Performance Analysis Of A Certain Type Of Multi-class Queueing Networks[1]

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Abstract: Queueing models of re-entrant lines with semi-infinite buffer capacities are developed under buffer priority scheduling policies. On the assumption of Poisson arrival and exponential processing times, the models can be formulated as a standard form of Quasi-birth-and-death(QBD) type. Then non-linear matrix equations can be used to solve the static performances of the systems. Not only can numerical results be worked out but also a preferable method for comparison between disciplines is available for such systems.

Keywords: multi-class queueing network, performance analysis, re-entrant lines, modelling

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1. Introduction

Re-entrant lines are common in semi-conductor manufacturing. During the cycle time parts of them may visit more than once the same machine, thus making it difficult to analyse the static performances. Only queueing networks which are of product form such as Jackson network could be explicitly solved so far. Most re-entrant lines are not of product form [1].

Scheduling policies and stability of the system have been extensively studied by many researchers. L.M.Wein listed twelve disciplines in [2]. Comparison among disciplines is mainly through simulations as we lack quantitative theories.

Both linear programming (LP) and fluid model methods have been used in studying the conditions of stability. By LP method one can work out upper and lower bound of a certain performance measure [3]. Fluid model is an asymptotic model of systems in the long-run. On the basis of the law of Large Numbers and the Central Limit theory, diffusion between fluid model and the original traffic equations can be approximated. Dai and Chen[4,5] have developed sufficient and necessary conditions of stability for two-station networks by means of the concept of virtual station and push start. Regretfully neither of them can lead to numerical results for performances.

The matrix-geometric solution method was proposed by M.F.Neuts late 70's [6]. This method has been extensively studied and widely used in communication networks, manufacturing systems and computer networks. We introduce it to re-entrant lines to show that it is preferable in solving complex Markovian networks.

Buffer capacities are assumed to be infinite in [3,4,5]. In practice they are always finite. We assume that all but the first buffer capacities are finite.

We also assume that the network is Markovian. So it can be modelled as QBD type under Poisson arrival rate and exponential processing times. The numerical results of static performances can be worked out through such models. The results also provide a criterion of comparison between scheduling disciplines.

The paper is organized as follows: re-entrant queueing network is illustrated in Section 2. QBD type model under random scheduling (RS) is developed in Section 3 together with the condition of stability. In Section 4 we model the network under last buffer first serve (LBFS) policy and analyse its stability. A comparison between RS and LBFS is made in Section 5. Final conclusions are drawn in Section 6.

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2. Illustration of the Network

Consider the system shown in Figure 1. Its behaviour under clear-a-fraction policy has been studied by Kumar [7]. Deterministic form under FIFO is studied by Whitt[8]. A dynamic model under static buffer priority queueing disciplines is studied by Dai [5]. The conditions of stability are discussed. No numerical results of static performances are available so far.

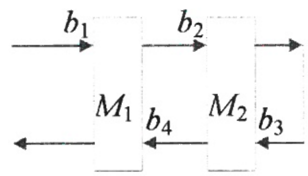

Figure 1

There are two stations in Figure 1 named M_1 and M_2 . The route of parts is $M_1 \to M_2 \to M_2 \to M_1$. The successive buffers visited are b_1, b_2, b_3, b_4 respectively. Parts arrive at b_1 as Poisson distribution of rate λ. Processing times in each buffer are of exponential distribution of rate μ_i. Let c_i denote the capacity of b_i.

Set of states is denoted by $J=\{(i,j,m,n),\ i \le c_1,\ j \le c_2,\ m \le c_3,\ n \le c_4\}$. i, j, m, n are numbers of parts in b_1, b_2, b_3, b_4 at the sample time. Space of states is $(0,1,\ldots,c_1)\mathrm{X}(0,1,\ldots,c_2)\mathrm{X}(0,1,\ldots,c_3)\mathrm{X}(0,1,\ldots,c_4)$.

States are listed in a lexicographical order.

Let $p_{(i,j,m,n)(i',j',m',n')}$ denote the transition probability from state (i,j,m,n) to state (i',j',m',n'). $\boldsymbol{P}$ is the matrix of transition probability of the system. $\boldsymbol{X}=[\ X_0, X_1, \ldots, X_i, \ldots\ ]$ stands for the static invariant probability vector of $\boldsymbol{P}$.

The systems are scheduled according to a buffer priority policy. It can be seen from Figure 1 that both parts in b_1 and b_4 need be processed in M_1, while only one part can be processed at a time. Apparently resource constraints exist. Order of processing is decided by the policy. If RS is adopted, then both b_1 and b_4 are selected with the probability 0.5. On the other hand, if LBFS policy is adopted, then b_1 may not be processed as long as b_4 is not empty. Situation about b_2 and b_3 is the same as that of b_1 and b_4.

Throughout the paper semi-infinite buffer capacities are assumed. That is, all capacities but c_1 are finite. Set $c_1=\infty$, $c_2=c_3=c_4=H-1$.

Without any loss of generality, set $\lambda + \sum_{i=1}^{4} \mu_i = 1$.

This is a method of sampling a continuous time system to obtain a discrete time system with the same steady -state behaviour. Under the above assumptions, transition probability of the states is only dependent on the current state and has nothing to do with the states before. So the process is a Markovian one.

In the next two Sections we will discuss the models under RS and LBFS respectively.

3. Modelling Networks Under RS

Transition probabilities over infinitesimal time slot Δt can be derived as follows:

$$p_{(i,j,m,n),(i+1,j,m,n)}(\Delta t) = \lambda \Delta t + o(\Delta t)$$

When b_4 is empty, b_1 can be processed as long as b_2 is not full.

$$p_{(i,j,m,0),(i-1,j+1,m,0)}\big|_{i>0,j<H-1}(\Delta t) = \mu_1 \Delta t + o(\Delta t)$$

If neither b_1 nor b_4 is empty, then b_1 can be processed with the probability 0.5.

$$p_{(i,j,m,n),(i-1,j+1,m,n)}\big|_{i>0,j<H-1}(\Delta t) = 0.5\mu_1 \Delta t + o(\Delta t)$$

When b_4 is full, b_3 may not be processed for the sake of preventing blocking. So b_2 can be processed.

$$p_{(i,j,m,H-1),(i,j-1,m+1,H-1)}\big|_{j>0,m<H-1}(\Delta t) = \mu_2 \Delta t + o(\Delta t)$$

Other transitions are similar to the above.

We can construct $\boldsymbol{P}$ as follows:

Define:

$$\Lambda_1 = \begin{bmatrix} \mu_1 & & & \\ & 0.5\mu_1 & & \\ & & \ddots & \\ & & & 0.5\mu_1 \end{bmatrix}_{H \times H},$$

$$\Lambda^0{}_2 = \begin{bmatrix} \mu_2 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_2 \end{bmatrix}_{H\times H},$$

$$\Lambda^1{}_2 = \begin{bmatrix} 0.5\mu_2 & & & \\ & \ddots & & \\ & & 0.5\mu_2 & \\ & & & \mu_2 \end{bmatrix}_{H\times H},$$

$$\Lambda^0{}_3 = \begin{bmatrix} 0 & \mu_3 & & \\ & 0 & \ddots & \\ & & \ddots & \mu_3 \\ & & & 0 \end{bmatrix}_{H\times H},$$

$$\Lambda^1{}_3 = \begin{bmatrix} 0 & 0.5\mu_3 & & \\ & 0 & \ddots & \\ & & \ddots & 0.5\mu_3 \\ & & & 0 \end{bmatrix}_{H\times H},$$

$$\Lambda^0{}_4 = \begin{bmatrix} * & & & \\ \mu_4 & * & & \\ & \ddots & \ddots & \\ & & \mu_4 & * \end{bmatrix}_{H\times H},$$

$$\Lambda^1{}_4 = \begin{bmatrix} * & & & \\ 0.5\mu_4 & * & & \\ & \ddots & \ddots & \\ & & 0.5\mu_4 & * \end{bmatrix}_{H\times H}$$

where * represents the diagonal elements of $\boldsymbol{P}$, keeping the sum of each line being one.

Then define:

$$D_1 = \begin{bmatrix} \Lambda_1 & & & \\ & \Lambda_1 & & \\ & & \ddots & \\ & & & \Lambda_1 \end{bmatrix}_{H\times H},$$

$$D_2 = \begin{bmatrix} 0 & \Lambda^0{}_2 & & & \\ & & \Lambda^1{}_2 & & \\ & & & \ddots & \\ & & & & \Lambda^1{}_2 \\ & & & & 0 \end{bmatrix}_{H\times H},$$

$$D^{00}{}_{34} = \begin{bmatrix} \Lambda^0{}_4 & & & \\ \Lambda^0{}_3 & \ddots & & \\ & \ddots & \ddots & \\ & & \Lambda^0{}_3 & \Lambda^0{}_4 \end{bmatrix}_{H\times H},$$

$$D^{01}{}_{34} = \begin{bmatrix} \Lambda^0{}_4 & & & & \\ \Lambda^1{}_3 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \Lambda^1{}_3 & \ddots & \\ & & & \Lambda^0{}_3 & \Lambda^0{}_4 \end{bmatrix}_{H\times H},$$

$$D^{10}{}_{34} = \begin{bmatrix} \Lambda^1{}_4 & & & \\ \Lambda^0{}_3 & \ddots & & \\ & \ddots & \ddots & \\ & & \Lambda^0{}_3 & \Lambda^1{}_4 \end{bmatrix}_{H\times H},$$

$$D^{11}{}_{34} = \begin{bmatrix} \Lambda^1{}_4 & & & & \\ \Lambda^1{}_3 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \Lambda^1{}_3 & \ddots & \\ & & & \Lambda^0{}_3 & \Lambda^1{}_4 \end{bmatrix}_{H\times H}$$

and

$$A_1 = \begin{bmatrix} D^{10}{}_{34} & & & & \\ D_2 & D^{11}{}_{34} & & & \\ & \ddots & \ddots & & \\ & & \ddots & D^{11}{}_{34} & \\ & & & D_2 & D^{10}{}_{34} \end{bmatrix}_{H\times H}$$

$$A_2 = \begin{bmatrix} 0 & D_1 & & \\ & \ddots & \ddots & \\ & & \ddots & D_1 \\ & & & 0 \end{bmatrix}_{H\times H},$$

$$B_0 = \begin{bmatrix} D^{00}{}_{34} & & & \\ D_2 & D^{01}{}_{34} & & \\ & \ddots & \ddots & \\ & & D_2 & D^{01}{}_{34} \end{bmatrix}_{H\times H},$$

$A_0=\lambda I$,

I is the unit matrix of the corresponding dimensions.

Finally we get the canonical form of QBD type:

$$P = \begin{bmatrix} B_0 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \quad (1)$$

$\boldsymbol{P}$ is stochastic. That is:

$$\begin{cases} A_0 e + A_1 e + A_2 e = e \\ B_0 e + A_0 e = e \end{cases} \quad (2)$$

e is the all 1 column vector of the corresponding dimensions.

Define

$$A = A_0 + A_1 + A_2 \quad (3)$$

The stability of **P** is largely dependent on A.

First several properties of P and A can be given.

Property 3.1 Set of states J of the queueing network in Figure 1 is irreducible under RS.

Proof: P is irreducible if and only if all the states of it are communicating[9].

First we show that state (0,0,0,0) is reachable from an arbitrary state in J.

$$p_{(i,j,m,n)(0,0,0,0)} > p_{(i,j,m,n)(i,j,m,0)} \mathrm{X} p_{(i,j,m,0)(i,j,0,m)} \mathrm{X} p_{(i,j,0,m)(i,j,0,0)} \mathrm{X} p_{(i,j,0,0)(i,0,j,0)}$$

$$\mathrm{X} p_{(i,0,0,0)(i,0,0,0)} \mathrm{X} p_{(i,0,0,0)(0,0,0,0)}$$

$$>(0.5\mu_4)^n(0.5\mu_3)^m(0.5\mu_4)^m(0.5\mu_2)^j(0.5\mu_3)^j(0.5\mu_4)^j (0.5\mu_2)^i{\mu_3}^i{\mu_4}^i$$

$$>0$$

So we can take (0,0,0,0) as an intermediate state. Let us prove that the arbitrary state (i,j,m,n) can next be reached from (0,0,0,0).

$$p_{(0,0,0,0)(i,j,m,n)} > p_{(0,0,0,0)(i+j+m+n,0,0,0)} \mathrm{X} p_{(i+j+m+n,0,0,0)(i+j+m,n,0,0)} \mathrm{X} p_{(i+j+m,n,0,0)(i+j+m,0,n,0)}$$

$$\mathrm{X} p_{(i+j+m,0,n,0)(i+j+m,0,0,n)} \mathrm{X} p_{(i+j+m,0,0,n)(i+j,m,0,n)} \mathrm{X} p_{(i+j,m,0,n)(i+j,0,m,n)}$$

$$\mathrm{X} p_{(i+j,0,m,n)(i,j,m,n)}$$

$$>\lambda^{i+j+m+n}(\mu_1)^n(0.5\mu_2)^n(\mu_3)^n(0.5\mu_1)^m(0.5\mu_2)^m(0.5\mu_1)^j$$

$>0.$ □

Property 3.2 Matrix A is irreducible and stochastic.

Proof: by defining A, we can explain it as the transition probability matrix of the system in Figure 2.

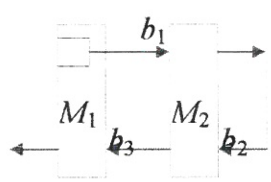

Figure 2

There are two stations and three buffers. Infinite parts are assumed to be stored in M_1 and only through M_1 can they reach b_1 . M_1 has two functions. One function is to release parts to b_1 at the mean rate of μ_1 , the other one is to process parts in b_3. When b_3 is not empty, M_1 releases parts to b_1 with the probability 0.5, and processes parts in b_3 with the probability 0.5. M_2 is similar to that in Figure 1. Their processing rates are μ_2, μ_3, μ_4 respectively.

Set of states of this system is $G=\{(j,m,n), j\leq c_2, m\leq c_3, n\leq c_4\}$. We can see that every state in G can reach (0,0,0). Assume h denotes the transition probability.

$$h_{(j,m,n)(0,0,0)} > h_{(j,m,n)(j,m,0)} \mathrm{X} h_{(j,m,0)(j,0,0)} \mathrm{X} h_{(j,0,0)(0,0,0)}$$

$$>(0.5\mu_4)^n(0.5\mu_3)^m(0.5\mu_4)^m(0.5\mu_2)^j(0.5\mu_3)^j(0.5\mu_4)^j$$

$$>0$$

And the arbitrary state (j,m,n) can be reached from (0,0,0).

$$h_{(0,0,0)(j,m,n)} > h_{(0,0,0)(n,0,0)} \mathrm{X} h_{(n,0,0)(0,n,0)} \mathrm{X} h_{(0,n,0)(0,0,n)} \mathrm{X}$$

$$h_{(0,0,n)(m,0,n)} \mathrm{X} h_{(m,0,n)(0,m,n)} \mathrm{X} h_{(0,m,n)(j,m,n)}$$

$$>{\mu_1}^n(0.5\mu_2)^n{\mu_3}^n(0.5\mu_1)^m(0.5\mu_2)^m(0.5\mu_1)^j$$

$$>0$$

So A is irreducible.

Since P is stochastic, from (2) and (3) we have $Ae=e$. That is, A is stochastic. □

If P is positive recurrent, the invariant probability vector $X=\{X_0, X_1, \ldots, X_i, \ldots\}$ (X_i is of dimension of H^3) of P satisfies

$$X_i=X_0R^i, \qquad i>=0 \tag{4}$$

where R is the minimal non-negative solution of quadratic non-linear matrix equation

$$R^2A_2+RA_1+A_0=R \tag{5}$$

Theorem 3.1 If A is irreducible and stochastic, it has the invariant measure $\pi>0$, satisfying

$$\pi A=\pi, \ \pi e=1. \tag{6}$$

Proof: Since A is stochastic, we have $Ae=e$. That is $(A-I)e=0$. Then (6) has non-zero solutions $\pi\neq 0$.

It is well- known that all the states are positive recurrent provided one state is positive recurrent in an irreducible system. Since $\pi\neq 0$, then $\pi>0$.□

Corollary 3.1 [6] If A is irreducible and stochastic, then irreducible $\boldsymbol{P}$ is positive recurrent if and only if

$$\pi A_2 e > \pi A_0 e \tag{7}$$

and equation

$$\begin{cases} X_0(B_0 + RA_2) = X_0 \\ X_0(I - R)^{-1}e = 1 \end{cases} \tag{8}$$

has positive solutions X_0, where π is the invariant measure of A.

Theorem 3.2 If A is irreducible and R is positive, then $B[R]:=B_0+RA_2$ is irreducible. If $(I-R)$ is invertible, $B[R]$ is stochastic. If both conditions exist, Eq (8) has a strictly positive solution.

Proof: From their definitions, we can write A and $B[R]$ as :

$$A = \begin{bmatrix} D^{10}{}_{34} + \lambda I & D_1 & & & \\ D_2 & D^{11}{}_{34} + \lambda I & D_1 & & \\ & D_2 & \ddots & \ddots & \\ & & \ddots & D^{11}{}_{34} + \lambda I & D_1 \\ & & & D_2 & D^{01}{}_{34} + \lambda I \end{bmatrix}$$

$$B[R] = \begin{bmatrix} D^{00}{}_{34} & R_{1,1}D_1 & R_{1,2}D_1 & \cdots & R_{1,H-1}D_1 \\ D_2 & D^{01}{}_{34} + R_{2,1}D_1 & R_{2,2}D_1 & \cdots & R_{2,H-1}D_1 \\ & D_2 & D^{01}{}_{34} + R_{3,2}D_1 & \ddots & R_{3,H-1}D_1 \\ & & \ddots & \ddots & \vdots \\ & & & D_2 & D^{01}{}_{34} + R_{H,H-1}D_1 \end{bmatrix}$$

where R_{ij} is the iXjth submatrix of R after dividing R into HXH submatrices. If comparing the two matrices, one can see that: 1). Elements under the diagonal are the same. 2). Since R_{ii} is nonnegative, so $R_{ii}D_1$ has the same sign(+ or 0) as D_1. So above the diagonal, $B[R]$ has positive elements no less than A. 3). The elements on the diagonal have the same sign.

So $B[R]$ is irreducible if A and R are positive.

The second part of Theorem 3.2 can be found in [6].

From Theorem 3.1, when $B[R]$ is irreducible and stochastic, $B[R]$ has an invariant measure and (8) has positive solutions. □

Theorem 3.3 The queueing system in Figure 1 with semi-infinite buffer capacities is positive recurrent under RS if and only if

$$(\pi_1+\pi_2+\ldots+\pi_{H-1})*D_1 e > \lambda \tag{9}$$

Proof: Invariant measure of A can be written as $\pi=[\pi_1,\pi_2,\ldots,\pi_H]$. If substituting A_2 and A_0 in (7), together with $\pi e=e$, we get (9).

From Theorem 3.1 A is irreducible and stochastic, so (8) has positive solution by Theorem 3.2. The second part of Corollary 3.1 is satisfied. Then Theorem 3.3 is proved. □

Given a set of parameters, we can judge the stability of the system by means of Theorem 3.3.

The numerical algorithm to compute R is as follows. You may find it in [6].

Set $R_0=0$

$$R_k=(A_0+R_{k-1}^2A_2)(I-A_1)^{-1} \tag{10}$$

Stop substitution when the deviation between R_k and R_{k-1} is less than ε (10^{-8} will be enough for common usage).

Invariant probabilities of $\boldsymbol{P}$ can be solved by (8) provided $\boldsymbol{P}$ is positive recurrent. Other static performances of the system can be solved on this basis.

Fortunately $\boldsymbol{P}$ is irreducible under RS. But this is not always the case. In the next Section we will model a system which is reducible.

4. Modelling Network Under LBFS

Similarly to the procedure in Section 3, we can get transition probabilities of this system under LBFS in an infinitesimal time interval Δt.

$$p_{(i,j,m,n),(i+1,j,m,n)}(\Delta t)=\lambda\Delta t+o(\Delta t)$$

Only when b_4 is empty can b_1 be processed.

$$p_{(i,j,m,0),(i-1,j+1,m,0)}\big|_{i>0,j<H-1}(\Delta t)=\mu_1\Delta t+o(\Delta t)$$

When b_4 is full, b_3 may not be processed for the sake of preventing blocking. Nor can b_2 be processed since b_3 is not empty. Only b_4 can be processed at this time.

$$p_{(i,j,m,H-1),(i,j,m,H-2)}\big|_{m>0}(\Delta t)=\mu_4\Delta t+o(\Delta t)$$

Before constructing the transition probability matrix, some properties of it can be given.

Property 4.1 State (0,0,0,0) can be reached from the arbitrary state in $\boldsymbol{J}$ when the system is under LBFS.

Property 4.2 State (i,j,m,n) cannot be reached from (0,0,0,0) if either of the following conditions is satisfied.

1.$m>1$, 2.$j+m+n>H$.

Proof: First we notice that the system has the following characteristics:

C1. When $n>0$, b_1 may not be processed. That is, it is impossible that the number of parts in b_2 increases if $n>0$.

C2. When $m>0$, b_2 may not be processed. So it is impossible that the number of parts in b_3 increases if $m>0$.

From C2 we know that m may not further increase as soon as it increases from 0 to 1. That is, $m\le 1$ provided $m=0$ in the beginning.

From C1 we know that j may not increase as soon as n increases from 0 to 1, which is only possibly companioned with m decreasing from 1 to 0. Then $j+m+n$ may not further increase. The maximal number of parts in b_2, b_3, b_4 is

H-1+1=H. □

Theorem 4.1 $G:=\{(i,j,m,n),\ m\le 1,\ j+m+n\le H\}$, which is a subset of $\boldsymbol{J}$, is absorbing and irreducible.

Proof: Denote $\boldsymbol{G'}$ as a complementary set of $\boldsymbol{G}$ in $\boldsymbol{J}$.

In order to prove that $\boldsymbol{G}$ is absorbing we need to demonstrate that states in $\boldsymbol{G}$ cannot reach $\boldsymbol{G'}$. If this does not hold, there will be some state $y\in\boldsymbol{G}$ which can reach $g\in\boldsymbol{G'}$ with the probability $h>0$. Since y is reachable from (0,0,0,0), then g is also reachable from (0,0,0,0). This is a conflict to Property 4.2. So $\boldsymbol{G}$ is absorbing.

For simplicity, we shall write 'A→B' when B is reachable from A in the following context.

Obviously (0,0,0,0) is reachable from the arbitrary state $y\in\boldsymbol{G}$.

On the other hand, we have:

when $m=0$,
$(0,0,0,0)\to(n+j,0,0,0)\to(j,n,0,0)\to(j,n-1,1,0)\to(0,j+n-1,1,0)\to(0,j+n-1,0,1)\to(0,j,0,n)$

when $m=1$,
$(0,0,0,0)\to(n+j+1,0,0,0)\to(j+1,n,0,0)\to(j+1,n-1,1,0)\to(0,j+n,1,0)\to(0,j+n,0,1)\to(0,j+n-1,1,1)\to(0,j+n-1,0,2)\to(0,j,1,n)$

So $\boldsymbol{G}$ is irreducible. □

The states of the system will converge to $\boldsymbol{G}$ with the probability 1 regardless of the original states. So only $\boldsymbol{G}$ needs be considered when the static performances are concerned. Next we will construct the transition probability matrix $\boldsymbol{F}$ consisting only of subset $\boldsymbol{G}$.

Define

$$\Lambda_1=\begin{bmatrix}\mu_1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0\end{bmatrix}_{H\times H},$$

$$\Lambda_2=\begin{bmatrix}\mu_2 & 0 & & \\ 0 & \mu_2 & & \\ & & \ddots & \\ & & & \mu_2\end{bmatrix}_{H\times H},$$

$$\Lambda_3 = \begin{bmatrix} 0 & \mu_3 & & \\ & 0 & \ddots & \\ & & \ddots & \mu_3 \\ & & & 0 \end{bmatrix}_{H\times H},$$

$$\Lambda_4 = \begin{bmatrix} * & & & \\ \mu_4 & * & & \\ & \ddots & \ddots & \\ & & \mu_4 & * \end{bmatrix}_{H\times H}$$

where * represent non-zero diagonal elements. And

$$D^0{}_{34} = \begin{bmatrix} \Lambda_4 & 0 \\ \Lambda_3 & \Lambda_4 \end{bmatrix},$$

$$D^i{}_{34} = \begin{bmatrix} \Lambda_4(H-i+1, H-i+1) & \\ \Lambda_3(H-i, H-i+1) & \Lambda_4(H-i, H-i) \end{bmatrix},$$

$i=1,\ldots,H-1$.

$$D^0{}_1 = \begin{bmatrix} \Lambda_1(H,H) & 0 \\ 0 & \Lambda_1(H,H-1) \end{bmatrix},$$

$$D^i{}_1 = \begin{bmatrix} \Lambda_1(H-i+1, H-i) & 0 \\ 0 & \Lambda_1(H-i, H-i+1) \end{bmatrix}$$

$i=1,\ldots,H-2$.

$$D_2 = \begin{bmatrix} H \quad columns \quad of \quad 0 & \Lambda_2 \\ 0 & 0 \end{bmatrix},$$

$$D^i{}_2 = \begin{bmatrix} (H-i+2) \quad columns \quad of \quad 0 & \Lambda_2(H-i+1) \\ 0 & 0 \end{bmatrix},$$

$i=2,\ldots,H-1$.

where $\Lambda_i(l,k)$ represents the submatrix of the first l lines and the first k columns of Λ_i.

Then define

$$A_2 = \begin{bmatrix} 2H \quad columns \quad of \quad 0 & D^0{}_1 & & \\ & & \ddots & \\ & & & D^{H-2}{}_1 \\ & & & 0 \end{bmatrix}_{T\times T}$$

$$A_1 = \begin{bmatrix} D^0{}_{34} & & & \\ D^1{}_2 & D^1{}_{34} & & \\ & \ddots & \ddots & \\ & & D^{H-1}{}_2 & D^{H-1}{}_{34} \end{bmatrix}_{T\times T},$$

where

$$T = H + H + H + H - 1 + H - 1 + H - 2 + H - 2 + \ldots + 1 = H^2 + 2H - 1$$

and $A_0=\lambda I$, I is the unit matrix of the corresponding dimensions.

Then the transition matrix $\boldsymbol{F}$ of subset $\boldsymbol{G}$ has the canonical form of :

$$F = \begin{bmatrix} B_0 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \tag{11}$$

B_0 is the same as A_1 except for the diagonal elements determined by

$$\begin{cases} A_0 e + A_1 e + A_2 e = e \\ B_0 e + A_0 e = e \end{cases}$$

Theorem 3.1 and Theorem 3.2 are also suitable for this Section. As their Proof follows the same method as adopted in Section 3, it is omitted here.

Similarly we can have the following theorem.

Theorem 4.2 The queueing system in Figure 1 with semi-infinite buffer capacities is positive recurrent under LBFS iff

$$\pi A_2 e > \lambda \tag{12}$$

An equivalent expression of it is

$$\mu_1 \sum_{i\in\sigma} \pi_i > \lambda, \quad \text{where}$$

$$\sigma = \left\{ k \,\middle|\, \begin{matrix} k = j(2H+1) - 2\sum_{l=1}^{j-1} l, & H(2j+1) - \\ 2\sum_{l=1}^{j-1} l + 1, & j = 1,\ldots,H-1 \end{matrix} \right\}$$

where A_2 is defined in Section 4 and π is the invariant measure of A in Section 4.

Proof: Substituting the expression of A_2 in (7) leads to the result. □

5. Examples

After constructing the transition matrices, the system stability should be judged according to Theorem 3.4 and Theorem 4.2. Then R should be computed according to Eq (10). If the system is stable, its invariant measure can be computed by (8).

In queueing networks, the mean queue length is significant. In this context the mean queue length can be evaluated as:

$$L_1 = \sum_{i=0}^{\infty} i * \sum_{j=0}^{H-1} \sum_{m=0}^{H-1} \sum_{n=0}^{H-1} x_k$$

$$L_2 = \sum_{j=0}^{H-1} j * \sum_{i=0}^{\infty} \sum_{m=0}^{H-1} \sum_{n=0}^{H-1} x_k$$

where x_k is the kth element of X, $k=i*H^3+j*H^2+m*H+n$. Formulas of L_3 and L_4 are similar to L_2's.

Example 1. Set the parameters as $\mu_1=\mu_2=\mu_3=\mu_4=2.5/11$, $\lambda=1/11$, $H=4$.

RS: the system is stable.

LBFS: the system is unstable.

Example 2. Set the parameters as $\mu_1=\mu_2=\mu_3=\mu_4=3/13$, $\lambda=1/13$, H=3. Both systems under RS and LBFS are stable with these parameters. And further we have:

RS: L=[3.2910,0.8154,0.7435,0.6763]

LBFS: L=[5.4860,0.8032,0.3444,0.3908].

Example 3. Under RS, set the parameters as $\mu_1=\mu_2=2.6/11.2$, $\mu_3=\mu_4=2.5/11.2$, $\lambda=1/11.2$.

When H=3, the system is unstable.

When H=4, the system is stable. The mean queue length is

L=[8.6010,1.4158,1.3125,1.1581].

Example 4. Under LBFS, set the parameters the same as those in Example 2.

When H=3, the system is unstable.

When H=4, the system is stable. The mean queue length is

L=[43.7438,1.4252,0.4032,0.5139].

Numerical results show that

1. Performances under LBFS are lower than those under RS. In Example 1, with the same parameters, the system is stable under RS but unstable under LBFS. In Example 2, also with the same parameters, the mean queue length under RS is less than that under LBFS.

We conjecture that with the system in Figure 1 under LBFS, the capacity of b_3 is equivalent to be restricted to 1, so parts in b_2 are often delayed. This may prolong the delay of parts.

But this is not always the case. In another system with the route $M_1 \rightarrow M_2 \rightarrow M_1 \rightarrow M_2$, we have shown that LBFS is better than RS. So we can safely say that the scheduling effects of policies also depend on the structure of the system.

2. The system is apter to be unstable when there are lower capacities.

In Example 3 and Example 4, the systems are stable when H=4, but unstable when H=3. These results show that the systems with insufficient capacities are liable to be unstable.

We also conjecture that static performances get always improved when the spectral radius of R is smaller. To prove this, further research should be done.

6. Concluding Remarks

In this paper, we developed queueing models of re-entrant lines under RS and LBFS respectively. Conditions of stability have been given and an algorithm for static performances was derived.

It is for the first time in the study of re-entrant lines that numerical results of performances can be explicitly worked out. This demonstrates the effectiveness of non-linear matrix equation theory in solving complex stochastic networks.

Also numerical results show that the scheduling effects of policies largely depend on the structure of the system. Static performances have something to do with the capacities of buffers and the arrival rate together with the processing rates. Further results need a more elaborate consideration. This method will certainly highlight the control of various complex networks.

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