 2.3 and 3.1 this implies that for all $x_1, x_2 \in X$:

(1) $e_X(x_1, x_2) \leq e_Y(f(x_2), f(x_1))$;

(2) $d_Y(f(x_2), f(x_1)) \leq d_X(x_1, x_2)$.

Let $\nu \in \mathcal{M}[\mathcal{Y}]$ i.e. $\nu : Y \to A$ is a function such that for all $y_1, y_2 \in Y$:

(3) $\nu(y_1) \otimes e_Y(y_1, y_2) \leq \nu(y_2)$;

(4) $\nu(y_2) \leq \nu(y_1) \oplus d_Y(y_1, y_2)$.

Suppose that $x_1, x_2 \in X$. Using (3) and (4) and the properties of the operations ⊗ and ⊕ in **A**, from (1) and (2) it follows that:

$$\nu(f(x_1)) \otimes e_X(x_1, x_2) \leq \nu(f(x_1)) \otimes e_Y(f(x_1), f(x_2))$$
$$\leq \nu(f(x_2));$$
$$\nu(f(x_2)) \leq \nu(f(x_1)) \oplus d_Y(f(x_1), f(x_2))$$
$$\leq \nu(f(x_1)) \oplus d_X(x_1, x_2).$$

Therefore, $\nu \cdot f \in \mathcal{M}[\mathcal{X}]$. This shows that the condition 3.5 (ii) holds.

(ii) ⇒ (i). Suppose that 3.5 (ii) holds. Using Lemma 3.4 it follows that for all $x_1 \in X$, the functions

$$e_Y[f(x_1)] : Y \to A$$

and

$$d_Y[f(x_1)] : Y \to A$$

are monotone **A**-subsets of $\mathcal{Y}$. From 3.5 (ii) it follows that

$$e_Y[f(x_1)] \cdot f : X \to A$$

and

$$d_Y[f(x_1)] \cdot f : X \to A$$

are monotone **A**-subsets of $\mathcal{X}$. This implies that the following relations hold for all $x_1, x_2 \in X$:

(5) $e_X(x_1, x_2) = e_X(x_1, x_2) \otimes 1$
$= e_X(x_1, x_2) \otimes e_Y(f(x_1), f(x_1))$
$= e_X(x_1, x_2) \otimes (e_Y[f(x_1)] \cdot f)(x_1)$
$\leq (e_Y[f(x_1)] \cdot f)(x_2)$

$= e_Y(f(x_1), f(x_2))$.

(6) $d_Y(f(x_1), f(x_2)) = (d_Y[f(x_1)] \cdot f)(x_2)$
$\leq d_X(x_1, x_2) \oplus (d_Y[f(x_1)] \cdot f)(x_1)$
$= d_X(x_1, x_2) \oplus d_Y(f(x_1), f(x_1))$
$= d_X(x_1, x_2) \oplus 0$
$= d_X(x_1, x_2)$.

From (5) and (6) it follows that f verifies the conditions from Definition 3.1, i.e. 3.5 (i) holds. This completes the proof. □

### 3.6 Example

Let **2** be a biresiduated algebra having precisely two elements 0 and 1. Suppose that $R \subseteq X \times X$ is an equivalence relation on X. Let

$$\mathcal{X}[R] = (X, e_R, d_R)$$

be the standard **2**-valued space associated with R defined as in example 2.4. Then $(X, d_R)$ *is a metric space* such that $D \subseteq X$ is *open* iff D is R-*monotone* i.e. the following condition holds for all $x, y \in X$:

$$x \in D \text{ and } (x, y) \in R \text{ imply } y \in D.$$

Using Definition 3.3, it follows that the following condition holds for all $D \subseteq X$:

$$D \text{ is R-monotone iff } \mu_D \in \mathcal{M}[\mathcal{X}[R]],$$

where $\mu_D : X \to \{0, 1\}$ is the characteristic function of D defined for all $x \in X$ by

$$\mu_D(x) = \begin{cases} 1, \text{if } x \in D \\ 0, \text{if } x \notin D \end{cases}.$$

Therefore, the set of all *monotone* **2**-*subsets of* $\mathcal{X}[R]$ can be identified with the set of all *open sets of the metric space* $(X, d_R)$.

Suppose that

$$\mathcal{Y}[R'] = (Y, e_{R'}, d_{R'})$$

is another **2**-valued space associated with an equivalence relation R' on Y. From Proposition 3.5 it follows that the following conditions are equivalent for any function $f : X \to Y$:

(i) $f \in Hom(\mathcal{X}[R], \mathcal{Y}[R'])$;

(ii) f is a *continuous function* from the metric space $(X, d_R)$ to the metric space $(Y, d_{R'})$.

Therefore, the category $\underline{\mathcal{S}}$[**A**] includes all **2**-valued spaces associated with equivalence relations on sets such that the notion of morphism between **A**-valued spaces is an extension of the notion of continuous functions between standard metric spaces associated with these equivalence relations.

### 3.7 Example

Let $\mathcal{X}_1 = (X, e_1, d_1)$ and $\mathcal{X}_2 = (X, e_2, d_2)$ be two **A**-valued spaces having the same carrier set X and $1_X : X \to X$ be the identity mapping on X. Define $\mathcal{X}_1 < \mathcal{X}_2$ iff $1_X \in Hom(\mathcal{X}_1, \mathcal{X}_2)$ i.e. the following conditions hold for all $x, y \in X$:

$$e_1(x, y) \leq e_2(x, y);$$
$$d_2(x, y) \leq d_1(x, y).$$

Let $\mu : X \to A$ be any function. Define two functions

$$e_\mu, e^*_\mu : X \times X \to A \otimes 1$$

such that for all $x, y \in X$:

$$e_\mu(x, y) = e_A(\mu(x), \mu(y));$$

$$e^*_\mu(x, y) = i\big((\mu(x) \to \mu(y)) \wedge (\mu(y) \to \mu(x))\big),$$

where $e_A$ is the equivalence function on **A** defined as in Example 2.5 and

$$i : A \to A \otimes 1$$

is *an interior operator on the poset* $(A, \leq)$ [2,10] defined by

$$i(a) = a \otimes 1, \forall a \in A.$$

Define also two functions

$$d_\mu, d^*_\mu : X \times X \to A \oplus 0$$

such that for all $x, y \in X$:

$$d_\mu(x, y) = d_A(\mu(x), \mu(y));$$

$$d^*_\mu(x, y) = c\big((\mu(x) - \mu(y)) \vee (\mu(y) - \mu(x))\big),$$

where $d_A$ is the distance function on **A** defined as in Example 2.5 and

$$c : A \to A \oplus 0$$

is *a closure operator on the poset* $(A, \leq)$ [2,10] defined by

$$c(a) = a \oplus 0, \forall a \in A.$$

Then $\mathcal{X}_\mu = (X, e_\mu, d_\mu)$ and $\mathcal{X}^*_\mu = (X, e^*_\mu, d^*_\mu)$ are two **A**-valued spaces with the same carrier set X such that the following conditions hold:

- $\mu \in \mathcal{M}[\mathcal{X}_\mu] \cap \mathcal{M}[\mathcal{X}^*_\mu]$;
- $\mathcal{X}_\mu < \mathcal{X}^*_\mu$;
- $\mathcal{X} \in \underline{\mathcal{S}}[\mathbf{A}]$ and $\mu \in \mathcal{M}[\mathcal{X}]$ imply $\mathcal{X} < \mathcal{X}^*_\mu$.

The next Lemma presents conditions for a morphism to be monomorphism, epimorphism or isomorphism of the category $\underline{\mathcal{S}}$[**A**].

### 3.8 Lemma

Let $\mathcal{X} = (X, e_X, d_X)$, $\mathcal{Y} = (Y, e_Y, d_Y) \in \mathcal{S}[\mathbf{A}]$. Then the following conditions hold in the category $\mathcal{S}[\mathbf{A}]$ for all $f \in Hom(\mathcal{X}, \mathcal{Y})$:

(i) f is a *monomorphism* iff f is *injective*.

(ii) If the function f is *surjective* then f is an *epimorphism*.

(iii) If the function f is *bijective* then f is an *isomorphism* iff for all $x_1, x_2 \in X$:

(I1) $e_X(x_1, x_2) = e_Y(f(x_1), f(x_2))$;
(I2) $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$.

### *Proof*

The condition 3.8 (ii) and the fact that if f is an injective function then f is a monomorphism are clear. Suppose that f is a monomorphism, but f is not an injective function i.e. for some $x_1, x_2 \in X$, we have $f(x_1) = f(x_2)$ and $x_1 \neq x_2$. Let $Z = \{ x_1, x_2 \}$ and $g, h : Z \to X$ such that $g(z) = z$ and $h(z) = x_1$, $\forall z \in Z$. This implies that $f \cdot g = f \cdot h$. Let $\mathcal{Z}[\Delta] = (Z, e_\Delta, d_\Delta)$ be the standard structure of **A**-valued space associated with the identity relation $\Delta$ on Z defined as in example 2.4. Using definition 3.1 one obtains that $g, h \in Hom(\mathcal{Z}[\Delta], \mathcal{X})$. Then, from the property $f \cdot g = f \cdot h$ and the condition that f is a monomorphism it follows that $g = h$, but the definitions of g and h imply $g \neq h$, contradiction. Thus, 3.8 (i) and 3.8(ii) hold. The condition 3.8(iii) follows from the fact that if f is a bijective function then the relations 3.8(I1) and 3.8 (I2) hold iff $f^{-1} \in Hom(\mathcal{Y}, \mathcal{X})$, where $f^{-1} : Y \to X$ is the inverse function of f.

□

Now we show that the finite direct limits exist in the category $\mathcal{S}[\mathbf{A}]$. Concrete constructions of finite products and equalizers are given.

### 3.9 Finite direct limits

Let I be a nonempty finite set and

$$(\mathcal{X}_i = (X_i, e_i, d_i))_{i \in I}$$

be a family of objects in $\mathcal{S}[\mathbf{A}]$. Let

$$X = \prod_{i \in I} X_i$$

be the cartesian product set of $(X_i)_{i \in I}$ and

$$(\pi_i : X \to X_i)_{i \in I}$$

be the family of canonical projections of X. Define two functions

$$e_X : X \times X \to A \otimes 1$$

and

$$d_X : X \times X \to A \oplus 0$$

such that for all $x, y \in X$:

$$e_X(x, y) = \bigotimes_{i \in I} e_i(\pi_i(x), \pi_i(y))$$

and

$$d_X(x, y) = \bigoplus_{i \in I} d_i(\pi_i(x), \pi_i(y)).$$

Then the triplet

$$\mathcal{X} = (X, e_X, d_X)$$

is an object in $\mathcal{S}[\mathbf{A}]$ such that

$$(\forall i \in I)\ \pi_i \in Hom(\mathcal{X}, \mathcal{X}_i).$$

*The system* $(\mathcal{X}, (\pi_i)_{i \in I})$ *is a direct product of the family* $(\mathcal{X}_i)_{i \in I}$ *in the category* $\mathcal{S}[\mathbf{A}]$. This property is a consequence of the fact that for all $\mathcal{Y} = (Y, e_Y, d_Y) \in \mathcal{S}[\mathbf{A}]$, if

$$(\forall i \in I)\ p_i \in Hom(\mathcal{Y}, \mathcal{X}_i)$$

then there exists an unique function

$$f : Y \to X$$

defined by

$$(\forall y \in Y)(\forall i \in I)\ f(y)(i) = p_i(y)$$

such that

$$f \in Hom(\mathcal{Y}, \mathcal{X})$$

and

$$(\forall i \in I)\ \pi_i \cdot f = p_i.$$

Let $\mathcal{I}[\Delta] = (\mathcal{I}, e_\Delta, d_\Delta)$ be an object of $\mathcal{S}[\mathbf{A}]$ associated with the identity relation $\Delta$ on a set $\mathcal{I}$ having precisely one element. Then $\mathcal{I}[\Delta]$ *is a final object for* $\mathcal{S}[\mathbf{A}]$ i.e. the set $Hom(\mathcal{X}, \mathcal{I}[\Delta])$ has precisely one element, for each $\mathcal{X} \in \mathcal{S}[\mathbf{A}]$. It follows that the following condition holds:

(1) *Finite products exist in the category* $\mathcal{S}[\mathbf{A}]$.

Now we show that:

(2) *Equalizers exist in the category* $\mathcal{S}[\mathbf{A}]$.

Suppose that

$$f, g \in Hom(\mathcal{X}, \mathcal{Y}),$$

where

$$\mathcal{X} = (X, e_X, d_X), \mathcal{Y} = (Y, e_Y, d_Y) \in \mathcal{S}[\mathbf{A}].$$

Define a subset Z of X by

$$Z = \{ x \in X / f(x) = g(x) \}$$

and two functions

$$e_Z : Z \times Z \to A \otimes 1$$

and

$$d_Z : Z \times Z \to A \oplus 0$$

such that for all $u, v \in Z$:

$$e_Z(u, v) = e_X(u, v);$$
$$d_Z(u, v) = d_X(u, v).$$

Let $\sigma : Z \to X$ be the inclusion map of Z in X i.e. $(\forall z \in Z)\ \sigma(z) = z$. Then

$$\mathcal{Z} = (Z, e_Z, d_Z) \in \mathcal{S}[\mathbf{A}]$$

and the system $(\mathcal{Z}, \sigma)$ is an equalizer of the couple (f, g) in the category $\mathcal{S}[\mathbf{A}]$.

From the properties (1) and (2) it follows that the following conditions hold:

(3) *Finite direct limits and pullbacks exist in the category* $\underline{\mathcal{S}}[\mathbf{A}]$.

A *complete biresiduated algebra* is any system $\mathbf{A} \in \underline{BR}$ such that $(A, \wedge, \vee, 0, 1)$ is a complete lattice.

The next result was established in [9] and it presents in a synthetical form new properties of the category $\underline{\mathcal{S}}[\mathbf{A}]$ if **A** is a complete D-algebra.

### 3.10 Theorem

If $\mathbf{A} \in \underline{D}^{*}$ is a biresiduated algebra associated with a *complete D-algebra* then $\underline{\mathcal{S}}[\mathbf{A}]$ *is a cartesian closed category.*

Theorem 3.10 follows from 3.9(3) and the fact that for all $\mathcal{X}, \mathcal{Y} \in \mathcal{S}[\mathbf{A}]$, if **A** is associated with a complete D-algebra then there exists a standard structure of **A**-valued space on the set $Hom(\mathcal{X}, \mathcal{Y})$ and one can define a functor

$$E^{\mathcal{X}} = (\ \ )^{\mathcal{X}} : \underline{\mathcal{S}}[\mathbf{A}] \to \underline{\mathcal{S}}[\mathbf{A}]$$

as follows, for all $\mathcal{Y} \in \mathcal{S}[\mathbf{A}]$, $g \in Hom(\mathcal{Y}, \mathcal{Z})$ and $f \in E^{\mathcal{X}}(\mathcal{Y})$):

- $E^{\mathcal{X}}(\mathcal{Y}) = Hom(\mathcal{X}, \mathcal{Y})$;
- $E^{\mathcal{X}}(g)(f) = g \cdot f$

such that the following condition holds:

- $E^{\mathcal{X}}$ is *the exponentiation functor by* $\mathcal{X}$.

The precedent condition means that $E^{\mathcal{X}}$ *is a right adjoint of the functor direct product by* $\mathcal{X}$,

$$\Pi[\mathcal{X}] = (\ \ ) \times \mathcal{X} : \underline{\mathcal{S}}[\mathbf{A}] \to \underline{\mathcal{S}}[\mathbf{A}].$$

If **A** is associated with a complete D-algebra one can prove also that the *inverse limits exist in the category* $\underline{\mathcal{S}}[\mathbf{A}]$.

In order to elaborate a new interesting and comprehensive mathematical theory of fuzzy sets having an unitary logical foundation, different new ways to make the class $\mathcal{S}[\mathbf{A}]$ into a category can be considered. For this purpose, a good source of inspiration is the elementary toposes theory together with complete $\Omega$-sets theory, where $\Omega$ is a complete Heyting algebra [5]. A first goal is to identify an adequate class of many-valued mathematical models including sheaves. A completeness theorem with respect to these models also must be proved.

## REFERENCES


1. BALBES, R. and DWINGER, PH., **Distributive Lattices**, UNIVERSITY OF MISSOURI PRESS, 1974.
2. BOICESCU, V., FILIPOIU, A., GEORGESCU, G. and RUDEANU S., **Lukasiewicz-Moisil Agebras**, NORTH-HOLLAND, 1991.
3. CHANG, C. C., **Algebraic Analysis of Many-valued Logics,** TRANS. AMER. MATH. SOC. 88, 1958, pp. 467-490.
4. CIGNOLI, R., D'OTTAVIANO L. M. and MUNDICI, D., **Algebraic Foundations of Many-valued Reasoning**, KLUWER ACADEMIC PUBL., DORDRECHT.
5. FOURMAN, M. P. and SCOTT, D. A., **Sheaves and Logic**, Lectures Notes in Mathematics, Vol.753, 1979, pp. 302-401.
6. GOGUEN, J. A., **L-fuzzy sets**, JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **18** (1967) 145-174.
7. NEGOITA, C. V. and RALESCU, D. A., **Simulation, Knowledge-base Computing and Fuzzy Statistics**, VAN NOSTRAND, 1987.
8. PONASSE, D. and CARREGA, J. C., **Algèbre et topologie booléennes**, MASSON, Paris, 1979.
9. SULARIA, M., **Contributions to the Study of Some Formal Systems from the Point of View of Algebraic Logic**, Ph. D dissertation, Babes-Bolyai University of Cluj-Napoca, Romania, 1986.
10. SULARIA, M., **A Logical System for Multicriteria Decision Analysis**, Studies in Informatics and Control, Vol. 7, No.3, September 1998, pp. 237-260.
11. TURUNEN, E., **Residuated Lattices in Fuzzy Logical Systems**, in T. Terano, M. Sugeno , M. Mukaidono and K. Shigemashu (Eds.) Fuzzy Engineering, Proceedings of the International Fuzzy Engineering Symposium'91, pp.60-69.
12. ZADEH, L.A., **Fuzzy Sets**, INFORMATION AND CONTROL, No. 8, 1965, pp.338-353.