A Contribution to the Filter Design for a Class of Linear Singular Delayed Systems: The Discrete Time Case

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Abstract: This paper proposes a filter scheme for linear singular systems. The considered models are affected by an unknown input vector and a bounded disturbance vector which are present in both dynamic and output equations. To that, a variable-time delay also affects the state vector and the known input vector. This paper sets forth an unbiased H_∞ filter dynamics based on the Lyapunov-Krasovskii method and using a Linear Matrix Inequality (LMI) approach. The designed estimator manages to reconstruct both the functional state vector and the functional unknown input vector despite the effect of the bounded disturbance. A numerical example is provided with the purpose of highlighting the effectiveness of the proposed approach.

Keywords: Functional filter, Unknown inputs, Singular systems, Discrete-time case, Variable-time delay.

1. Introduction

The potency of singular systems is due to their ability to introduce real dynamics in the presence of algebraic constraints (Hamzaoui, Khadhraoui & Messaoud, 2020a; Hamzaoui, Khadhraoui & Messaoud, 2020b; Kchaou et al., 2014). Despite their complex mathematical structure, descriptor models are widely used for estimation and control purposes (Meddeb, Jmii & Chebbi, 2018; Ezzine et al., 2011b).

Moreover, unknown input topics have been considered as the main focus of a large part of research (Hamzaoui, Khadhraoui & Messaoud, 2020c; Zarrougui et al., 2020). In fact, such modelling invites researchers to analyse the fault detection subjects and the enhancement of the dynamics of the controller and observer (Larroque, 2008; Ezzine et al., 2011a).

Furthermore, including bounded disturbance in the state equations is quite necessary. Many researchers have suggested to employ a set of algorithms for the controller and filter design (Johnson, 1975; Ezzine et al., 2011a) beside the effects of the considered disturbance according to a H_∞ and/or H_2 criterion (Gu et al., 2020).

The employed criteria which are aimed at attenuating the disturbance effect on considered dynamics are usually added to stability constraints such as the Lyapunov functional method in (Krasovskii, 1963) in order to obtain both a stable and disturbance-immune dynamics.

In view of these facts, this paper makes a contribution to the filter design systems destressed by unknown inputs and bounded disturbances both with regard to dynamic and output equations of the considered state representation.

Variable-time delay is also presented in both state and known input vectors. This additional parameter is highly important, at it introduces the dynamic system input propagation, and thus, its major effect on the system stability (Sassi et al., 2017; Hamzaoui, Khadhraoui & Messaoud, 2020a).

The presented filter estimates a state functional and an unknown input functional for system with variable time delay based on H_∞ criteria. The given filter scheme is developed in a discrete time domain and related to the results of the continuous time domain (Hamzaoui, Khadhraoui & Messaoud, 2020a) The suggested stable unbiased dynamics has to be determined by choosing a suitable Lyapunov function added to a H_∞ criterion bounded by a fixed value γ .

This work is based on unknown input reconstruction (Zasadzinski et al., 1998) and continuous time filter design (Hamzaoui, Khadhraoui & Messaoud, 2020a). When considering variable time delay, the authors proposed the delay dependent technique in (Hamzaoui, Khadhraoui & Messaoud, 2020a) whereas this paper presents an independent method for the discrete-time case.

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The remainder of this paper is structured as follows. Section 2 presents the difference between this paper and recently published works. H_∞ filter design problem is formulated in Section 3 and it leads to certain assumptions. Section 4 sets forth the time domain design by formulating an unbiased estimation error and a H_∞ stable dynamics according to the Lyapunov–Krasovskii theory (Krasovskii, 1963). Section 5 presents the functional filter design. Section 6 proves the efficiency of the proposed approach by providing a numerical example. Finally, section 7 includes the conclusion of this paper

2. Problem Formulation

Let's consider the following discrete time linear, singular and variable delay systems presented as follows:

$$Ex(k+1) = Ax(k) + A_d x(k-\tau(k)) + Bu(k) + B_d u(k-\tau(k)) + F_1 v(t) + G_1 w(k) \quad (1)$$

$$y(k) = Cx(k) + F_2 v(k) + G_2 w(k) \quad (2)$$

$$z(k) = Lx(k) \quad (3)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the known input vector, $y \in \mathbb{R}^p$ is the output vector and then $z \in \mathbb{R}^r$ is a state functional vector. $v(t) \in \mathbb{R}^q$ and $w \in \mathbb{R}^k$ represent the unknown input vector and the bounded disturbances vector affecting the dynamic and the output equations.

n, m, p, r, q, k are the dimensions of the state vector, the known input vector, the output vector, the functional vector, the unknown input vector and the bounded disturbances vector respectively.

$E, A, A_d, B, B_d, F_1, F_2, G_1, G_2, L$ and C are previously known matrices of the appropriate dimensions.

For simplicity and without loss of generality, it can be assumed that the variable delays $\tau(k)$ at the level of state and input vectors are identical. These delays meet the following:

$$0 \le \tau(k) \le \tau_m \quad (4)$$

Hypothesis 1: Thus, the following conditions are obtained:

1. $rank(E) = \bar{n} < n$
2. $rank\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = q$
3. $rank\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = k$
4. $rank[F_2] = \bar{q} < q$

Based on Zasadzinski et al. (1998) there exists an orthogonal matrix, namely $R \in \mathbb{R}^{p\times p}$ and a non-singular matrix namely $S \in \mathbb{R}^{q\times q}$, such that:

$$R^T F_2 S = \begin{pmatrix} I_{\bar{q}} & 0 \\ 0 & 0 \end{pmatrix} \quad (5)$$

By multiplying equation (2) above by R^T, the system in equations (1)-(3) is equivalent to:

$$Ex(k+1) = \bar{A}x(k) + A_d x(k-\tau(k)) + Bu(k) + B_d u(k-\tau(k)) + \bar{G}_1 w(k) + F_{11} y_1(t) + F_{12} v_2(k) \quad (6)$$

$$y_1(k) = C_1 x(k) + v_1(k) + G_{21} w(k) \quad (7)$$

$$y_2(k) = C_2 x(k) + G_{22} w(k) \quad (8)$$

$$z(k) = Lx(k) \quad (9)$$

where

$$R^T y(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} \quad (10)$$

with $y_1(k) \in \mathbb{R}^{\bar{q}}$ and $y_2(k) \in \mathbb{R}^{p-\bar{q}}$

$$R^T C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \quad (11)$$

with $C_1 \in \mathbb{R}^{\bar{q}\times n}$ and $C_2 \in \mathbb{R}^{(p-\bar{q})\times n}$

$$R^T G_2 = \begin{bmatrix} G_{21} \\ G_{22} \end{bmatrix} \quad (12)$$

with $G_{21}(k) \in \mathbb{R}^{\bar{q}\times k}$ and $G_{22} \in \mathbb{R}^{(p-\bar{q})\times k}$

$$S^{-1} v(k) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (13)$$

with $v_1(k) \in \mathbb{R}^{\bar{q}}$ and $v_2(k) \in \mathbb{R}^{(p-\bar{q})}$

$$F_1 S = \begin{bmatrix} F_{11} \\ F_{12} \end{bmatrix} \quad (14)$$

$$\bar{A} = A - F_{11} C_1 \quad (15)$$

and

$$\bar{G} = G_1 - F_{11} G_{21} \quad (16)$$

The aim of this research is to develop a H_∞ unknown input filter in a discrete time domain for a singular linear system. The given system is stirred by a bounded disturbance acting on output equations as well as on dynamic aquations. The proposed filter will estimate the functional state $z(k)$ and the unknown functional input $v_1(k)$ derived from (7).

Hereby, the reconstruction of a new state vector is suggested:

$$\bar{z}(t)=\begin{bmatrix}Lx(t)\\ v_1(t)\end{bmatrix}\in\mathbb{R}^{(r+\bar{q})} \quad (17)$$

Using equations (7) and (9), $\bar{z}(k)$ can be written as:

$$\bar{z}(k)=\bar{L}x(k)+\bar{G}_{21}w(k)+\bar{I}y_1(k) \quad (18)$$

with

$$\bar{L}=\begin{bmatrix}L\\ -C_1\end{bmatrix},\bar{D}_1=\begin{bmatrix}0_{r\times m}\\ -D_1\end{bmatrix},\bar{G}_{21}=\begin{bmatrix}0_{r\times k}\\ -G_{21}\end{bmatrix},\bar{I}=\begin{bmatrix}0_{r\times\bar{q}}\\ I_{\bar{q}}\end{bmatrix} \quad (19)$$

$rank(\bar{L})=r+\bar{q}<n$ and $\bar{L}\in\mathbb{R}^{(r+\bar{q})\times n}$

Based on Hamzaoui, Khadhraoui & Messaoud, (2020a) and Khadhraoui et al. (2014), it is supposed that:

$$rank\begin{pmatrix}E\\ C_2\\ \bar{L}\end{pmatrix}=rank\begin{pmatrix}E\\ C_2\end{pmatrix} \quad (20)$$

As a result, a matrix T is non-singular:

$$T=\begin{pmatrix}a & b\\ c & d\end{pmatrix} \quad (21)$$

such that

$$aE+bC_2=\bar{L} \quad (22)$$

$$cE+dC_2=0 \quad (23)$$

with $a\in\mathbb{R}^{(r+\bar{q})\times n}$, $b\in\mathbb{R}^{(r+\bar{q})\times(p-\bar{q})}$, $c\in\mathbb{R}^{(p-\bar{q})\times n}$ and $d\in\mathbb{R}^{(p-\bar{q})\times(p-\bar{q})}$.

3. Time Domain Design

The main goal of this paper is to propose the design of a functional filter for the system defined by equations (6)-(9) of the form:

$$\begin{aligned}\lambda(k+1)&=M\lambda(k)+M_d\lambda(k-\tau(k))+Hu(k)\\ &+H_du(k-\tau(k))+N_1y_1(k)+N_2y_2(k)\\ &+N_dy_1(k-\tau(k))\end{aligned} \quad (24)$$

$$\hat{\bar{z}}(k)=\lambda(k)+L_1y_1(k)+L_2y_2(k) \quad (25)$$

Here, $\lambda(k)\in R^{r+\bar{q}}$ represents the state vector of the filter and $\hat{\bar{z}}(t)$ is the estimation of the functional state $\bar{z}(t)\in R^{r+\bar{q}}$.

The following matrices $M,M_d,H,H_d,N_1,N_2,N_d,L_1$ and L_2 will be reached using the LMI approach.

3.1 Conditions of the Functional Filter Synthesis

$e(k)$ will be defined as a time estimation error:

$$e(k)=\bar{z}(k)-\hat{\bar{z}}(k) \quad (26)$$

Using (18) and (25), $e(k)$ is given by:

$$\begin{aligned}e(k)&=(\bar{I}-L_1)y_1(k)+(\bar{L}-L_2C_2)x(k)\\ &+(\bar{G}_{21}-L_2G_{22})w(k)-\lambda(k)\end{aligned} \quad (27)$$

It is assumed that:

$$L_1=\bar{I} \quad (28)$$

So, equation (27) is equivalent to:

$$\begin{aligned}e(k)&=(\bar{L}-L_2C_2)x(k)-\lambda(k)\\ &+(\bar{G}_{21}-L_2G_{22})w(k)\end{aligned} \quad (29)$$

Using (22) and (23) the following can be obtained:

$$\begin{aligned}e(k)&=(aE+bC_2-L_2C_2)x(k)+\beta(cE+dC_2)x(k)\\ &+(\bar{G}_{21}-L_2G_{22})w(k)-\lambda(k)\\ &=(a+\beta c)Ex(k)+(b+\beta d-L_2)C_2x(k)\\ &+(\bar{G}_{21}-L_2G_{22})w(k)-\lambda(k)\end{aligned} \quad (30)$$

where $\beta\in\mathbb{R}^{(r+\bar{q})\times(p-\bar{q})}$

It is assumed that:

$$L_2=b+\beta d \quad (31)$$

Equation (30) can be expressed as follows:

$$e(k)=\varphi Ex(k)-\lambda(k)+(\bar{G}_{21}-L_2G_{22})w(k) \quad (32)$$

with

$$\varphi=a+\beta c\ ;\ \varphi\in\mathbb{R}^{(r+\bar{q})\times n} \quad (33)$$

The following matrices, namely M,M_d,H,H_d, N_1,N_2,N_d,L_1 and L_2 can be determined as:

1. $\lim_{k\to\infty}e(k)=0$, if $w=0$
2. The filter error as in (32) must be asymptotically stable if it satisfies the H_∞ performance given by equation (34).

The H_∞ criterion is then:

$$0<\|H_{ew}\|=\sup_{w\neq0}\frac{\|e\|_\infty}{\|w\|_\infty}<\gamma \quad (34)$$

with $H_{ew}(s)=\dfrac{e(s)}{w(s)}$ as a transfer matrix and γ as a positive scalar.

Therefore, the following theorem is proposed:

Theorem 1: The unbiased estimation error presented in (32) as relative to the system expressed by equations (1) - (3) and functional H_∞ filter expressed by equations (24) and (25) are confirmed as follows:

$$\begin{aligned} e(k+1) &= Me(k) + M_d e(k-\tau(k)) + \alpha w(k) \\ &\quad + \varsigma w(k-\tau(k)) + \delta w(k+1) \end{aligned} \tag{35}$$

with

$$\alpha = \varphi\bar{G} - M(\bar{G}_{21} - L_2 G_{22}) N_2 G_{22} \tag{36}$$

$$\varsigma = -M_d(\bar{G}_{21} - L_2 G_{22}) N_d G_{22} \tag{37}$$

$$\delta = \bar{G}_{21} - L_2 G_{22} \tag{38}$$

if and only if the following equations are confirmed:

i. $\varphi\bar{A} - M\varphi E + N_2 C_2 = 0$

ii. $\varphi A_d - M_d \varphi E + N_d C_2 = 0$

iii. $\varphi B - H = 0$

iv. $\varphi B_d - H_d = 0$

v. $N_1 - \varphi F_{11} = 0$

vi. $\varphi F_{12} = 0$

Proof: The time derivative of $e(t)$ as it is given in (32) will be:

$$\begin{aligned} e(k+1) &= \varphi Ex(k+1) - \lambda(k+1) \\ &\quad + (\bar{G}_{21} - L_2 G_{22}) w(k+1) \end{aligned} \tag{39}$$

By replacing $Ex(k+1)$ and $\lambda(k+1)$ by their expressions given by (6) and (24) respectively, the following is obtained:

$$\begin{aligned} e(k+1) &= Me(k) + M_d e(k-\tau(k)) \\ &\quad + [\varphi\bar{G} - M(\bar{G}_{21} - L_2 C_{22}) N_2 G_{22}] w(k) \\ &\quad + [-M_d(\bar{G}_{21} - L_2 G_{22}) N_d G_{22}] w(k-\tau(k)) \\ &\quad + [\varphi\bar{A} - M\varphi E - N_2 C_2] x(k) \\ &\quad + [\varphi A_d - M_d \varphi E - N_d C_2] x(k-\tau(k)) \\ &\quad + [\varphi B - H] u(k) + [\varphi B_d - H_d] u(k-\tau(k)) \\ &\quad + [\varphi F_{11} - N_1] y_1(k) + \varphi F_{12} v_2(k) \\ &\quad + [\bar{G}_{21} - L_2 G_{22}] w(k+1) \end{aligned} \tag{40}$$

Using the conditions (i) - (vi) of Theorem 1, and assuming that $w(k) = 0$, $e(k+1) = Me(k) + M_d e(k-\tau(k))$, the following is obtained: $\lim_{k\to\infty} e(k) = 0$.

At this point, the proposed functional filter can be designed independently from the state delay using the Lyapunov-Krasovskii stability theory expressed in (24) and a LMI approach. The estimated functional state $\hat{\bar{z}}(k)$ converges asymptotically to $\bar{z}(k)$ with condition satisfaction as in (34). So, by replacing φ as it is given by (33) for conditions i), ii) and vi) of Theorem 1, after using (22), the following is obtained:

$$a\bar{A} = MaE + JC_2 - \beta c\bar{A} \tag{41}$$

$$aA_d = M_d aE + J_d C_2 - \beta c A_d \tag{42}$$

$$aF_{12} = -\beta c F_{12} \tag{43}$$

with

$$J = N_2 - M\beta d \tag{44}$$

$$J_d = N_d - M_d \beta d \tag{45}$$

Equations (41) - (43) can be expressed in the following matrix form:

$$\Gamma = X\Theta \tag{46}$$

$$X = [M \quad M_d \quad J \quad J_d \quad \beta] \tag{47}$$

$$\Theta = \begin{pmatrix} aE & 0 & 0 \\ 0 & aE & 0 \\ C_2 & 0 & 0 \\ 0 & C_2 & 0 \\ -c\bar{A} & -cA_d & -cF_{12} \end{pmatrix} \tag{48}$$

$$\Gamma = \begin{bmatrix} a\bar{A} & aA_d & aF_{12} \end{bmatrix} \tag{49}$$

Thus, the general solution of (46) exists if and only if:

$$rank \begin{bmatrix} \Theta \\ \Gamma \end{bmatrix} = rank(\Theta) \tag{50}$$

Based on condition (50), the following is obtained:

$$X = \Gamma\Theta^+ - Z(I - \Theta\Theta^+) \tag{51}$$

Θ^+ is a general inverse of matrix Θ but Z is an arbitrary matrix. The matrix Z will be determined later on by using the LMI approach.

The matrix M is presented as:

$$M = X \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = M_1 - ZM_2 \tag{52}$$

By replacing the given X from (51) in (52), one obtains:

$$M_1 = \Gamma\Theta^+\begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad M_2 = (I - \Theta\Theta^+)\begin{pmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{53}$$

The matrix M_d is determined:

$$M_d = X\begin{pmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \end{pmatrix} = M_{d1} - ZM_{d2} \tag{54}$$

with

$$M_{d1} = \Gamma\Theta^+\begin{pmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad M_{d2} = (I - \Theta\Theta^+)\begin{pmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{55}$$

Similarly, J is obtained as:

$$J = X\begin{pmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \end{pmatrix} = J_1 - ZJ_2 \tag{56}$$

with

$$J_1 = \Gamma\Theta^+\begin{pmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \end{pmatrix}, \quad J_2 = (I - \Theta\Theta^+)\begin{pmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \end{pmatrix} \tag{57}$$

J_d is obtained as:

$$J_d = X\begin{pmatrix} 0 \\ 0 \\ 0 \\ I \\ 0 \end{pmatrix} = J_{d1} - ZJ_{d2} \tag{58}$$

with

$$J_{d1} = \Gamma\Theta^+\begin{pmatrix} 0 \\ 0 \\ 0 \\ I \\ 0 \end{pmatrix}, \quad J_{d2} = (I - \Theta\Theta^+)\begin{pmatrix} 0 \\ 0 \\ 0 \\ I \\ 0 \end{pmatrix} \tag{59}$$

The matrix β can be expressed as:

$$\beta_d = X\begin{pmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \beta_1 - Z\beta_2 \tag{60}$$

with

$$\beta_1 = \Gamma\Theta^+\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{pmatrix}, \quad \beta_2 = (I - \Theta\Theta^+)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{pmatrix} \tag{61}$$

Using equations (44), (52) and (60), N_2G_{22} can be expressed as:

$$\begin{aligned} N_2G_{22} &= (J + M\beta d)G_{22} \\ &= JG_{22} + (M_1 - ZM_2)(\beta_1 - Z\beta_2)dG_{22} \\ &= JG_{22} + M_1(\beta_1 - Z\beta_2)dG_{22} \\ &= (ZM_2\beta_1 - ZM_2Z\beta_2)dG_{22} \end{aligned} \tag{62}$$

Similarly using equations (45), (53) and (60), N_dG_{22} can be written as:

$$\begin{aligned} N_dG_{22} &= (J_d + M_d\beta d)G_{22} \\ &= J_dG_{22} + (M_{d1} - ZM_{d2})(\beta_1 - Z\beta_2)dG_{22} \\ &= J_dG_{22} + M_{d1}(\beta_1 - Z\beta_2)dG_{22} \\ &= (ZM_{d2}\beta_1 - ZM_{d2}Z\beta_2)dG_{22} \end{aligned} \tag{63}$$

The matrices N_2G_{22} and N_dG_{22} have a bilinearity marked by (ZM_2Z) and $(ZM_{d2}Z)$. To avoid this bilinearity, it should be assumed that the gain matrix Z meets the following relation:

$$Z\beta_2 = 0 \tag{64}$$

So there always exists a matrix Z_1 such that:

$$Z = Z_1(I - \beta_2\beta_2^+) = Z_1\Delta \tag{65}$$

β_2^+ is the general inverse matrix of β_2, $\Delta = I - \beta_2\beta_2^+$.

Then, M, M_d, J, J_d and β can be defined as:

$$M = M_1 - Z_1\Delta M_2 \tag{66}$$

$$M_d = M_{d1} - Z_1\Delta M_{d2} \tag{67}$$

$$J = J_1 - Z_1\Delta J_2 \tag{68}$$

$$J_d = J_{d1} - Z_1\Delta J_{d2} \tag{69}$$

$$\beta = \beta_1 \tag{70}$$

If equations (36) - (38) are rewritten by using equations (65) - (70), it can be inferred that:

$$\alpha = \alpha_1 - Z_1\Delta\alpha_2 \tag{71}$$

$$\varsigma = \varsigma_1 - Z_1\Delta\varsigma_2 \tag{72}$$

$$\delta = \bar{G}_{21} - bG_{22} - \beta_1 dG_{22} \tag{73}$$

with

$$\alpha_1 = \varphi\bar{G} - M_1(\bar{G}_{21} - bG_{22}) - J_1 G_{22} \quad (74)$$

$$\alpha_2 = -M_2(\bar{G}_{21} - bG_{22}) - J_d G_{22} \quad (75)$$

$$\varsigma_1 = M_{d1}(\bar{G}_{21} - bG_{22}) - J_{d1} G_{22} \quad (76)$$

$$\varsigma_2 = M_{d2}(\bar{G}_{21} - bG_{22}) - J_{d2} G_{22} \quad (77)$$

The filter matrices can only be determined if matrix Z_1 is determined.

The dynamic error given by (35) contains a time derivative of $w(k)$. This problem should generally be solved by adding a constraint to the filter matrices or by selecting a new norm for the respective system.

Based on Khadhraoui et al. (2017), equation (35) could be reformulated into a singular form, as follows:

$$\begin{bmatrix} I & -\delta \\ 0 & 0 \end{bmatrix}\begin{bmatrix} e(k+1) \\ w(k+1) \end{bmatrix} = \begin{bmatrix} M & \alpha \\ 0 & -I \end{bmatrix}\begin{bmatrix} e(k) \\ w(k) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w(k) + \begin{bmatrix} M_d & \delta \\ 0 & 0 \end{bmatrix}\begin{bmatrix} e(k-\tau(k)) \\ w(k-\tau(k)) \end{bmatrix} \quad (78)$$

Let's consider that:

$$\varepsilon(k) = \begin{bmatrix} e(k) \\ w(k) \end{bmatrix} \quad (79)$$

Then, the following can be obtained:

$$\rho\varepsilon(k+1) = \bar{M}\varepsilon(k) + \bar{M}_d\,\varepsilon(k-\tau(k)) + \bar{K}w(k) \quad (80)$$

where

$$\rho = \begin{bmatrix} I & -\delta \\ 0 & 0 \end{bmatrix},\ \bar{M} = \begin{bmatrix} M & \alpha \\ 0 & -I \end{bmatrix} \quad (81)$$

$$\bar{M}_d = \begin{bmatrix} M_d & \delta \\ 0 & 0 \end{bmatrix},\ \bar{K} = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad (82)$$

The LMI approach was proposed for the following theorem in order to determine how the gain Z_1 parametrizes the H_∞ filter matrices.

Theorem 2: The system given in (24)-(25) is thus a functional filter for the system expressed by (1)-(3) if there exist matrices such as $P = P^T > 0$, P_2, $Q = Q^T > 0$ and γ which are the solutions for the following LMI that should be given as $\gamma > 0$:

$$P\rho = \rho^T P^T > 0 \quad (83)$$

$$P = \begin{bmatrix} P_1 & P_2^T \\ P_2 & P_3 \end{bmatrix} \quad (84)$$

$$P_2 = -\delta^T P_1 \quad (85)$$

$$Q = \begin{bmatrix} Q_1 & Q_2^T \\ Q_2 & Q_3 \end{bmatrix} \quad (86)$$

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} & \Pi_{16} & \Pi_{17} \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} & \Pi_{26} & \Pi_{27} \\ * & * & \Pi_{33} & \Pi_{34} & \Pi_{35} & \Pi_{36} & \Pi_{37} \\ * & * & * & \Pi_{44} & \Pi_{45} & \Pi_{46} & \Pi_{47} \\ * & * & * & * & \Pi_{55} & \Pi_{56} & \Pi_{57} \\ * & * & * & * & * & \Pi_{66} & \Pi_{67} \\ * & * & * & * & * & * & \Pi_{77} \end{pmatrix} < 0 \quad (87)$$

where δ and ρ are given by (73) and (81) and the matrix Π can be further expressed as:

$$\Pi_{11} = -P_1 \quad (88)$$

$$\Pi_{12} = \Pi_{16} = -\Pi_{17} = -P_1\delta \quad (89)$$

$$\Pi_{13} = P_1 M_1 = -YM_2 \quad (90)$$

$$\Pi_{14} = P_1\alpha_1 - Y\alpha_2 + P_1\delta \quad (91)$$

$$\Pi_{15} = P_1 M_{d1} - YM_{d2} \quad (92)$$

$$\Pi_{22} = -\Pi_{27} = -P_3 \quad (93)$$

$$\Pi_{23} = -\delta^T P_1 M_1 + \delta^T YM_2 \quad (94)$$

$$\Pi_{24} = -\delta^T P_1\alpha_1 + \delta^T Y\alpha_2 - P_3 \quad (95)$$

$$\Pi_{25} = -\delta^T P_1 M_{d1} + \delta^T YM_{d2} \quad (96)$$

$$\Pi_{26} = -\delta^T P_1\delta \quad (97)$$

$$\Pi_{33} = Q_1 - P_1 + I \quad (98)$$

$$\Pi_{34} = Q_1\delta + P_1\delta \quad (99)$$

$$\Pi_{35} = \Pi_{36} = \Pi_{37} = \Pi_{45} = \Pi_{46} = \Pi_{47} = \Pi_{57} = \Pi_{67} = 0 \quad (100)$$

$$\Pi_{44} = -\delta^T Q_1\delta - \delta^T P_1\delta + I \quad (101)$$

$$\Pi_{55} = -Q_1 \quad (102)$$

$$\Pi_{56} = Q_1\delta \quad (103)$$

$$\Pi_{66} = -\delta^T Q_1\delta \quad (104)$$

$$\Pi_{77} = \gamma^2 \quad (105)$$

The gain Z_1 is given by:

$$Z_1 = P^{-1}Y \quad (106)$$

Proof: In order to prove Theorem 2, the following Lyapunov-Krasovskii function should be considered:

$$V(k) = \varepsilon^T(k)P\rho\,\varepsilon(k) + \sum_{i=k-\tau_m}^{k-1} \varepsilon^T(i)\rho^T Q\,\varepsilon(k) \quad (107)$$

where P and Q are Positive Definite Matrices.

Defining $\Delta V(k) = V(k+1) - V(k)$ yields:

$$\begin{aligned}\Delta V(k) = &\ \varepsilon^T(k)[\bar{M}^T P\bar{M} + \rho^T(P-Q)\rho]\varepsilon(k)\\ &+ \varepsilon^T(k)\bar{M}^T P\bar{M}_d\ \varepsilon(k-\tau_m)\\ &+ \varepsilon^T(k-\tau_m)\bar{M}_d^T P\bar{M}\varepsilon(k) + w^T(k)\bar{K}^T P\bar{K}w(k)\\ &+ \varepsilon^T(k-\tau_m)[\bar{M}^T P\bar{M}_d - \rho^T Q\rho]\varepsilon(k-\tau_m)\\ &+ \varepsilon^T(k)\bar{M}^T P\bar{K}\ w(k) + \varepsilon^T(k-\tau_m)\bar{M}_d^T P\bar{K}\ w(k)\\ &+ w^T(k)\bar{K}^T P\bar{M}\ \varepsilon(k) + w^T(k)\bar{K}^T P\bar{M}_d\ \varepsilon(k-\tau_m)\end{aligned} \quad (108)$$

To provide the necessary conditions for the existence of (24) and (25) and according to H_∞ criterion in (34), the following inequality should be recalled (Khadhraoui et al., 2017):

$$H(\varepsilon,k) = \Delta V(k) + \varepsilon^T(k)\varepsilon(k) - \gamma^2 w^T(k)w(k) < 0 \quad (109)$$

which can be written as follows:

$$\begin{bmatrix}\varepsilon^T(k) & \varepsilon^T(k-\tau_m) & w^T(k)\end{bmatrix}\Psi\begin{bmatrix}\varepsilon(k)\\ \varepsilon(k-\tau_m)\\ w(k)\end{bmatrix} < 0 \quad (110)$$

$$\Psi = \begin{pmatrix}\bar{Q} & \bar{M}^T P\bar{M}_d & \bar{M}^T P\bar{K}\\ * & \bar{M}^T P\bar{M}_d - \rho^T Q\rho & \bar{M}_d^T P\bar{K}\\ * & * & \bar{K}^T P\bar{K}\end{pmatrix} \quad (111)$$

$$\bar{Q} = \bar{M}^T P\bar{M} + \rho^T(P-Q)\rho + I \quad (112)$$

To get rid of the quadratic form presented in equation (112), this equation can be rewritten as follows:

$$\Psi = U - \Phi^T \Xi^{-1}\Phi \quad (113)$$

$$\Phi = \begin{pmatrix}\bar{M} & \bar{M}_d & \bar{K}\end{pmatrix} \quad (114)$$

$$\Xi = -P^{-1} \quad (115)$$

$$U = \begin{pmatrix}\rho^T(P-Q)\rho + I & 0 & 0\\ 0 & \rho^T Q\rho & 0\\ 0 & 0 & -\gamma^2\end{pmatrix} \quad (116)$$

According to the Schur lemma (Hamzaoui, Khadhraoui & Messaoud, 2020a), $\Psi < 0$ and $\Xi < 0$ if and only if:

$$\Lambda = \begin{pmatrix}\Xi & \Phi\\ \Phi^T & U\end{pmatrix} < 0 \quad (117)$$

Here, a congruence transformation (Hamzaoui, Khadhraoui & Messaoud, 2020b) is applied to Λ such that:

$$\Sigma^T \Lambda \Sigma < 0 \quad (118)$$

where Σ is a non-singular matrix given as:

$$\Sigma = \begin{pmatrix}P & 0 & 0 & 0\\ * & I & 0 & 0\\ * & * & I & 0\\ * & * & * & I\end{pmatrix} \quad (119)$$

If Λ and Σ are replaced in equation (118) by their expressions given by (117) end (119), respectively, Theorem 2 becomes obvious.

4. Algorithm for Functional Filter Design

Task 1) Checking if conditions 1), 2), 3) and 4) of Hypothesis 1 are evident.

Task 2) Calculating the matrix T by using (22) and (23).

Task 3) Calculating the matrices Θ and Γ by using (48) and (49).

Task 4) Deducing the values of matrices M_1, M_2, M_{d1} and M_{d2} from (53) and (55).

Task 5) Checking if condition (50) is evident.

Task 6) Solving the proposed LMI (87) in order to obtain matrix Z_1.

Task 7) Calculating M and M_d by using (52) and (54).

Task 8) Calculating N and N_d by using (44) and (45).

Task 9) Obtaining matrices L_1 and L_2 by using (28) and (31).

Task 10) Obtaining matrices H, H_d and N_1 by using iii), iv) and v) from Theorem 1.

5. A Numerical Example

Considering system (1)-(3) such that:

$$E = \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix},\ A = \begin{pmatrix}-7 & 0\\ 0 & -5\end{pmatrix},\ A_d = \begin{pmatrix}-1 & 0\\ 0 & -0.5\end{pmatrix},$$

$$B = \begin{pmatrix}3\\ -2\end{pmatrix},\ B_d = \begin{pmatrix}2\\ 3\end{pmatrix},\ F_1 = \begin{pmatrix}3 & 0\\ 5 & -1\end{pmatrix}, G_1 = \begin{pmatrix}-1\\ 1\end{pmatrix},$$

$$C = \begin{pmatrix}1 & 1\\ 1 & -2\end{pmatrix},\ G_2 = \begin{pmatrix}1\\ 0\end{pmatrix},\ L = \begin{pmatrix}3 & -1\end{pmatrix}$$

the variable state and the input delay have a sinusoid form, namely:

$$\tau(k) = 2 + 1.5\sin(k)$$

According to (4), the following is obtained: $0.5 \le \tau(k) \le 3.5$.

The known input $u(k)$, the unknown input $v(k)$ and the bounded disturbance $w(k)$ are successively illustrated in Figure 1, Figure 2 and Figure 3.

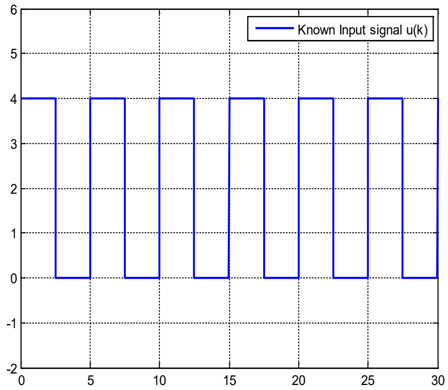

Figure 1. Known Input Signal u(k)

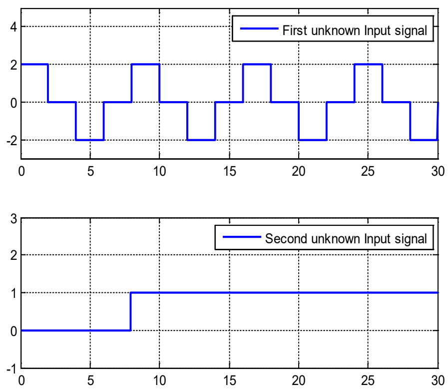

Figure 2. Unknown Input Signal v(k)

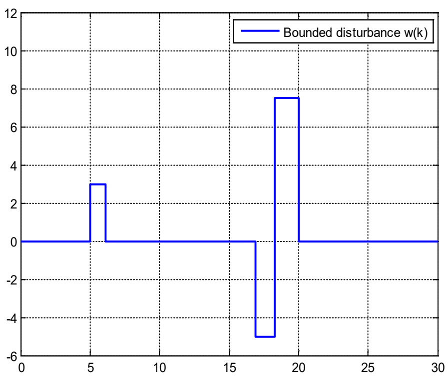

Figure 3. Bounded Disturbance Signal w(k)

Following the previously mentioned procedure, the following was obtained:

$$R = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix},\ S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The resolution of LMI (87) gives the following matrices:

$$P_1 = \begin{pmatrix} 0.028 & 0.056 \\ 0.056 & 0.108 \end{pmatrix},\ P_3 = 0.0161$$

$$Z_1 = \begin{pmatrix} 2.833 & 4.722 & 0.507 & 0.846 & 0 & 0 & 0 \\ -1.608 & -2.680 & -0.191 & -0.319 & 0 & 0 & 0 \end{pmatrix}$$

Thus, the values of the functional filter matrices are given as follows:

$$M = \begin{pmatrix} -9.634 & -0.641 \\ 5.689 & 0.232 \end{pmatrix},\ M_d = \begin{pmatrix} -1.243 & -0.405 \\ 0.632 & 0.054 \end{pmatrix}, L_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$H_d = \begin{pmatrix} 5 \\ -3 \end{pmatrix},\ H = \begin{pmatrix} 7.5 \\ -4.5 \end{pmatrix}, L_2 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix},\ N_1 = \begin{pmatrix} 3.75 \\ -2.25 \end{pmatrix}$$

$$N_2 = \begin{pmatrix} 1.875 \\ -1.125 \end{pmatrix},\ N_d = 10^{-15} \begin{pmatrix} 0.666 \\ -0.444 \end{pmatrix}$$

Figure 4 illustrates the difference between the real and the estimated functional state vector $\overline{z}(k)$. Similarly, Figure 5 shows the difference between the real and the estimated functional unknown input vector $v_1(k)$.

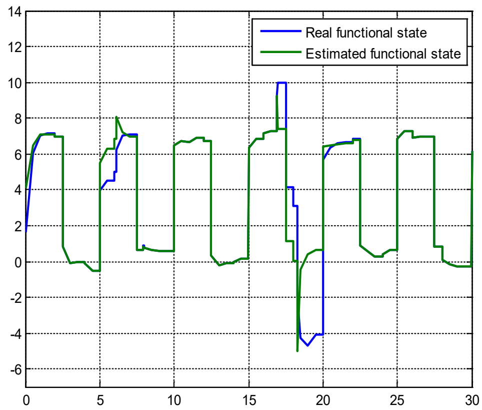

Figure 4. Functional state vector Z(k)

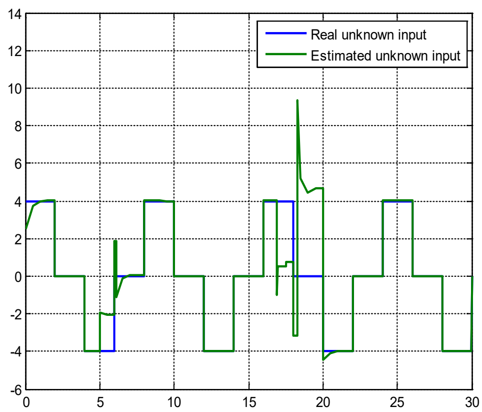

Figure 5. Functional unknown input vector $v_1(k)$

It is easy to notice the strong effect of the considered disturbance on the estimation error of the functional state vector $z(k)$ and functional unknown input vector $v_1(k)$ in Figures 4 and 5, respectively. This effect is quite clear in the transitory phase between the instants of 15s and 20s with important variation in the bounded disturbance signal $w(k)$ as it is shown in Figure 3. This effect is controlled by the H_∞ criterion and its value.

Then, the estimation error for the functional state vector $z(k)$ and the unknown functional input $v_1(k)$ are illustrated in Figures 6 and 7.

It can be noticed that the filter convergence has been reached with respect to the H_∞ disturbance attenuation criterion. This confirms the efficiency of the proposed approach. The disturbance effect on the estimation error is evaluated as:

$$\|H_{ew}\|_\infty = 0.987 < 1.2$$

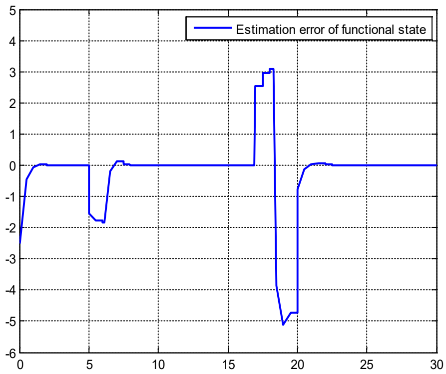

Figure 6. Estimation error for the functional state

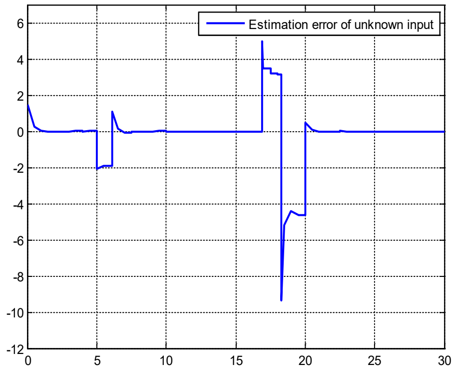

Figure 7. Estimation error for the functional unknown input vector

6. Conclusion

This paper presents the design procedure for a H_∞ filter. The proposed estimator has proved its efficiency through a given numerical example and manages to track the functional state vector and the considered unknown functional input vector, in spite of the effect of the difference between the initial conditions for real and estimated vectors and the transitory phase caused by the presence of the bounded disturbance. The proposed filter scheme is independent from the considered variable-time delay and was designed by choosing a suitable Lyapunov function in order to provide a stable estimation error.

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