A Logical System for Multicriteria Decision Analysis

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Abstract: Starting from some logical aspects regarding the use of fuzzy sets in the representation of multicriteria decision problems, this paper defines the structure of biresiduated algebra together with a corresponding logical system. Different properties are presented.

Keywords: fuzzy set, multicriteria decision problem, algebraic logic, MV-algebra, D-algebra, residuated lattice, biresiduated algebra, distance function, equivalence function, aggregating mapping

Introduction

Decision analysis implies knowledge acquisition, realization of mathematical models, computer simulation and action [10, 11]. This paper considers some logical aspects regarding the use of fuzzy sets in the representation of multicriteria decision problems.

It is accepted here that a multicriteria decision problem is represented by a system (X, g, d), where X is the set of alternatives, g is a vector of goals and d is a decision set representing acceptable alternatives to some decision-making process [3, 8].

Any goal is associated with an attribute of the alternatives. A domain of its possible values is associated with each attribute. Then the alternatives are evaluated in the domain associated with the corresponding attribute and the goal is expressed in terms of values of this attribute of the alternatives. Suppose that the domain D_A associated with any involved attribute A is a *bounded ordered set* called *membership grades scale*. Each goal associated with an attribute A is represented by a mapping g_A from X to D_A and consists in determining a maximum Pareto point for g_A on X.

An attribute is called elementary if its membership grades scale is a bounded chain (e.g. the attribute "height" with respect to some set of persons is elementary). Any attribute can, in some respect, be considered as a logical combination of elementary attributes.

The representation of multicriteria decision problems using fuzzy sets, as they were introduced by Zadeh [3, 8, 22], implies the use of a unique membership grades scale, namely, the complete bounded chain $L = [0, 1] \subseteq \mathbf{R}$ of positive subunitary real numbers. This helps equally describe the vector of goals by a finite collection of fuzzy sets on X and the decision set as a fuzzy set on X obtained from the goals through an aggregating mapping, which was defined using different combination operators. Considering both the algebraic structure and the topological structure of the standard chain [0, 1], Dubois and Prade [8] provide interesting solutions to the problem of choosing a suitable combination operator capable to build the decision set.

In the above mentioned approach any involved attribute is considered to be an elementary attribute and the decision set is defined using operators from various logical systems. In order to obtain a more comprehensive *conceptual framework for the representation of attributes* and *the construction of a decision set in multicriteria decision problems*, a major problem is to identify a *standard logical system* including features common to various logical systems. Turunen [21] shows that *adjoint couples and residuated lattices* happen very often together. Goguen [12] considers that the *algebra of inexact concepts* is a residuated lattice. Pultr [16] introduces the concept of *L-space*, with L a residuated lattice, and shows that *L-fuzzy sets* and *L-fuzzy sets with equality* can be represen-ted by a system N = ((P, ν) ; K, R), where (P, ν) is an L-space and K, R are subsets of P, called *L-nebula*. In Negoita and Ralescu [14] several notions of *generalized sets* are presented.

Different structures have been defined as algebraic counterparts of various systems of logic (e.g. *Boolean algebra* [15], *Heyting algebra* and *Brouwer algebra* [1,2,7,13,17,18], *Heyting-Brouwer (semi-Boolean) algebra* [19], *D-algebra* [20]).

Starting from some properties of the structure of *complete D-algebra* [20] and given the results obtained by Turunen [21] and Pultr [16], we define the structure of *biresiduated lattice* with a view at getting a suitable *membership grades scale* for the representation of attributes. A biresiduated lattice is defined by a system

$$(A, \oplus, \otimes, \leq, 0, 1),$$

where $\oplus$ and $\otimes$ are binary operations on the set A, $\leq$ is an order relation on A and $0, 1 \in A$ such that $(A, \oplus, \otimes)$ is an ordered bisemigroup and $(A, \leq, 0, 1)$ is a complete lattice satisfying some distributivity conditions.

Different examples together with some basic properties of the biresiduated lattice structure are presented. A biresiduated lattice can be associated with each complete D-algebra. Several biresiduated lattice structures are also defined on the chain $([0, 1], \leq, 0, 1)$. Then a description of the biresiduated lattice structure as a system $\mathbf{A} = (A, \oplus, -, \otimes, \rightarrow, \leq, 0, 1)$ is given, such that $(A, \leq, 0, 1)$ is a complete lattice equipped with an adjoint couple $(\otimes, \rightarrow)$ and a dual adjoint couple $(\oplus, -)$. Some specific relations are also derived.

The *Heyting structure*, the *Brouwer structure* and the *Lukasiewicz structure* associated with the complete chain $([0, 1], \leq, 0, 1)$ will further be considered as basic biresiduated lattices on $[0, 1]$. The Lukasiewicz structure on $[0, 1]$ may be associated with a standard structure of *MV-algebra* [5, 6]. The class of D-algebras [20] is a minimal extension of the union between the class of Heyting algebras and the class of Brouwer algebras. Then, in order to develop a logical system for multicriteria decision analysis, the definition of a structure including both the structure of *D-algebra* and the structure of *MV-algebra* should be considered. A solution of the precedent problem will be given by the notion of *biresiduated algebra,* defined as a system

$$\mathbf{A} = (A, \wedge, \vee, \oplus, -, \otimes, \rightarrow, \neg, 0, 1)$$

of type (2, 2, 2, 2, 2, 2, 1, 0, 0) which satisfies some equations. Let R̲ be the class of *residuated algebras*, $\underline{R}^{o}$ be the class of *dual residuated algebras* and B̲R̲ be the class of *biresiduated algebras*. We prove that B̲R̲ is a variety generated by $\underline{R} \cup \underline{R}^{o}$.

The notions of *distance function* and *equivalence function* for a biresiduated algebra are introduced and applied to construct *homomorphic images* using *strong ideals* and *strong filters*. Then the notions of *formula, valuation, valid formula* and *invalid formula* over each class C̲ of biresiduated algebras are presented. The *word problem for free algebras over* C̲ is also defined.

The concept of *set over a complete biresiduated algebra* including the notion of set over a complete Heyting algebra defined in Fourman and Scott [9], is introduced.

We also present the concept of an aggregating mapping in multicriteria decision analysis.

1. Biresiduated Lattices

1.1 Terminology and Notations

Let A be a set together with two binary operations $\oplus$ (addition) and $\otimes$ (multiplication) on A, an order relation $\leq$ on A and two constants $0, 1 \in A$ such that $(A, \leq, 0, 1)$ is a bounded poset with the minimum element 0 and the maximum element 1. Then the system $\mathbf{A} = (A, \oplus, \otimes, \leq, 0, 1)$ will be called a *bounded ordered bigrupoid*. Let Ord-BG [0,1] be the class of bounded ordered bigrupoids. A mapping $f : A \rightarrow B$ is called a *morphism in* Ord-BG[0,1] from **A** to $\mathbf{B} = (B, \oplus, \otimes, \leq, 0, 1)$ if f is a *morphism of bigrupoids* from $(A, \oplus, \otimes)$ to $(B, \oplus, \otimes)$ and f is an *order-preserving mapping of posets* from $(A, \leq, 0, 1)$ to $(B, \leq, 0, 1)$ i.e. for all $x, y \in A$,

$$f(x \oplus y) = f(x) \oplus f(y); \quad f(x \otimes y) = f(x) \otimes f(y);$$
$$x \leq y \Rightarrow f(x) \leq f(y); \quad f(0) = 0 \text{ and } f(1) = 1.$$

If the bounded poset $(A, \leq, 0, 1)$ is a *complete lattice* and X is a subset of the set A, then let supX for the supremum (join) of X and infX for the infimum (meet) of X. For $x, y \in A$ and for every family $(y_i)_{i \in I}$ of elements of A, let

$$x \wedge y = \inf \{x, y\}; \quad x \vee y = \sup \{x, y\};$$
$$\bigwedge_{i \in I} y_i = \inf \{y_i / i \in I\}; \bigvee_{i \in I} y_i = \sup \{y_i / i \in I\}.$$

A complete lattice $(A, \leq, 0, 1)$ will be called *finite distributive* if A together with the binary meet and the binary join on A, $(A, \wedge, \vee)$, is a distributive lattice. If $(A, \leq, 0, 1)$ and $(B, \leq, 0, 1)$ are complete lattices then a mapping $f : A \rightarrow B$ is called a *homomorphism of complete lattices* if for every family $(y_i)_{i \in I}$ of elements of A,

$$f(\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} f(y_i);\ f(\bigvee_{i \in I} y_i) = \bigvee_{i \in I} f(y_i).$$

The use of the notions of closure operator and interior operator on a poset and the notions of modal operator and dual modal operator of a lattice will be in accordance with the terminology introduced in [4].

A *closure operator* on the poset (A, $\leq$) is a mapping $c : A \to A$ such that $x \leq c(x)$, $c(c(x)) = c(x)$ and $x \leq y \Rightarrow c(x) \leq c(y)$, for every x, y $\in$ A.

An *interior operator* on the poset (A, $\leq$) is a mapping $i : A \to A$ such that $i(x) \leq x$, $i(i(x)) = i(x)$ and $x \leq y \Rightarrow i(x) \leq i(y)$, for every x, y $\in$ A.

A *modal operator* on a lattice (A, $\wedge$, $\vee$, 0, 1) is a mapping $p : A \to A$ such that it satisfies :

$$p(0) = 0;$$
$$p(x \vee y) = p(x) \vee p(y).$$

A *dual modal operator* on (A, $\wedge$, $\vee$, 0, 1) is a mapping $q : A \to A$ such that it satisfies :

$$q(1) = 1;$$
$$q(x \wedge y) = q(x) \wedge q(y).$$

Now the definitions of some known structures, to be used in the sequel, are presented.

A *Boolean lattice* is a bounded distributive lattice (A, $\wedge$, $\vee$, 0, 1) such that every element x of A is complemented, i.e there is an element y such that $x \wedge y = 0$ and $x \vee y = 1$, and in this case let $y = \neg x$.

A *Heyting lattice* is a relatively pseudo-complemented lattice (A, $\wedge$, $\vee$, 0) with 0, i.e. a lattice such that for every x, y $\in$ A, $0 \leq x$ and the relative pseudocomplement of x with respect to y exists, namely, there is an element $x \to y \in A$ such that it is the greatest element z $\in$ A which verifies the relation $z \wedge x \leq y$. Every Heyting lattice is a bounded distributive lattice. A *Heyting algebra* is a system **H** = (H, $\wedge$, $\vee$, $\to$, 0, 1) associated with a Heyting lattice, where $\wedge$ (meet), $\vee$ (join), $\to$ (relative pseudocomplementation) are binary operations and 0, 1 $\in$ H with $1 = 0 \to 0$. Let $\neg x = x \to 0$, for every x $\in$ H.

A Heyting algebra is called *complete* if it is a complete lattice.

A *Brouwer lattice* is a lattice (A, $\wedge$, $\vee$, 1) such that it is a dual Heyting lattice, i.e. the system (A, $\wedge^o$, $\vee^o$, 0^o) is a Heyting lattice, with $0^o = 1$, $x \wedge^o y = x \vee y$, $x \vee^o y = x \wedge y$, for all x, y $\in$ A. The relative pseudocomplement of y with respect to x in the Heyting lattice (A, $\wedge^o$, $\vee^o$, 0^o) is denoted by $x - y$. This implies that $x - y$ is the least element z $\in$ A such that $x \leq y \vee z$. A *Brouwer algebra* is a system

$$\mathbf{Br} = (Br, \wedge, \vee, -, 0, 1)$$

associated with a Brouwer lattice, where $\wedge$ (meet), $\vee$ (join), $-$ (relative pseudosubtraction) are binary operations and 0, 1 $\in$ Br with $0 = 1 - 1$. Let $\neg x = 1 - x$, for every x $\in$ Br.

A Brouwer algebra is called *complete* if it is a complete lattice.

1.2 Definition

A *biresiduated lattice* is a system **A** = (A, $\oplus$, $\otimes$, $\leq$, 0, 1) such that **A** is a bounded ordered bigrupoid and the following conditions hold:

(i) (A, $\leq$, 0, 1) is a *finite distributive complete lattice;*

(ii) (A, $\oplus$) and (A, $\otimes$) are *commutative semigroups,* i.e. for all x, y, z $\in$ A,

(1) $x \oplus y = y \oplus x$;
$x \otimes y = y \otimes x$;

(2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
$(x \otimes y) \otimes z = x \otimes (y \otimes z)$.

(iii) For all x, y, z $\in$ A and for every family $(y_i)_{i \in I}$ of elements of A:

(3) $0 \oplus 0 = 0$;
$1 \otimes 1 = 1$;

(4) $x \wedge (x \oplus y) = x$;
$x \vee (x \otimes y) = x$;

(5) $(x \oplus y) \oplus 0 = x \oplus y$;
$(x \otimes y) \otimes 1 = x \otimes y$;

(6) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$;
$x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$;

(7) $x \oplus \bigwedge_{i \in I} (y_i \oplus 0) = \bigwedge_{i \in I} (x \oplus y_i)$;

$x \otimes \bigvee_{i \in I} (y_i \otimes 1) = \bigvee_{i \in I} (x \otimes y_i)$.

1.3 Examples

(1) *Complete Boolean lattices*

Let (B, $\leq$, 0, 1) be a complete Boolean lattice [15]. If for all x, y $\in$ B, we define $x \oplus y = x \vee y$ and $x \otimes y = x \wedge y$ then (B, $\oplus$, $\otimes$, $\leq$, 0, 1) is a biresiduated lattice such that for every x $\in$ B,

$$x \oplus 0 = x \text{ and } x \otimes 1 = x.$$

(2) *Complete Heyting algebras*

Let **H** = (H, $\wedge$, $\vee$, $\rightarrow$, 0, 1) be a complete Heyting algebra. For every x, y $\in$ H, one defines $x \oplus y = \neg\neg(x \vee y)$ and $x \otimes y = x \wedge y$. It follows that (H, $\oplus$, $\otimes$, $\leq$, 0, 1) is a biresiduated lattice such that $x \otimes 1 = x$, for all x $\in$ H. If **H** is not a Boolean lattice, the relation $x \oplus 0 = x$ can be false for some x $\in$ H. It follows that the complete lattice of the open sets of a topological space has a natural structure of biresiduated lattice, namely, if X is a topological space, (O(X), $\subseteq$, $\varnothing$, X) is the complete Heyting lattice of the open sets in X and one defines,

$$O_1 \oplus O_2 = \mathrm{int}(X \setminus \mathrm{int}(X \setminus (O_1 \cup O_2)));$$
$$O_1 \otimes O_2 = O_1 \cap O_2,$$

for all open sets $O_1, O_2 \in O(X)$, then the system (O(X), $\oplus$, $\otimes$, $\subseteq$, $\varnothing$, X) is a biresiduated lattice.

(3) *Complete Brouwer algebras*

Let **Br** = (Br, $\wedge$, $\vee$, $-$, 0, 1) be a complete Brouwer algebra. For every x, y $\in$ Br, one defines $x \oplus y = x \vee y$ and $x \otimes y = \neg\neg(x \wedge y)$. Then (Br, $\oplus$, $\otimes$, $\leq$, 0, 1) is a biresiduated lattice such that $x \oplus 0 = x$, for every x $\in$ Br. If **Br** is not a Boolean lattice, the relation $x \otimes 1 = x$ can be false for some x $\in$ Br. In particular, it follows that the complete lattice of the closed sets of a topological space has a natural structure of biresiduated lattice, namely, if X is a topological space, (F(X), $\subseteq$, $\varnothing$, X) is the complete Brouwer lattice of the closed sets in X and one defines,

$$F_1 \oplus F_2 = F_1 \cup F_2;$$
$$F_1 \otimes F_2 = \mathrm{adh}(X \setminus \mathrm{adh}(X \setminus (F_1 \cap F_2))),$$

for all closed sets $F_1, F_2 \in F(X)$,then the system (F(X), $\oplus$, $\otimes$, $\subseteq$, $\varnothing$, X) is a biresiduated lattice.

(4) *Heyting-Brouwer algebras*

Suppose that (A, $\wedge$, $\vee$, $\rightarrow$, $-$, 0, 1) is a complete Heyting-Brouwer algebra [19], i.e. (A, $\wedge$, $\vee$, $\rightarrow$, 0, 1) is a complete Heyting algebra and (A, $\wedge$, $\vee$, $-$, 0, 1) is a complete Brouwer algebra.

Two operations of addition $\oplus$ on the set A can then be defined as follows, for every x, y $\in$ A:

(A_1) *Brouwer addition*
$$x \oplus y = x \vee y.$$
(A_2) *Heyting addition*
$$x \oplus y = [(x \vee y) \rightarrow 0] \rightarrow 0.$$

One can also define two operations of multiplication $\otimes$ on the set A as follows, for every x, y $\in$ A:

(M_1) *Brouwer multiplication*
$$x \otimes y = 1 - [1 - (x \wedge y)].$$
(M_2) *Heyting multiplication*
$$x \otimes y = x \wedge y.$$

Thus, *it will be possible to associate four structures of biresiduated lattice* (A, $\oplus$, $\otimes$, $\leq$, 0, 1) with (A, $\wedge$, $\vee$, $\rightarrow$, $-$, 0, 1) if one defines addition $\oplus$ and multiplication $\otimes$ by (A_i) and (M_j) respectively, for every i, j = 1, 2, where $x \leq y$ iff $x = x \wedge y$.

Let (C, $\leq$, 0, 1) be a *complete chain*. Then there is a standard structure of complete Heyting-Brouwer algebra (C, $\wedge$, $\vee$, $\rightarrow$, $-$, 0, 1), defined as follows, for every x, y $\in$ C:

$x \wedge y = \min(x, y)$; $\quad x \vee y = \max(x, y)$;

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}; \quad x - y = \begin{cases} 0, & \text{if } x \leq y \\ x, & \text{if } x > y \end{cases}.$$

Thus, *four structures of biresiduated lattice can be associated with* (C, $\leq$, 0, 1) as above.

(5) *Complete D-algebras*

If **H** = (H, $\wedge$, $\vee$, $\rightarrow$, 0, 1) is a Heyting algebra then we define (H, $\wedge$, $\vee$, $\rightarrow$, $-$, 0, 1), such that $x - y = \neg(x \rightarrow y)$, for every x, y $\in$ H.
If **Br** = (Br, $\wedge$, $\vee$, $-$, 0, 1) is a Brouwer algebra then we define (Br, $\wedge$, $\vee$, $\rightarrow$, $-$, 0, 1), such that $x \rightarrow y = \neg(x - y)$, for every x, y $\in$ A.

Suppose that **A** = (A, $\wedge$, $\vee$, $\rightarrow$, $-$, 0, 1) is a complete D-algebra [20], i.e. (A, $\wedge$, $\vee$, 0, 1) is a complete lattice and **A** is isomorphic to a subdirect product between two structures associated with both a Heyting algebra and a Brouwer algebra as above.

Let ν : A $\rightarrow$ A be the mapping defined by $\nu(x) = 1 \rightarrow x$, for every x $\in$ A. Then ν is an interior operator on the poset (A, $\leq$) associated with **A** and ν is a modal operator on the lattice (A, $\wedge$, $\vee$, 0, 1) such that 0 and 1 are fixed points for ν.

Let $\overline{f}$: A → A be the mapping defined by $\overline{f}x = x - 0$, for every x ∈ A. Then $\overline{f}$ is a closure operator on the poset (A, ≤) and $\overline{f}$ is a dual modal operator on the lattice associated with **A** such that 0, 1 are fixed points for $\overline{f}$.

Any Heyting (Brouwer) algebra can be defined as a D-algebra such that the interior operator ν (closure operator $\overline{f}$) is the identity mapping on A. A Boolean algebra is a D-algebra such that the mappings ν and $\overline{f}$ coincide with the identity mapping.

We define on A the binary operations ⊕ (addition) and ⊗ (multiplication) by

$x \oplus y = \overline{f}(x \vee y)$;

$x \otimes y = \nu(x \wedge y)$,

for all x, y ∈ A.

The following conditions hold:

- (νA, ⊗, ∨, →, 0, 1) is a complete Heyting algebra;
- ($\overline{f}$A, ∧, ⊕, −, 0, 1) is a complete Brouwer algebra;
- (A, ⊕, ⊗, ≤, 0, 1) is a biresiduated lattice such that the D-algebra **A** is isomorphic to a subdirect product of the family of two D_algebras associated with both the Heyting algebra (νA, ⊗, ∨, →, 0, 1) and the Brouwer algebra ($\overline{f}$A, ∧, ⊕, −, 0, 1).

(6) *Structures of biresiduated lattice associated with the chain* [0, 1]

Let ([0, 1], ≤, 0, 1) be the complete chain of positive subunitary real numbers. Consider the standard structure of Heyting-Brouwer algebra ([0, 1], ∧, ∨, →, −, 0, 1) defined as in Example 1.3 (4). Using common operations of addition +, subtraction −, and multiplication · on R, we define on [0, 1] four operations of addition ⊕ as follows, for every x, y ∈ [0, 1]:

(A_1) *Brouwer addition*

$x \oplus y = \max(x, y)$.

(A_2) *Heyting addition*

$$x \oplus y = \begin{cases} 1, \text{ if } x \vee y \neq 0 \\ 0, \text{ if } x \vee y = 0 \end{cases}.$$

(A_3) *Lukasiewicz addition*

$x \oplus y = \min(1, x + y)$.

(A_4) *Gaines addition*

$x \oplus y = x + y - x \cdot y$.

Four operations of multiplication ⊗ will be defined on [0, 1] as follows:

(M_1) *Brouwer multiplication*

$$x \otimes y = \begin{cases} 1, \text{ if } x \wedge y = 1 \\ 0, \text{ if } x \wedge y \neq 1 \end{cases}.$$

(M_2) *Heyting multiplication*

$x \otimes y = \min(x, y)$.

(M_3) *Lukasiewicz multiplication*

$x \otimes y = \max(0, x + y - 1)$.

(M_4) *Gaines multiplication*

$x \otimes y = x \cdot y$.

Therefore, sixteen biresiduated lattice structures

([0, 1], ⊕, ⊗, ≤, 0, 1)

can be associated with the chain ([0, 1], ≤, 0, 1) if one defines the operations of addition ⊕ and multiplication ⊗ by (A_i) and (M_j) respectively, for every i, j = 1, 2, 3, 4.

(7) *Homomorphic images*

Let **A** = (A, ⊕, ⊗, ≤, 0, 1) and **B** = (B, ⊕, ⊗, ≤, 0, 1) be two biresiduated lattices. A mapping f : A → B is called *homomorphism* from **A** to **B** if f is a *morphism of bounded ordered bigrupoids* from **A** to **B** such that f is a *homomorphism of complete lattices* from (A, ≤, 0, 1) to (B, ≤, 0, 1), i.e.

$f(x \oplus y) = f(x) \oplus f(y)$; $f(x \otimes y) = f(x) \otimes f(y)$

$f(\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} f(y_i)$; $f(\bigvee_{i \in I} y_i) = \bigvee_{i \in I} f(y_i)$,

for all x, y ∈ A and for every family $(y_i)_{i \in I}$ of elements of A.

We call **B** a homomorphic image of **A** if there is a surjective homomorphism from **A** onto **B**. The class of homomorphic images of a biresiduated lattice **A** is determined by the quotient structures of **A** with respect to the congruences of **A**.

A congruence of **A** is an equivalence relation R on A compatible with arbitrary meet and join and with the operations of addition and multiplication of **A**. Then the set A/R has a natural structure of biresiduated lattice **A/R** called the *R-quotient* of **A**.

Then **B** is a homomorphic image of **A** iff **B** is isomorphic to a quotient of **A**, i.e. there is a congruence R of **A** such that **B** and the R-quotient **A/R** are isomorphic biresiduated lattices.

Using this notion will result in many new examples of biresiduated lattices (e.g.

homomorphic images of a biresiduated lattice associated with a D-algebra as in Example 1.3 (5) can be obtained using some couple of filters and ideals [20]).

(8) *Biresiduated sublattices*

Let **A** = (A, ⊕, ⊗, ≤, 0, 1) be a biresiduated lattice and B a *subuniverse* of **A,** i.e. B is a subset of the set A such that the following conditions hold :

- B is closed with respect to addition and multiplication, i.e. for every x, y ∈ B,
x ⊕ y ∈ B and x ⊗ y ∈ B;

- the poset (B, ≤, 0, 1) is a complete sublattice of (A, ≤, 0, 1), i.e. for every subset X of B, sup X ∈ B and inf X ∈ B.

Then **B** = (B, ⊕, ⊗, ≤, 0, 1) is a *biresiduated lattice called the biresiduated sublattice of A induced on the subuniverse B*. Intersection of every family of subuniverses of **A** is a subuniverse of **A**. This implies that for every subset X of the set A, the set <X> defined by
<X> = ∩(B ⊆ A / B is a subuniverse of **A** and X ⊆ B)
is a subuniverse of **A**. The biresiduated sublattice of **A** induced on the subuniverse <X> is called the *biresiduated sublattice of A generated by X*.

For example, let **P**(X) = (P(X), ⊕, ⊗, ⊆, ∅, X) be the biresiduated lattice associated with the complete Boolean lattice (P(X), ∩, ∪, ∅, X) of the subsets of the set X as in Example 1.3 (1), i.e. A ⊕ B = A ∪ B and A ⊗ B = A ∩ B, for every A, B ∈ P(X), where ∩ and ∪ are the set-theoretical binary intersection and the binary union of subsets of the set X respectively. Then K ⊆ P(X) is a subuniverse of **P**(X) iff K is a *Kripke family*, i.e. *K* is a family of subsets of X such that K is closed with respect to arbitrary set-theoretical intersections and unions.

(9) *Direct products*

From Definition 1.2 it follows that the direct product of every family of biresiduated lattices is a biresiduated lattice. In particular, it follows that for every topological space X, a biresiduated lattice (O(X) × F(X), ⊕, ⊗, ≤, 0, 1) will be obtained, if one considers the direct product of two biresiduated lattices (O(X), ⊕, ⊗, ⊆, ∅, X) and (F(X), ⊕, ⊗, ⊆, ∅, X) defined as in Examples 1.3 (2) and 1.3 (3). Therefore, for all (O, F), (O', F') ∈ O(X) × F(X), the following relations hold:
(O, F) ⊕ (O', F') = (O ⊕ O', F ⊕ F');
(O, F) ⊗ (O', F') = (O ⊗ O', F ⊗ F');
(O, F) ≤ (O', F') iff O ⊆ O' and F ⊆ F';
0 = (∅, ∅); 1 = (X, X).

1.4 The Duality Principle

Let **A** = (A, ⊕, ⊗, ≤, 0, 1) be a biresiduated lattice. We associate with **A** a system $\mathbf{A}^{o}$ = (A^{o}, $\oplus^{o}$, $\otimes^{o}$, $\leq^{o}$, 0^{o}, 1^{o}) such that A^{o} = A, $\oplus^{o} = \otimes$, $\otimes^{o} = \oplus$, $x \leq^{o} y$ iff $y \leq x$, $0^{o} = 1$ and $1^{o} = 0$.

Then $\mathbf{A}^{o}$ is a biresiduated lattice which can be called dual to **A**. Let φ be a statement about all biresiduated lattices. We associate with φ a second dual statement φ^{o} obtained from φ if replacing ⊕, ⊗, ≤, 0, 1 by $\oplus^{o}$, $\otimes^{o}$, $\leq^{o}$, 0^{o}, 1^{o} respectively . A *duality principle* follows from Definition 1.2:

$$\varphi \text{ is valid} \Rightarrow \varphi^{o} \text{ is valid.}$$

Let **A** = (A, ⊕, ⊗, ≤, 0, 1) be a biresiduated lattice. The following results present some first consequences of Definition 1.2. It will be shown that the class of biresiduated lattices includes several classes of residuated lattices.

1.5 Lemma

Define a mapping c : A → A by c(x) = x ⊕ 0, for every x ∈ A. Then 0 and 1 are fixed points for *c*, *c* is a dual modal operator on the lattice associated with A and *c* is a closure operator on the poset (A, ≤), i.e. the following conditions hold, for every x, y ∈ A:

(i) *c*(0) = 0; c(1) = 1;
(ii) c(x ∧ y) = c(x) ∧ c(y);
(iii) x ≤ c(x);
(iv) c(c(x) = c(x);
(v) x ≤ y ⇒ c(x) ≤ c(y).

Proof

(i) $c(0) = 0 \oplus 0 = 0$;
$c(1) = 1 \oplus 0 = 1 \wedge (1 \oplus 0) = 1.$

(ii) $c(x \wedge y) = (x \wedge y) \oplus 0 = 0 \oplus (x \wedge y) = (0 \oplus x) \wedge (0 \oplus y) = c(x) \wedge c(y).$

(iii) $x \wedge c(x) = x \wedge (x \oplus 0) = x$.

(iv) $c(c(x)) = (x \oplus 0) \oplus 0$
$= x \oplus (0 \oplus 0)$
$= x \oplus 0 = c(x)$;

(v) $x \leq y \Rightarrow c(x) = c(x \wedge y) = c(x) \wedge c(y)$
$\Rightarrow c(x) \leq c(y)$. □

1.5° Lemma

Define a mapping $i : A \to A$ by $i(x) = x \otimes 1$, for every $x \in A$. Then 0 and 1 are fixed points for i, i is a modal operator on the lattice associated with **A** and i is an interior operator on the poset $(A, \leq)$, i.e. the following conditions hold, for all $x, y \in A$:

(i) $i(0) = 0$; $i(1) = 1$;
(ii) $i(x \vee y) = i(x) \vee i(y)$;
(iii) $i(x) \leq x$;
(iv) $i(i(x) = i(x)$;
(v) $x \leq y \Rightarrow i(x) \leq i(y)$.

Proof
Lemma 1.5° follows from Lemma 1.5 and the duality principle 1.4. □

The next properties follow from lemmas 1.5 and 1.5°.

1.6 Consequence

Let $c(A) = A \oplus 0$ be the image of the mapping c defined as in Lemma 1.5, i.e.
$c(A) = \{c(x) / x \in A\} = \{x \oplus 0 / x \in A\} = A \oplus 0$.
The following conditions hold:

(i) $(c(A), \oplus, 0)$ is a *commutative monoid* such that $a \oplus 0 = a$, for every $a \in c(A)$;

(ii) $(c(A), \leq, 0, 1)$ is a complete lattice such that it is an *inf-complete sublattice* of
$(A, \leq, 0, 1)$,
i.e. $\inf X \in c(A)$ for every subset X of $c(A)$;

(iii) in the complete lattice
$(c(A), \leq, 0, 1)$,
the supremum of every subset X of the set $c(A)$ is defined by the following relation:
$$\sup_{c(A)} X = c(\sup_A X).$$

1.6° Consequence

Let $i(A) = A \otimes 1$ be the image of the mapping i defined as in Lemma 1.5°, i.e.
$i(A) = \{i(x) / x \in A\} = \{x \otimes 1 / x \in A\} = A \otimes 1$.
The following conditions hold:

(i) $(i(A), \otimes, 1)$ is a *commutative monoid* such that $a \otimes 1 = a$ for every $a \in i(A)$;

(ii) $(i(A), \leq, 0, 1)$ is a complete lattice such that it is a *sup-complete sublattice* of $(A, \leq, 0, 1)$, i.e. $\sup X \in i(A)$ for every subset X of $i(A)$;

(iii) in the complete lattice
$(i(A), \leq, 0, 1)$,
the infimum of every subset X of the set $i(A)$ is defined by the following relation:
$$\inf_{i(A)} X = i(\inf_A X).$$

1.7 Lemma

(i) The operations of addition and multiplication are increasing mappings in both variables, i.e. for every $x, y, z \in A$,
$x \leq y \Rightarrow x \oplus z \leq y \oplus z$ and $z \oplus x \leq z \oplus y$;
$x \leq y \Rightarrow x \otimes z \leq y \otimes z$ and $z \otimes x \leq z \otimes y$.

(ii) For every $x, y \in A$, $x \vee y \leq x \oplus y$ and $x \otimes y \leq x \wedge y$.

Proof
(i) $x \leq y \Rightarrow x = x \wedge y$
$\Rightarrow x \oplus z = (x \wedge y) \oplus z$
$= (x \oplus z) \wedge (y \oplus z)$
$\Rightarrow x \oplus z \leq y \oplus z$
$\Rightarrow z \oplus x \leq z \oplus y$.
Thus the first implication from (i) holds. The duality principle will also yield the second implication from (i).

(ii) From the finite distributivity of the lattice $(A, \leq)$ and 1.2(4) it follows that
$(x \vee y) \wedge (x \oplus y) =$
$= [x \wedge (x \oplus y)] \vee [y \wedge (x \oplus y)]$
$= x \vee y$.
By duality $(x \wedge y) \vee (x \otimes y) = x \wedge y$ is obtained.
Thus the relations (ii) hold. □

The following results establish a connection between the biresiduated lattices and the residuated lattices.

1.8 Lemma

Define a binary operation $-$ on A called *dual residuation with respect to* $\oplus$ by
$$a - b = \wedge(c \in c(A) \ / \ a \leq b \oplus c),$$
for every $a, b \in A$, where $c(A) = A \oplus 0$ is defined as in Consequence 1.6.
The following conditions hold:

(i) for every $a, b \in A$, $a \leq b \oplus (a - b)$;

(ii) the pair $(\oplus, -)$ is a *dual adjoint couple on* $c(A)$, i.e. for every a, b, c $\in c(A)$,

$$a - b \le c \text{ iff } a \le b \oplus c;$$

(iii) for every a, b, c $\in$ A,

$$a \le b \Rightarrow c - b \le c - a.$$

Proof

From Consequence 1.6 (ii) it follows that

$$x - y \in c(A),$$

for every x, y $\in$ A.

(i) Let a, b $\in$ A. Using the first Equation 1.2(7) and the definition of $-$, it follows that:

$$b \oplus (a - b) = b \oplus [\Lambda(c \in c(A) \ / \ a \le b \oplus c)] = L(b \oplus c \ / \ c \in c(A) \text{ and } a \le b \oplus c),$$

which implies $a \le b \oplus (a - b)$.

(ii) Let a, b, c $\in c(A)$. From the definition of $-$ it follows that $a \le b \oplus c$ implies

$$a - b = L(c' \in c(A) \ / \ a \le b \oplus c') \le c.$$

Suppose now that $a - b \le c$. From 1.7(i) results that $b \oplus (a - b) \le b \oplus c$. Using 1.8 (i), it follows that $a \le b \oplus (a - b)$, but $\le$ is transitive, therefore $a \le b \oplus c$. This completes the proof of 1.8 (ii).

(iii) Let a, b, c $\in$ A with $a \le b$. From 1.7(i) results that $a \oplus (c - a) \le b \oplus (c - a)$, but using 1.8(i) we have $c \le a \oplus (c - a)$, therefore $c \le b \oplus (c - a)$. Using 1.8 (ii) it follows that $c - b \le c - a$. □

1.8° Lemma

Define a binary operation $\to$ on A called *residuation with respect to* $\otimes$ by

$$a \to b = V(c \in i(A) \ / \ a \otimes c \le b),$$

for every a, b $\in$ A, where $i(A) = A \otimes 1$ is defined as in Consequence 1.6°.

The following conditions hold:

(i) for every a, b $\in$ A, $a \otimes (a \to b) \le b$;

(ii) the pair $(\otimes, \to)$ is an *adjoint couple on* $i(A)$, i.e. for every a, b, c $\in i(A)$,

$$c \le a \to b \text{ iff } a \otimes c \le b;$$

(iii) for every a, b, c $\in$ A,

$$a \le b \Rightarrow b \to c \le a \to c.$$

Proof

Lemma 1.8° follows from Lemma 1.8 and the duality principle 1.4.
□

The following theorem provides a description of the biresiduated lattice structure as a complete lattice $(A, \le, 0, 1)$ equipped with two specific couples of binary operations on A, a dual adjoint couple $(\oplus, -)$ and an adjoint couple $(\otimes, \to)$.

1.9 Theorem

If $\mathbf{A} = (A, \oplus, \otimes, \le, 0, 1) \in$ Ord-BG[0,1], then the following are equivalent:

(i) **A** is a biresiduated lattice;

(ii) there is a binary operation $-$ on A called *dual residuation with respect to* $\oplus$ and there is a binary operation $\to$ on A called *residuation with respect to* $\otimes$ such that for every x, y, z $\in$ A, the system

$$(A, \oplus, -, \otimes, \to, \le, 0, 1)$$

verifies 1.2(i), 1.2(1)-(6) and:

(1) $(x - y) \oplus 0 = x - y$;

(1°) $(x \to y) \otimes 1 = x \to y$;

(2) $x - y \le z \oplus 0 \Leftrightarrow x \le y \oplus z$;

(2°) $z \otimes 1 \le x \to y \Leftrightarrow x \otimes z \le y$.

Proof

(i) $\Rightarrow$ (ii): suppose (i). Let $-$ and $\to$ be the two binary operations on A defined respectively as in Lemmas 1.8 and 1.8°. Relations (1) and (2) follow from 1.6(ii) and Lemma 1.8(ii). Relations (1°) and (2°) follow from 1.6°(ii) and Lemma 1.8°(ii).

(ii) $\Rightarrow$ (i): suppose (ii). We verify that the two Equations 1.2(7) hold. For x $\in$ A and for a family $(y_i)_{i \in I}$ of elements of A we have

$$(\forall j \in I) \bigwedge_{i \in I}(y_i \oplus 0) \le y_j \oplus 0.$$

From Lemma 1.7 (i) and Equation 1.2(5) it follows that

$$(\forall j \in I)\, x \oplus \bigwedge_{i \in I}(y_i \oplus 0) \le x \oplus y_j,$$

thus

$$(*)\, x \oplus \bigwedge_{i \in I}(y_i \oplus 0) \le \bigwedge_{i \in I}(x \oplus y_i).$$

Also we have $(\forall j \in I) \bigwedge_{i \in I}(x \oplus y_i) \le x \oplus y_j$.

Then $(\forall j \in I) \bigwedge_{i \in I}(x \oplus y_i) - x \le y_j \oplus 0$, because of 1.9(2). Thus

$$\bigwedge_{i \in I}(x \oplus y_i) - x \le \bigwedge_{i \in I}(y_i \oplus 0).$$

Lemma 1.7(i) implies

$$[\bigwedge_{i \in I}(x \oplus y_i) - x] \oplus 0 \le [\bigwedge_{i \in I}(y_i \oplus 0)] \oplus 0.$$

Then

$$\bigwedge_{i \in I}(x \oplus y_i) - x \le [\bigwedge_{i \in I}(y_i \oplus 0)] \oplus 0,$$

because of 1.9(1). From 1.9(2) it follows that

$$(**)\ \bigwedge_{i\in I}(x\oplus y_i)\le x\oplus\bigwedge_{i\in I}(y_i\oplus 0).$$

The first Equation 1.2(7) follows from (*) and (**). Similarly, the second Equation 1.2(7) follows from 1.9(1°) and 1.9(2°). Therefore, **A** verifies all the conditions from Definition 1.2, i.e. 1.9(i) holds.
This completes the proof. □

From Lemmas 1.8 and 1.8° it follows that any biresiduated lattice **A** = (A, ⊕, ⊗, ≤, 0, 1) can associate two structures, a *complete inf-sublattice with dual residuation with respect to* ⊕, *c*(**A**) = (*c*(A), ⊕ , −, ≤, 0, 1), and a *complete sup-sublattice with residuation with respect to* ⊗, *i*(**A**) = (*i*(A), ⊗, →, ≤, 0, 1), called the *algebra of closed elements* and the *algebra of open elements of A* respectively.

This shows that the theory of biresiduated lattices can be viewed as an extended theory of residuated lattices [12, 16].

The next results present other equations which hold in *c*(**A**) and *i*(**A**).

1.10 Lemma

Dual residuation − with respect to ⊕ is a *sup-morphism in the first variable* and a *dual inf-morphism in the second variable on the lattice* *c*(A) = A ⊕ 0, i.e. for every family $(b_i)_{i\in I}$ of elements in *c*(A) and for a ∈ *c*(A), there is

$$(\mathrm{i})\ c(\bigvee_{i\in I} b_i)-a=c\left[\bigvee_{i\in I}(b_i-a)\right];$$

$$(\mathrm{ii})\ a-\bigwedge_{i\in I} b_i=c\left[\bigvee_{i\in I}(a-b_i)\right].$$

Proof

Let $(b_i)_{i\in I}$ be a family of elements in *c*(A) and an element a ∈ *c*(A).

(i) The following relations hold, for every j ∈ I:

$$b_j\le(\bigvee_{i\in I} b_i)\oplus 0\le$$

$$\bigwedge(a\oplus b\ /\ b\in c(A)\text{ and }(\bigvee_{i\in I} b_i)\oplus 0\le a\oplus b)$$

$$=a\oplus[\bigwedge(b\in c(A)\ /\ (\bigvee_{i\in I} b_i)\oplus 0\le a\oplus b)]$$

$$=a\oplus[((\bigvee_{i\in I} b_i)\oplus 0)-a].$$

Using Lemma 1.8 (ii) it follows that

$$(\forall j\in I)b_j-a\le[(\bigvee_{i\in I} b_i)\oplus 0]-a.$$

Therefore

$$(*)\ [\bigvee_{i\in I}(b_i-a)]\oplus 0\le[(\bigvee_{i\in I} b_i)\oplus 0]-a.$$

The following relation holds

$(\forall j\in I)b_j-a\le[\bigvee_{i\in I}(b_i-a)]\oplus 0$. From Lemma 1.7(i) it follows that

$$(\forall j\in I)(b_j-a)\oplus a\le[[\bigvee_{i\in I}(b_i-a)]\oplus 0]\oplus a,$$

but $(\forall j\in I)\ b_j\le(b_j-a)\oplus a$, therefore

$$(\forall j\in I)\ b_j\le[[\bigvee_{i\in I}(b_i-a)]\oplus 0]\oplus a.$$

The precedent relation and a ⊕ 0 = a imply

$$(\bigvee_{i\in I} b_i)\oplus 0\le[[\bigvee_{i\in I}(b_i-a)]\oplus 0]\oplus a.$$

Using Lemma 1.8 (ii) it follows that

$$(**)\ [(\bigvee_{i\in I} b_i)\oplus 0]-a\le[\bigvee_{i\in I}(b_i-a)]\oplus 0.$$

From (*) and (**) here results Equation 1.10(i).

(ii) We have $(\forall j\in I)\bigwedge_{i\in I} b_i\le b_j$. From Lemma 1.8(iii) it follows that

$$(\forall j\in I)\ a-b_j\le a-\bigwedge_{i\in I} b_i,$$

which implies

$$(*)\ [\bigvee_{i\in I}(a-b_i)]\oplus 0\le a-\bigwedge_{i\in I} b_i.$$

Let

$$b=[\bigvee_{i\in I}(a-b_i)]\oplus 0.$$

Then $(\forall i\in I)\ a-b_i\le b$. Lemma 1.8 (ii) implies $(\forall i\in I)\ a\le b_i\oplus b$. Using 1.2 (7) it follows that

$$a\le\bigwedge_{i\in I}(b_i\oplus b)=b\oplus(\bigwedge_{i\in I} b_i).$$

From 1.8 (ii) it follows that

$$(**)\ a-\bigwedge_{i\in I} b_i\le b=[\bigvee_{i\in I}(a-b_i)]\oplus 0.$$

Relations (*) and (**) show that property 1.10(ii) is satisfied. □

1.10° Lemma

Residuation → with respect to ⊗ is an *inf-morphism in the second variable* and a *dual sup-morphism in the first variable* on *the lattice* *i*(A) = A ⊗ 1, i.e. for every family $(b_i)_{i\in I}$ of elements in *i*(A) and for a ∈ *i*(A), we have

$$(\mathrm{i})\ a\to i(\bigwedge_{i\in I} b_i)=i\left[\bigwedge_{i\in I}(a\to b_i)\right];$$

$$(\mathrm{ii})\ \bigvee_{i\in I} b_i\to a=i\left[\bigwedge_{i\in I}(b_i\to a)\right].$$

Proof

Lemma 1.10° follows from Lemma 1.10 and the duality principle 1.4.
□

The next results present other relations derived from properties 1.9, 1.10 and 1.10°.

1.11 Consequence

In each biresiduated lattice **A**, for all x, y, z ∈ A:

(i) $x \oplus 0 = x - 0$;

(ii) $x - y = 0 \Leftrightarrow x \oplus 0 \leq y \oplus 0$;

(iii) $(x \vee y) \oplus 0 = [(x \oplus 0) \vee y] \oplus 0$;

(iv) $x - y \leq x \oplus 0$;

(v) $y \oplus (x - y) \leq x \oplus y$;

(vi) $(x \vee y) \oplus 0 \leq y \oplus (x - y)$;

(vii) $(x - y) - z = x - (y \oplus z)$;

(viii) $x - y = x - (y \oplus 0)$;

(ix) $x - y = (x \oplus 0) - y$;

(x) $x \oplus [(x \oplus y) - (x \oplus z)] \leq x \oplus (y - z)$;

(xi) $(x \vee y) - (x \vee z) = y - (x \vee z)$;

(xii) $[x \vee (y - z)] \oplus 0 \leq x \oplus [(x \vee y) - (x \vee z)]$.

Proof

(i) We have

$$x - 0 = \Lambda(y \oplus 0 \ / \ x \leq y \oplus 0)$$

and $x \leq x \oplus 0$, thus $x - 0 \leq x \oplus 0$. Relation

$$x \oplus 0 \leq x - 0$$

holds because, for every y ∈ A,

$$x \leq y \oplus 0 \Rightarrow x \oplus 0 \leq y \oplus 0.$$

(ii) Suppose $x - y = 0$. From 1.2(3) here results $x - y \leq 0 \oplus 0$. Using 1.9(2) implies $x \leq y \oplus 0$. From 1.7(i), 1.2(2) and 1.2(3) it follows that

$$x \oplus 0 \leq y \oplus 0.$$

Suppose now $x \oplus 0 \leq y \oplus 0$. We have

$$x \leq x \oplus 0,$$

therefore $x \leq y \oplus 0$. From 1.10(2) it follows that $x - y \leq 0 \oplus 0 = 0$, which implies $x - y = 0$.

(iii) $x \leq x \oplus 0$ and $y \leq y \Rightarrow x \vee y \leq (x \oplus 0) \vee y$, thus

$$(*) \ (x \vee y) \oplus 0 \leq [(x \oplus 0) \vee y] \oplus 0.$$

We also have

$$y \leq x \vee y \leq (x \vee y) \oplus 0$$

and

$$x \oplus 0 \leq (x \vee y) \oplus 0,$$

which implies

$$(**) \ [(x \oplus 0) \vee y] \oplus 0 \leq (x \vee y) \oplus 0.$$

Equation 1.11(iii) follows from (*) and (**).

(iv) Relation 1.11(iv) follows from $x \leq y \oplus x$ and 1.9(2).

(v) Lemma 1.7(i) and property 1.11(iv) imply

$$y \oplus (x - y) \leq y \oplus (x \oplus 0) = x \oplus y$$

because of 1.2 (1), 1.2(2) and 1.2(5). Thus 1.11(v) holds.

(vi) Lemmas 1.8(i) and 1.7(ii) imply

$$x \leq y \oplus (x - y) \text{ and } y \leq y \oplus (x - y).$$

From Lemma 1.7(i) and Definition 1.2(5) it follows that 1.11(vi).

(vii) From Lemma 1.8(i) it follows that

$$x \leq (y \oplus z) \oplus [x - (y \oplus z)] =$$
$$= y \oplus [z \oplus [x - (y \oplus z)]].$$

Relations 1.2(5), 1.9(1) and 1.9(2) imply

$$(*) \ (x - y) - z \leq x - (y \oplus z).$$

Lemma 1.8(i) implies $x - y \leq z \oplus [(x - y) - z]$, therefore, using 1.7(i) we have

$$y \oplus (x - y) \leq y \oplus [z \oplus [(x - y) - z]],$$

but $x \leq y \oplus (x - y)$, thus

$$x \leq (y \oplus z) \oplus [(x - y) - z].$$

From 1.9(1), 1.9(2) and the previous relation it follows that

$$(**) \ x - (y \oplus z) \leq (x - y) - z.$$

Then 1.11(vii) follows from (*) and (**).

(viii) We have

$$x \leq (y \oplus 0) \oplus [x - (y \oplus 0)] = y \oplus [x - (y \oplus 0)].$$

From 1.9(1) and 1.9(2) this implies

$$(*) \ x - y \leq x - (y \oplus 0).$$

But $x \leq y \oplus (x - y) = (y \oplus 0) \oplus (x - y)$, thus reusing 1.9(1) and 1.9(2) one obtains

$$(**) \ x - (y \oplus 0) \leq x - y.$$

Then 1.11(viii) follows from (*) and (**).

(ix) From 1.11(i), 1.11(vii) and 1.11(viii) it follows that

$$\begin{aligned} x - y &= x - (y \oplus 0) \\ &= x - (0 \oplus y) \\ &= (x - 0) - y \\ &= (x \oplus 0) - y, \end{aligned}$$

which proves 1.11(ix).

(x) Relation 1.2(7) and the definition of dual residuation imply:

$$(*) \ x \oplus (y - z) =$$
$$= \Lambda(x \oplus a \ / \ a \in c(A) \text{ and } y \leq z \oplus a)$$

$$(**) \ x \oplus [(x \oplus y) - (x \oplus z)] =$$
$$= \Lambda(x \oplus a \ / \ a \in c(A) \text{ and } x \oplus y \leq (x \oplus z) \oplus a).$$

Using Lemma 1.7(i) we have

$$(***) \quad (\forall a \in c(A))$$
$$[y \leq z \oplus a \Rightarrow x \oplus y \leq (x \oplus z) \oplus a].$$

Then 1.11(x) follows from (*), (**) and (***).

(xi) From 1.11(ii), 1.11(ix), Lemma 1.10(i) and Relation 1.9(1) it follows that

$$\begin{aligned} &(x \vee y) - (x \vee z) = \\ &= c(x \vee y) - (x \vee z) \\ &= c[[x - (x \vee z)] \vee [y - (x \vee z)]] \\ &= c[0 \vee [y - (x \vee z)]] \\ &= y - (x \vee z), \end{aligned}$$

which imply 1.11(11).

(xii) Using lemmas 1.7(ii), 1.11(viii) and 1.10(ii) we have

$y - (x \vee z) = y - [c(x \vee z) \wedge (x \oplus z)]$
$= c[[y - (x \vee z)] \vee [y - (x \oplus z)]].$

From 1.11(xi) and the precedent relation it follows that

$x \oplus [(x \vee y) - (x \vee z)] =$
$= x \oplus [[y - (x \vee z)] \vee [y - (x \oplus z)]],$

which implies

(*) $x \oplus [y - (z \oplus x)] \leq x \oplus [(x \vee y) - (x \vee z)].$

Using 1.11(vi) and 1.11(vii) we also have

$[(y - z) \vee x] \oplus 0 \leq x \oplus [(y - z) - x] =$
$= x \oplus [y - (z \oplus x)],$

thus

(**) $[x \vee (y - z)] \oplus 0 \leq x \oplus [y - (z \oplus x)].$

Then 1.11(xii) follows from (*) and (**). □

1. 11° Consequence

In each biresiduated lattice **A**, for $x, y, z \in A$:

(i) $x \otimes 1 = 1 \to x;$
(ii) $x \to y = 1 \Leftrightarrow x \otimes 1 \leq y \otimes 1;$
(iii) $(x \wedge y) \otimes 1 = [(x \otimes 1) \wedge y] \otimes 1;$
(iv) $y \otimes 1 \leq x \to y;$
(v) $x \otimes y \leq x \otimes (x \to y);$
(vi) $x \otimes (x \to y) \leq (x \wedge y) \otimes 1;$
(vii) $x \to (y \to z) = (x \otimes y) \to z;$
(viii) $x \to y = (x \otimes 1) \to y;$
(ix) $x \to y = x \to (y \otimes 1);$
(x) $x \otimes (y \to z) \leq x \otimes [(x \otimes y) \to (x \otimes z)];$
(xi) $(x \wedge y) \to (x \wedge z) = (x \wedge y) \to z;$
(xii) $x \otimes [(x \wedge y) \to (x \wedge z)] \leq [x \wedge (y \to z)] \otimes 1$

Proof

Consequence 1. 11° follows from Consequence 1.11 and the duality principle 1.4. □

This Section ends with mentioning the results which show that on each biresiduated lattice can be defined two new unary operations called ⊕-complementation and ⊗-complementation together with some specific properties.

1.12 Consequence

Let **A** = (A, ⊕, ⊗, ≤, 0, 1) be a biresiduated lattice, − be the dual residuation with respect to ⊕ and $c(A) = A \oplus 0$ be the set of closed elements of A defined as in 1.6. Define on A an unary operation $C_\oplus : A \to A$ called ⊕-*complementation* on A by

$$C_\oplus(x) = 1 - x,$$

for every $x \in A$. The following conditions hold:

(i) $C_\oplus(x)$ is the least closed element $y \in c(A)$ such that $x \oplus y = 1$, for any $x \in A$;

(ii) the restriction of $C_\oplus$ to $c(A) \subseteq A$ is a *meet-complete dual endomorphism of the complete lattice* $(c(A), \leq, 0, 1)$, i.e. for every subset X of set $c(A)$,

$$C_\oplus(\inf_{c(A)}X) = \sup_{c(A)}C_\oplus(X).$$

Proof

(i) Let $x \in A$. Then

$1 - x = \wedge(y \in c(A) / x \oplus y = 1)$;
[definition of − from Lemma 1.8]
$x \oplus (1 - x) = 1;$
[Lemma 1.8(i)]
$1 - x \in c(A),$
[Consequence 1.6(ii)]

which shows that 1.12(i) holds.

(ii) From Lemma 1.10 (ii) it follows that

$$1 - \bigwedge_{i \in I} x_i = c\left[\bigvee_{i \in I}(1 - x_i)\right],$$

for every family $(x_i)_{i \in I}$ of elements in $c(A)$. Using consequence 1.6 [(ii) and (iii)] it follows that 1.12(ii) holds. □

1.12° Consequence

Let **A** = (A, ⊕, ⊗, ≤, 0, 1) be a biresiduated lattice, → be the residuation with respect to ⊗ and $i(A) = A \otimes 1$ be the set of open elements of A defined as in 1.6°. Define on A an unary operation $C_\otimes : A \to A$ called ⊗-*complementation* on A by

$$C_\otimes(x) = x \to 0,$$

for every $x \in A$. The following conditions hold:

(i) $C_\otimes(x)$ is the greatest open element $y \in i(A)$ such that $x \otimes y = 0$, for any $x \in A$;

(ii) the restriction of $C_\otimes$ to $i(A) \subseteq A$ is a *join-complete dual endomorphism of the complete lattice* $(i(A), \leq, 0, 1)$, i.e. for every subset X of set $i(A)$,

$$C_\otimes(\sup_{i(A)}X) = \inf_{i(A)}C_\otimes(X).$$

Proof

Consequence 1. 12° follows from Consequence 1.12 and the duality principle 1.4. □

Starting from the previous properties, the following Section introduces the structure of *biresiduated algebra* as an algebra

$$\mathbf{A} = (A, \wedge, \vee, \oplus, -, \otimes, \rightarrow, \neg, 0, 1)$$

of type (2, 2, 2, 2, 2, 2, 1, 0, 0) including both the structure of *D-algebra* and the structure of *MV-algebra* such that each complete biresiduated algebra is a *biresiduated lattice with negation,* which satisfies some specific equations. Different structures of complete biresiduated algebras will be associated with the complete chain [0, 1].

2. Biresiduated Algebras

This Section makes first a presentation of the notion of *biresiduated lattice with negation* including both the structure of complete D_algebra and the structure of complete MV_algebra.

2.1 Definition

Let $\mathbf{A} = (A, \oplus, -, \otimes, \rightarrow, \leq, 0, 1)$ be a biresiduated lattice.

(i) We say that **A** is a *biresiduated lattice with negation* if *⊕-complementation* coincides with *⊗-complementation* (see Consequence 1.12 and Consequence 1.12°), i.e. for every $x \in A$, $C_{\oplus}(x) = 1 - x = x \rightarrow 0 = C_{\otimes}(x)$.

(ii) If **A** is a biresiduated lattice with negation then we define an unary operation on A, $\neg : A \rightarrow A$ called *negation* on A, by

$$\neg x = 1 - x,$$

or equivalently by

$$\neg x = x \rightarrow 0,$$

for every $x \in A$.

We present now new equations which hold in biresiduated lattices with negation.

2.2 Lemma

Let **A** be a biresiduated lattice with negation, $c(A) = A \oplus 0$ the set of closed elements of A defined as in 1.6, and $i(A) = A \otimes 1$ the set of open elements of A defined as in 1.6°. The following relations hold:

(i) $\neg\neg\neg x = \neg x$;

(ii) $\neg c(x) = \neg x$;

(iii) $\neg i(x) = \neg x$;

(iv) $\neg\neg(x \oplus 0) \leq x \oplus 0$ and $y \otimes 1 \leq \neg\neg(y \otimes 1)$;

(v) $c(A) \cap i(A) = \neg A$;

(vi) $\neg\left[\bigwedge_{i \in I} c(x_i)\right] = \left(\bigvee_{i \in I} \neg x_i\right) \oplus 0$;

(vii) $\neg\left[\bigvee_{i \in I} i(x_i)\right] = \left(\bigwedge_{i \in I} \neg x_i\right) \otimes 1$,

for all $x, y \in A$ and for every family $(x_i)_{i \in I}$ of elements of A, where

$$\neg A = \{\neg x \,/\, x \in A\}.$$

Proof

(i) We have

(1) $\neg x = 1 - x = (1 - x) \oplus 0 \in c(A)$;

(2) $\neg x \oplus \neg\neg x = \neg x \oplus (1 - \neg x) = 1$;

(3) $\neg\neg\neg x = 1 - \neg\neg x$
$= \bigwedge(y \in c(A) \,/\, 1 = y \oplus \neg\neg x)$.

From (1), (2) and (3) it follows that

$$(*)\ \neg\neg\neg x \leq \neg x.$$

We also have

(1') $\neg x = x \rightarrow 0 = (x \rightarrow 0) \otimes 1 \in i(A)$;

(2') $\neg x \otimes \neg\neg x = \neg x \otimes (\neg x \rightarrow 0) = 0$;

(3') $\neg\neg\neg x = \neg\neg x \rightarrow 0$
$= \bigvee(y \in i(A) \,/\, 0 = y \otimes \neg\neg x)$.

From (1'), (2') and (3') it follows that

$$(**)\ \neg x \leq \neg\neg\neg x.$$

Thus, 2.2(i) follows from (*) and (**).

(ii) Relation 2.2(ii) follows from

$$\neg c(x) = 1 - (x \oplus 0) = 1 - x = \neg x.$$

(iii) Relation 2.2(iii) follows from

$$\neg i(x) = (x \otimes 1) \rightarrow 0 = x \rightarrow 0 = \neg x.$$

(iv) From $(1 - x) \oplus x = 1$ and 1.9(2) it follows that $1 - (1 - x) \leq x \oplus 0$, but

$$\neg\neg(x \oplus 0) = 1 - [1 - (x \oplus 0)] = 1 - (1 - x),$$

thus $\neg\neg(x \oplus 0) \leq x \oplus 0$. From $y \otimes (y \rightarrow 0) = 0$ and 1.9(2°) it follows that

$$y \otimes 1 \leq (y \rightarrow 0) \rightarrow 0,$$

but

$$\neg\neg(y \otimes 1) = [(y \otimes 1) \rightarrow 0] \rightarrow 0 = (y \rightarrow 0) \rightarrow 0,$$

thus

$$y \otimes 1 \leq \neg\neg(y \otimes 1).$$

Therefore, 2.2(iv) holds.

(v) If $z \in \neg A$ then $z = \neg x$, for $x \in A$, but

$$\neg x = 1 - x = (1 - x) \oplus 0 \in c(A)$$

and

$$\neg x = x \rightarrow 0 = (x \rightarrow 0) \otimes 1 \in i(A),$$

therefore $z \in c(A) \cap i(A)$. It follows that

$$(*)\ \neg A \subseteq c(A) \cap i(A).$$

Now let $z \in c(A) \cap i(A)$ i.e. $z = c(x)$ and $z = i(y)$, for some $x, y \in A$. From 2.2(iv) it follows that

$$\neg\neg z = \neg\neg c(x) = \neg\neg(x \oplus 0) \leq x \oplus 0 = z$$
and
$$z = y \otimes 1 \leq \neg\neg(y \otimes 1) = \neg\neg i(y) = \neg\neg z.$$
This implies $z = \neg\neg z \in \neg A$. Therefore,
$$(^{**})\ c(A) \cap i(A) \subseteq \neg A.$$
Thus, 2.2(v) follows from (*) and (**).

(vi) Relation 2.2(vi) follows from Definition 2.1 and Consequence 1.12(ii).

(vii) Relation 2.2(vii) follows from Definition 2.1 and Consequence 1.12°(ii). □

A list of basic examples of biresiduated lattices with negation is given.

2.3 Examples

(E1) ***Complete D-algebras***

Let **A** = (A, ⊕, −, ⊗, →, ≤, 0, 1) be a biresiduated lattice associated with a *complete D-algebra* (A, ∧, ∨, →, −, 0, 1) as in Example 1.3(5), i.e. for every x, y ∈ A:

$$x \oplus y = (x \vee y) - 0 = \overline{f}\,(x \vee y);$$
$$x \otimes y = 1 \to (x \wedge y) = v(x \wedge y);$$

The following conditions hold:

(1) $x \oplus y = (x \vee y) \oplus 0 = c(x \vee y)$;

(2) the system c(**A**) = (c(A), ∧, ⊕, −, 0, 1) is a complete Brouwer algebra, i.e. it is a complete lattice such that:

$$a \leq b \text{ iff } a \oplus b = b;$$
$$a - b \leq c \text{ iff } a \leq b \oplus c,$$

for every a, b, c ∈ c(A) = A ⊕ 0;

(3) $x \otimes y = (x \wedge y) \otimes 1 = i(x \wedge y)$;

(4) the system i(**A**) = (i(A), ⊗, ∨, →, 0, 1) is a complete Heyting algebra, i.e. it is a complete lattice such that:

$$a \leq b \text{ iff } a \otimes b = a;$$
$$c \leq a \to b \text{ iff } c \otimes a \leq b,$$

for every a, b, c ∈ i(A) = A ⊗ 1.

Then **A** is a *biresiduated lattice with negation* such that **A** satisfies the *following determination principle*:

(5) $x \oplus 0 = y \oplus 0$ and $x \otimes 1 = y \otimes 1$ implies $x = y$

and the following specific relations hold:

(6) $x \to (y - z) = 1 - [x \wedge (y \to z)]$;

(7) $(x \to y) - z = [z \vee (x - y)] \to 0$,

for every x, y, z ∈ A.

Because the precedent structures are complete D-algebras, it follows that the structure of biresiduated lattice with negation includes complete Heyting algebras, complete Brouwer algebras, and complete Boolean algebras.

(E2) ***Complete MV-algebras***

Let (A, ⊕, ¬, 0) be an MV-algebra [6, Definition 1.1.1 pp.11], i.e. A is an algebra with a binary operation ⊕, an unary operation ¬, and a constant 0 such that the following Equations hold:

MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

MV2) $x \oplus y = y \oplus x$;

MV3) $x \oplus 0 = x$;

MV4) $\neg\neg x = x$;

MV5) $x \oplus \neg 0 = \neg 0$;

MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

From MV1)-MV3) it follows that (A, ⊕, 0) is an *abelian monoid*. We define a constant 1 and the operations ⊗, → and − together with a binary relation ≤ on A as follows, for any two elements x, y of A:

(1) $1 = \neg 0$;

(2) $x \otimes y = \neg(\neg x \oplus \neg y)$;

(3) $x \to y = \neg x \oplus y$;

(4) $x - y = x \otimes \neg y$;

(5) $x \leq y$ iff $\neg x \oplus y = 1$.

Then ≤ is an order relation which determines on A a structure of *distributive lattice with the smallest element 0 and the greatest element* 1, (A, ∧, ∨, 0, 1), such that:

(6) $x \vee y = (x \otimes \neg y) \oplus y$;

(7) $x \wedge y = \neg(\neg x \vee \neg y)$.

We say that A is a *complete MV-algebra* if A is an MV-algebra such that the ordered set

(A, ≤, 0, 1)

is a complete lattice.

The system

A = (A, ⊕, −, ⊗, →, ≤, 0, 1)

associated with every complete MV-algebra

(A, ⊕, ¬, 0)

and defined as above is a *biresiduated lattice with negation* such that the following specific relations hold:

(8) $x \vee y = (x - y) \oplus y$;

(9) $x \wedge y = x \otimes (x \to y)$;

(10) $\neg x = x \to 0 = 1 - x$.

(E3) ***The Brouwer structure on*** **[0, 1]**

Let $\oplus$ be the Brouwer addition 1.3(6)(A_1) and $\otimes$ the Brouwer multiplication 1.3(6)(M_1). Then ([0, 1], $\oplus$, $-$, $\otimes$, $\rightarrow$, $\leq$, 0, 1) is a biresiduated lattice with negation called the *Brouwer structure on [0, 1]*, where the binary operations $-$ and $\rightarrow$ are defined by:

$$x - y = \begin{cases} 0, & \text{if } x \leq y \\ x, & \text{if } x > y \end{cases};$$

$$x \rightarrow y = 1 - (x - y) = \begin{cases} 1, & \text{if } x \leq y \text{ or } y < x < 1 \\ 0, & \text{if } y < 1 = x \end{cases}$$

and the negation operator $\neg$ is defined by:

$$\neg x = 1 - x = x \rightarrow 0 = \begin{cases} 0, & \text{if } x = 1 \\ 1, & \text{if } x \neq 1 \end{cases},$$

for all x, y $\in$ [0, 1].

From the precedent relations it follows that the *Brouwer structure on [0, 1]* is a biresiduated lattice with negation associated with a complete D-algebra as in Example 2.3(E1) such that the following specific conditions hold:

- algebra of closed elements is the complete Brouwer algebra ([0, 1], $\vee$, $-$, $\leq$, 0, 1);
- algebra of open elements is the Boolean algebra with two elements: 0 and 1.

(E4) ***The Heyting structure on*** **[0, 1]**

Let $\oplus$ be the Heyting addition 1.3(6)(A_2) and $\otimes$ the Heyting multiplication 1.3(6)(M_2). Then ([0, 1], $\oplus$, $-$, $\otimes$, $\rightarrow$, $\leq$, 0, 1) is a biresiduated lattice with negation called the *Heyting structure* on [0, 1], where the binary operations $-$ and $\rightarrow$ are defined by:

$$x - y = (x \rightarrow y) \rightarrow 0 = \begin{cases} 0, & \text{if } x \leq y \text{ or } 0 < y < x \\ 1, & \text{if } 0 = y < x \end{cases};$$

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}$$

and the negation operator $\neg$ is defined by:

$$\neg x = x \rightarrow 0 = 1 - x = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases},$$

for all x, y $\in$ [0, 1].

From the precedent relations it follows that the *Heyting structure on* [0, 1] is a biresiduated lattice with negation associated with a complete D-algebra such that the following specific conditions hold:

- structure of closed elements is the Boolean algebra with two elements 0 and 1;
- structure of open elements is the complete Heyting algebra ([0, 1], $\wedge$, $\rightarrow$, $\leq$, 0, 1).

(E5) ***The Lukasiewicz structure on*** **[0, 1]**

Let $\oplus$ be the Lukasiewicz addition 1.3(6)(A_3) and $\otimes$ the Lukasiewicz multiplication 1.3(6)(M_3). The *Lukasiewicz structure on* [0, 1] is a *biresiduated lattice with negation*

([0, 1], $\oplus$, $-$, $\otimes$, $\rightarrow$, $\leq$, 0, 1)

associated with the standard structure of complete MV-algebra ([0, 1], $\oplus$, $\neg$, 0), where the operations $-$, $\rightarrow$ and $\neg$ are defined by:

$x - y = \max(0, x - y)$;

$x \rightarrow y = \min(1, 1 - x + y)$;

$\neg x = 1 - x$,

for all x, y $\in$ [0, 1].

(E6) ***The Heyting-Gaines structure on*** **[0, 1]**

Let $\oplus$ be the Heyting addition 1.3(6)(A_2) and $\otimes$ the Gaines multiplication 1.3(6)(M_4). Then the *Heyting-Gaines structure* on [0, 1] is a *biresiduated lattice with negation*

([0, 1], $\oplus$, $-$, $\otimes$, $\rightarrow$, $\leq$, 0, 1),

where dual residuation $-$ and negation operator $\neg$ on [0, 1] coincide respectively with the Heyting dual residuation $-$ and the Heyting negation $\neg$ defined as in Example 2.3 (E4), and residuation $\rightarrow$ is defined by:

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y \\ \dfrac{y}{x}, & \text{if } x > y \end{cases},$$

for all x, y $\in$ [0, 1].

(E7) ***The Gaines-Brouwer structure on*** **[0, 1]**

Let $\oplus$ be the Gaines addition 1.3(6)(A_4) and $\otimes$ the Brouwer multiplication 1.3(6)(M_1). Then the *Gaines-Brouwer structure* on [0, 1] is a *biresiduated lattice with negation*

([0, 1], $\oplus$, $-$, $\otimes$, $\rightarrow$, $\leq$, 0, 1),

where residuation $\rightarrow$ and negation operator $\neg$ on [0, 1] coincide respectively with the Brouwer residuation $\rightarrow$ and the Brouwer negation $\neg$ defined as in Example 2.3 (E3), and dual residuation $-$ is defined by:

$$x - y = \begin{cases} 0, & \text{if } x \leq y \\ \dfrac{x - y}{1 - y}, & \text{if } x > y \end{cases},$$

for all x, y $\in$ [0, 1].

Remark: Many of the structures from Example 1.3 (6) are not biresiduated lattices with negation. For example, the standard Heyting-Brouwer structure associated with the complete chain ([0, 1], $\leq$, 0, 1) is a biresiduated lattice ([0, 1], $\oplus$, $-$, $\otimes$, $\rightarrow$, $\leq$, 0, 1) which is without negation, where $\oplus$ is the binary join $\vee$, $\otimes$ is the binary meet $\wedge$, dual residuation $-$ is the Brouwer residuation defined as in Example 2.3 (E3) and residuation $\rightarrow$ is the Heyting residuation defined as in Example 2.3 (E4). Another example of biresiduated lattice which is without negation can be obtained if one defines the Gaines addition $\oplus$ by 1.3(6)(A_4), the Gaines multiplication $\otimes$ by 1.3(6)(M_4), dual residuation $-$ as in Example 2.3 (E7) and residuation $\rightarrow$ as in Example 2.3 (E6). □

The next definitions present three classes of algebras and the notion of *biresiduated algebra* including all biresiduated lattices with negation from Examples 2.3 (E1)-(E7).

2.4 The K, D and MV Classes

Let K be the class of algebras

$$\mathbf{A} = (A, \wedge, \vee, \oplus, -, \otimes, \rightarrow, \neg, 0, 1)$$

of type (2, 2, 2, 2, 2, 2, 1, 0, 0).

Let D be the class of algebras **A** of K such that $(A, \wedge, \vee, \rightarrow, -, 0, 1)$ is a D-algebra and for every $x, y \in A$,

$$x \oplus y = (x \vee y) - 0;$$
$$x \otimes y = 1 \rightarrow (x \wedge y);$$
$$\neg x = x \rightarrow 0 = 1 - x.$$

Let MV be the class of algebras **A** of K as above, associated with an MV-algebra

$$(A, \oplus, \neg, 0)$$

i.e. binary meet $\wedge$, binary join $\vee$, dual residuation $-$, multiplication $\otimes$, residuation $\rightarrow$, negation $\neg$ and $1 \in A$ are defined as in Example 2.3 (E2).

2.5 Definition

A *biresiduated algebra* is an algebra **A** of K as in 2.4 such that the following equations hold for all $x, y, z \in A$:

(1) $x \wedge y = y \wedge x;$
(1°) $x \vee y = y \vee x;$

(2) $x \wedge (y \wedge z) = (x \wedge y) \wedge z;$
(2°) $x \vee (y \vee z) = (x \vee y) \vee z;$

(3) $x \wedge (x \vee y) = x;$
(3°) $x \vee (x \wedge y) = x;$

(4) $x \wedge 1 = x;$
(4°) $x \vee 0 = x;$

(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z);$

(6) $x \oplus y = y \oplus x;$
(6°) $x \otimes y = y \otimes x;$

(7) $x \oplus (y \oplus z) = (x \oplus y) \oplus z;$
(7°) $x \otimes (y \otimes z) = (x \otimes y) \otimes z;$

(8) $0 \oplus 0 = 0;$
(8°) $1 \otimes 1 = 1;$

(9) $x \wedge (x \oplus y) = x;$
(9°) $x \vee (x \otimes y) = x;$

(10) $(x \oplus y) \oplus 0 = x \oplus y;$
(10°) $(x \otimes y) \otimes 1 = x \otimes y;$

(11) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z);$
(11°) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z);$

(12) $x \oplus (y \vee z) = [(x \oplus y) \vee (x \oplus z)] \oplus 0;$
(12°) $x \otimes (y \wedge z) = [(x \otimes y) \wedge (x \otimes z)] \otimes 1;$

(13) $x - 0 = x \oplus 0;$
(13°) $1 \rightarrow x = x \otimes 1;$

(14) $(x - y) \oplus 0 = x - y;$
(14°) $(x \rightarrow y) \otimes 1 = x \rightarrow y;$

(15) $x \oplus (y - x) = (x \vee y) \oplus 0;$
(15°) $x \otimes (x \rightarrow y) = (x \wedge y) \otimes 1;$

(16) $x - (y \oplus z) = (x - y) - z;$
(16°) $(x \otimes y) \rightarrow z = x \rightarrow (y \rightarrow z);$

(17) $x - (x \vee y) = 0;$
(17°) $(x \wedge y) \rightarrow x = 1;$

(18) $\neg x = 1 - x;$
(18°) $\neg x = x \rightarrow 0;$

(19) $(x \oplus y) \otimes 1 = (x \otimes 1) \oplus (y \otimes 1);$
(19°) $(x \otimes y) \oplus 0 = (x \oplus 0) \otimes (y \oplus 0);$

(20) $(x - y) \otimes 1 = (x \rightarrow y) \rightarrow 0;$
(20°) $(x \rightarrow y) \oplus 0 = 1 - (x - y);$

(21) $(x \oplus 0) \wedge \neg x = x \wedge \neg x;$
(21°) $(x \otimes 1) \vee \neg x = x \vee \neg x.$

The following results present new valid relations in biresiduated algebras to be used in establishing the link with the notion of residuated algebra [12, 16].

2.6 The Duality Principle

We associate with every biresiduated algebra **A** an algebra

$\mathbf{A^o} = (A^o, \wedge^o, \vee^o, \oplus^o, -^o, \otimes^o, \rightarrow^o, \neg^o, 0^o, 1^o)$

such that $A^o = A$ and for all $x, y \in A$, the following relations hold:

$x \wedge^o y = x \vee y$;	$x \vee^o y = x \wedge y$;
$x \oplus^o y = x \otimes y$;	$x \otimes^o y = x \oplus y$;
$x -^o y = y \rightarrow x$;	$x \rightarrow^o y = y - x$;
$0^o = 1$;	$1^o = 0$;
$\neg^o x = \neg x$.	

From the above definitions, it follows that $\mathbf{A^o}$ is a biresiduated algebra called dual to **A**.

With any statement φ about all biresiduated algebras one can associate a second dual statement φ^o obtained from φ if replacing $\wedge, \vee, \oplus, -, \otimes, \rightarrow, \neg$, 0 and 1 by $\wedge^o, \vee^o, \oplus^o, -^o, \otimes^o, \rightarrow^o, \neg^o, 0^o$ and 1^o respectively. From the precedent property, it follows that the following condition is satisfied:

φ is valid $\Rightarrow \varphi^o$ is valid. □

2.7 Lemma

The following conditions hold for every biresiduated algebra:

(1) $x \leq y$ implies $x \oplus z \leq y \oplus z$
(1^o) $x \leq y$ implies $x \otimes z \leq y \otimes z$

(2) $x - y = 0$ iff $x \oplus 0 \leq y \oplus 0$
(2^o) $x \rightarrow y = 1$ iff $x \otimes 1 \leq y \otimes 1$

(3) $x - y \leq z \oplus 0$ iff $x \leq y \oplus z$
(3^o) $z \otimes 1 \leq x \rightarrow y$ iff $z \otimes x \leq y$

(4) $(x \oplus 0) \wedge (x \vee \neg x) = x$
(4^o) $(x \otimes 1) \vee (x \wedge \neg x) = x$

(5) $(x \oplus 0) \wedge [(x \otimes 1) \vee \neg x] = x$
(5^o) $(x \otimes 1) \vee [(x \oplus 0) \wedge \neg x] = x$

(6) $\neg(x \oplus 0) = \neg x$
(6^o) $\neg(x \otimes 1) = \neg x$

(7) $x \oplus 0 = y \oplus 0$ and $x \otimes 1 = y \otimes 1$ implies $x = y$

(8) $(x \rightarrow y) - z = \neg[z \oplus (x - y)]$
(8^o) $x \rightarrow (y - z) = \neg[x \otimes (y \rightarrow z)]$

(9) $(x \oplus 0) \otimes 1 = \neg\neg x$
(9^o) $(x \otimes 1) \oplus 0 = \neg\neg x$

(10) $(x \oplus 0) \vee \neg\neg x = x \oplus 0$
(10^o) $(x \otimes 1) \wedge \neg\neg x = x \otimes 1$

(11) $x \oplus 0 = x$ iff $x \otimes 1 = \neg\neg x$
(11^o) $x \otimes 1 = x$ iff $x \oplus 0 = \neg\neg x$
(12) $(x \oplus y) - x = y - x$ iff $x \oplus y = (x \vee y) \oplus 0$
(12^o) $x \rightarrow (x \otimes y) = x \rightarrow y$ iff $x \otimes y = (x \wedge y) \otimes 1$

Proof

(1) $x \leq y \Rightarrow x = x \wedge y$
$\Rightarrow x \oplus z = (x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z)$
$\Rightarrow x \oplus z \leq y \oplus z$.

(2) We have $x - y = 0$
$\Rightarrow y \oplus (x - y) = y \oplus 0$ [2.5(15); 2.7(1)]
$\Rightarrow x \oplus 0 \leq (y \vee x) \oplus 0 = y \oplus 0$.

We also have
$x \oplus 0 \leq y \oplus 0$
$\Rightarrow y \oplus 0 = (x \vee y) \oplus 0$ [2.5(12-16)]

and

$x - y = (x - y) \oplus 0$
$= x - (y \oplus 0)$
$= x - [(x \vee y) \oplus 0]$
$= x - (x \vee y) = 0$.

Thus 2.7 (2) holds.

(3) Relation 2.7(3) follows from:

$x - y \leq z \oplus 0$
$\Leftrightarrow 0 = (x - y) - z$
$= x - (y \oplus z)$ [2.5(14, 16); 2.7(2)]
$\Leftrightarrow x \oplus 0 \leq y \oplus z$ [2.5(9, 10)]
$\Leftrightarrow x \leq y \oplus z$.

(4) Relation 2.7(4) follows from Definition 2.5(5, 11, 21):

$(x \oplus 0) \wedge (x \vee \neg x) =$
$= [(x \oplus 0) \wedge x] \vee [(x \oplus 0) \wedge \neg x]$
$= x \vee (x \wedge \neg x)$
$= x$.

(5) Relation 2.7(5) follows from 2.7(4) and 2.5(21^o).

(6) Relation 2.7(6) follows from Definition 2.5(14, 16, 18):

$\neg(x \oplus 0) = 1 - (x \oplus 0)$
$= (1 - x) \oplus 0$
$= 1 - x = \neg x$.

(7) If $x \oplus 0 = y \oplus 0$ and $x \otimes 1 = y \otimes 1$ then from 2.7(5, 6) it follows that $x = y$, which proves that 2.7 (7) holds.

(8) Relation 2.7(8) follows from Definition 2.5(14, 16, 18, 20°):

$$\begin{aligned}(x \to y) - z &= [(x \to y) - z] \oplus 0\\ &= [(x \to y) - z] - 0\\ &= (x \to y) - (z \oplus 0)\\ &= [(x \to y) \oplus 0] - z\\ &= [1 - (x - y)] - z\\ &= 1 - [(x - y) \oplus z]\\ &= \neg[z \oplus (x - y)].\end{aligned}$$

(9) Relation 2.7(9) follows from Definition 2.5(13, 18°, 20):

$$\begin{aligned}(x \oplus 0) \otimes 1 &= (x - 0) \otimes 1\\ &= (x \to 0) \to 0\\ &= \neg\neg x.\end{aligned}$$

(10) From 2.7(9) and 2.5(9°) it follows that

$(x \oplus 0) \vee \neg\neg x = (x \oplus 0) \vee [(x \oplus 0) \otimes 1] = x \oplus 0$,

thus 2.7(10) holds.

(11) From 2.7(7, 9) and 2.5(10) it follows that

$$\begin{aligned}x \oplus 0 = x &\Rightarrow x \otimes 1 = (x \oplus 0) \otimes 1 = \neg\neg x;\\ x \otimes 1 = \neg\neg x &\Rightarrow x \otimes 1 = (x \oplus 0) \otimes 1 \text{ and}\\ &\quad x \oplus 0 = (x \oplus 0) \oplus 0\\ &\Rightarrow x \oplus 0 = x.\end{aligned}$$

Therefore 2.7 (11) has been verified.

(12) From 2.5(9, 15) here results

$$\begin{aligned}(x \oplus y) - x = y - x &\Rightarrow [(x \oplus y) - x] \oplus x = (y - x) \oplus x\\ &\Rightarrow [x \vee (x \oplus y)] \oplus 0 = (x \vee y) \oplus 0\\ &\Rightarrow x \oplus y = (x \vee y) \oplus 0\end{aligned}$$

and from 2.5 (15) and 2.7 (3) here results

$$\begin{aligned}x \oplus y = (x \vee y) \oplus 0 &\\ &\Rightarrow x \oplus y = x \oplus (y - x) \text{ and}\\ &\quad y \le x \oplus y = x \oplus [(x \oplus y) - x]\\ &\Rightarrow (x \oplus y) - x \le y - x \text{ and}\\ &\quad y - x \le (x \oplus y) - x\\ &\Rightarrow (x \oplus y) - x = y - x.\end{aligned}$$

This proves that 2.7 (12) holds.

Relations 2.7(1°-12°) follow from 2.7 (1-12) and the duality principle 2.6. □

2.8 The $\underline{BR}$, $\underline{R}$ and $\underline{R}^{o}$ Classes

A biresiduated algebra **A** is called:

- *residuated algebra*, if $x \otimes 1 = x$, for every $x \in A$;
- *dual residuated algebra*, if $x \oplus 0 = x$, for every $x \in A$.

Let $\underline{BR}$ be the class of biresiduated algebras, $\underline{R}$ the class of residuated algebras and $\underline{R}^{o}$ the class of dual residuated algebras. It follows that

$$\underline{R} \cup \underline{R}^{o} \subseteq \underline{BR}.$$

Remark:

The following conditions hold:

- $\underline{R} \cap \underline{R}^{o} = \underline{MV}$;
- $\mathbf{A} \in \underline{D}$ iff $\mathbf{A} \in \underline{BR}$ and for all $x, y \in A$,

$x \oplus y = (x \vee y) \oplus 0$;

$x \otimes y = (x \wedge y) \otimes 1$. □

2.9 Definition

Let $\mathbf{A} \in \underline{BR}$. For all $x, y \in A$, define

$x \vee_c y = (x \vee y) \oplus 0$; $x \wedge_i y = (x \wedge y) \otimes 1$.

(i) *Algebra of open elements of* **A** is the system

$i(\mathbf{A}) = (i(A), \wedge_i, \vee, \oplus, -, \otimes, \to, \neg, 0, 1)$,

where $i(A) = A \otimes 1 = \{x \otimes 1 / x \in A\}$.

(ii) *Algebra of closed elements of* **A** is the system

$c(\mathbf{A}) = (c(A), \wedge, \vee_c, \oplus, -, \otimes, \to, \neg, 0, 1)$,

where $c(A) = A \oplus 0 = \{x \oplus 0 / x \in A\}$.

(iii) *Algebra of clopen elements of* **A** is the system

$\neg\mathbf{A} = (\neg A, \wedge_i, \vee_c, \oplus, -, \otimes, \to, \neg, 0, 1)$,

where $\neg A = c(A) \cap i(A) = \{\neg x / x \in A\}$.

Remark:

From Definition 2.5, the duality principle 2.6 and Lemma 2.7, it follows that:

$$i(\mathbf{A}) \in \underline{R};$$
$$c(\mathbf{A}) \in \underline{R}^{o};$$
$$\neg\mathbf{A} \in \underline{MV} = \underline{R} \cap \underline{R}^{o}.$$ □

This Section ends with the following result which shows that the class $\underline{BR}$ is a minimal extension of the class $\underline{R} \cup \underline{R}^{o}$, i.e. the theory of biresiduated algebras is an infimum between the theory of residuated algebras and the theory of dual residuated algebras.

2.10 Theorem

The class $\underline{BR}$ of biresiduated algebras is the variety of algebras of $\underline{K}$ which $\underline{R} \cup \underline{R}^{o}$ generate, i.e.

$\underline{BR} = \cap(\underline{V} \subseteq \underline{K} / \underline{V}$ is a variety and $\underline{R} \cup \underline{R}^{o} \subseteq \underline{V})$.

Proof

Let $<\underline{R} \cup \underline{R}^{o}>$ be the variety of algebras of $\underline{K}$ generated by $\underline{R} \cup \underline{R}^{o}$. As $\underline{BR}$ is a variety and $\underline{R} \cup \underline{R}^{o} \subseteq \underline{BR}$, it follows that

$$(*)\ <\underline{R} \cup \underline{R}^{o}> \subseteq \underline{BR}.$$

Let $\mathbf{A} \in \underline{BR}$ and $s : A \to i(A) \times c(A)$ be the mapping defined by

$$s(x) = (x \otimes 1, x \oplus 0),$$

for all x, y $\in$ A. Then s is a subdirect embedding of **A** into the direct product

$$i(\mathbf{A}) \times c(\mathbf{A})$$

of the couple of algebras

$$i(\mathbf{A}) \in \underline{R}$$

and

$$c(\mathbf{A}) \in \underline{R}^{o},$$

i.e. s is an injective homomorphism from **A** to $i(\mathbf{A}) \times c(\mathbf{A})$ such that $\pi_1(s(A)) = i(A)$ and $\pi_2(s(A)) = i(A)$, where

$$\pi_1 : i(A) \times c(A) \to i(A)$$

and

$$\pi_2 : i(A) \times c(A) \to c(A)$$

are the canonical projections of the direct product $i(A) \times c(A)$.

If $\underline{V} \subseteq \underline{K}$ is a variety such that $\underline{R} \cup \underline{R}^{o} \subseteq \underline{V}$, then $i(\mathbf{A}) \in \underline{V}$ and $c(\mathbf{A}) \in \underline{V}$, but **A** is isomorphic to a subalgebra of $i(\mathbf{A}) \times c(\mathbf{A}) \in \underline{V}$, thus

$$\mathbf{A} \in \underline{V}.$$

This implies

$$(^{**})\ \underline{BR} \subseteq \langle \underline{R} \cup \underline{R}^{o} \rangle.$$

Theorem 2.10 follows from (*) and (**). □

3. Distance and Equivalence

Suppose that **A** is a biresiduated algebra.

3.1 Definition

(i) The *distance function* d : A × A → A ⊕ 0 is defined by

$$d(x, y) = (x - y) \oplus (y - x).$$

(ii) The *equivalence function* e : A × A → A ⊗ 1 is defined by

$$e(x, y) = (x \to y) \otimes (y \to x).$$

3.2 Examples

(1) If $\mathbf{A} \in \underline{D}$ is associated with a *D-algebra*

$$(A, \wedge, \vee, \to, -, 0, 1)$$

then

$$d(x, y) = [(x - y) \vee (y - x)] \oplus 0;$$
$$e(x, y) = [(x \to y) \wedge (y \to x)] \otimes 1.$$

In particular, if $\mathbf{A} \in \underline{D}$ is associated with a *Boolean algebra* (A, ∧, ∨, ¬, 0, 1) then

$$d(x, y) = (x \wedge \neg y) \vee (y \wedge \neg x);$$
$$e(x, y) = (\neg x \vee y) \wedge (\neg y \vee x).$$

(2) If **A** is a *Lukasiewicz structure* on [0, 1] as defined in Example 2.3(E5) then

$$d(x, y) = |x - y|,$$

i.e. the distance function d is the *current distance* on $[0, 1] \subseteq \mathbb{R}$.

(3) If **A** is a *Heyting-Gaines structure* on [0, 1] as defined in Example 2.3 (E6) then

$$e(0, 0) = 1;$$

$$e(x, y) = \frac{\min(x, y)}{\max(x, y)} = 1 - \frac{|x - y|}{\max(x, y)},$$

if $x \neq 0$ or $y \neq 0$,

i.e. the equivalence function e is a *complement of the relative error of y with respect to x.* □

Next the basic properties of distance and equivalence functions will be presented.

3.3 Proposition

The following conditions are satisfied for all x, y, z, u, v $\in$ A:

(i) $d(x, y) = 0$ iff $x \oplus 0 = y \oplus 0$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq d(x, y) \oplus d(y, z)$;

(iv) $d(x \oplus 0, y \oplus 0) = d(x, y) \oplus 0 = d(x, y)$;

(v) $d(x \otimes 1, y \otimes 1) = d(x, y) \otimes 1 = d(\neg x, \neg y)$;

(vi) $d(x \oplus u, y \oplus v) \leq d(x, y) \oplus d(u, v)$;

(vii) $x \leq d(x, y) \oplus y$.

Proof

(i) Property 3.3 (i) follows from Definition 3.1 (i) using Lemma 2.7 (2) and the following relation:

$$(x - y) \vee (y - x) \leq (x - y) \oplus (y - x) = d(x, y).$$

(ii) Relation 3.3(ii) follows from the commutativity of addition ⊕.

(iii) To prove 3.3 (iii), notice that

(1) $x - z \leq (x - y) \oplus (y - z)$,

because from Lemma 2.7 (3) it follows that (1) is equivalent to the following relation:

(1') $x \leq z \oplus [(x - y) \oplus (y - z)]$

and using Definition 2.5 (6, 7, 10, 15) and Lemma 2.7(1), relation (1') derives from the following relations:

$$\begin{aligned} x \leq x \oplus 0 &\leq (x \vee y) \oplus 0 = (x - y) \oplus y \\ &\leq (x - y) \oplus [(y \vee z) \oplus 0] \\ &= (x - y) \oplus [(y - z) \oplus z] \\ &= z \oplus [(x - y) \oplus (y - z)]. \end{aligned}$$

We also have

(2) $z - x \le (z - y) \oplus (y - x)$.

Then 3.3 (iii) follows from (1), (2) and Lemma 2.7 (1).

(iv) Property 3.3 (iv) follows from Definition 2.5 (8, 10, 13, 14, 16).

(v) From Definition 2.5, Lemma 2.7 and 3.3 (iv) it follows that

$$
\begin{aligned}
d(x \otimes 1, y \otimes 1) &= d((x \otimes 1) \oplus 0, (y \otimes 1) \oplus 0) \\
&= d(\neg\neg x, \neg\neg y) \\
&= (\neg\neg x - \neg\neg y) \oplus (\neg\neg y - \neg\neg x) \\
&= \neg(\neg x \oplus \neg\neg y) \oplus \neg(\neg y \oplus \neg\neg x) \\
&= \neg(x \oplus \neg y) \oplus \neg(y \oplus \neg x) \\
&= (\neg x - \neg y) \oplus (\neg y - \neg x) \\
&= d(\neg x, \neg y); \\
d(x, y) \otimes 1 &= [(x - y) \oplus (y - x)] \otimes 1 \\
&= [(x - y) \otimes 1] \oplus [(y - x) \otimes 1] \\
&= \neg\neg(x - y) \oplus \neg\neg\, (y - x) \\
&= (\neg\neg x - \neg\neg y) \oplus (\neg\neg y - \neg\neg x) \\
&= d(\neg\neg x, \neg\neg y) \\
&= d(x \otimes 1, y \otimes 1),
\end{aligned}
$$

which implies 3.3(v).

(vi) From Definition 2.5 (15, 16) and Lemma 2.7(1, 2) it follows that

$$
\begin{aligned}
&[(x \oplus u) - (y \oplus v)] - [(x - y) \oplus (u - v)] \\
&\quad = (x \oplus u) - (y \oplus v) \oplus [(x - y)] \oplus (u - v)] \\
&\quad = (x \oplus u) - [y \oplus (x - y)] \oplus [v \oplus (u - v)] \\
&\quad = (x \oplus u) - (x \vee y) \oplus (u \vee v) = 0
\end{aligned}
$$

which implies

(3) $(x \oplus u) - (y \oplus v) \le (x - y) \oplus (u - v)$.

Here also results:

(4) $(y \oplus v) - (x \oplus u) \le (y - x) \oplus (v - u)$.

Then Relation 3.3(vi) follows from (3), (4) and Lemma 2.7(1).

(vii) Property 3.3 (vii) follows from Lemma 2.7(3) using the following relation:

$$
\begin{aligned}
x - y &\le (x - y) \oplus (y - x) \\
&= d(x, y) \\
&= d(x, y) \oplus 0.
\end{aligned}
$$

□

3.3° Proposition

The following conditions are satisfied for all $x, y, z, u, v \in A$:

(i) $e(x, y) = 1$ iff $x \otimes 1 = y \otimes 1$;

(ii) $e(x, y) = e(y, x)$;

(iii) $e(x, y) \otimes e(y, z) \le e(x, z)$;

(iv) $e(x \otimes 1, y \otimes 1) = e(x, y) \otimes 1 = e(x, y)$;

(v) $e(x \oplus 0, y \oplus 0) = e(x, y) \oplus 0 = e(\neg x, \neg y)$;

(vi) $e(x, y) \otimes e(u, v) \le e(x \otimes u, y \otimes v)$;

(vii) $x \otimes e(x, y) \le y$.

Proof

Proposition 3.3° follows from Proposition 3.3 and the duality principle 2.6. □

Remark: Define for every $x, y \in A$,

$x \equiv_c y$ iff $d(x, y) = 0$;

$x \equiv_i y$ iff $e(x, y) = 1$.

From Propositions 3.3 (i) and 3.3° (i) it follows that

- $x \equiv_c y$ iff $x \oplus 0 = y \oplus 0$;
- $x \equiv_i y$ iff $x \otimes 1 = y \otimes 1$.
- Both relation $\equiv_c$ and relation $\equiv_i$ are congruence relations on **A**.
- The quotient algebra $\mathbf{A}/\equiv_c$ is a residuated algebra isomorphic to the algebra $c(\mathbf{A})$ of closed elements of **A**.
- The quotient algebra $\mathbf{A}/\equiv_i$ is a dual residuated algebra isomorphic to the algebra $i(\mathbf{A})$ of open elements of **A**.
- $x = y$ iff $d(x, y) = 0$ and $e(x, y) = 1$ iff $x \equiv_c y$ and $x \equiv_i y$. □

The definition of the notions of *strong ideal* and *strong filter* is given, followed by the description of congruence relations on **A** associated with strong ideals and strong filters using distance and equivalence functions.

3.4 Definition

A *strong ideal* of **A** is a subset I of A such that the following conditions hold:

(I1) $I \ne \varnothing$;

(I2) $x \wedge y \in I$ for each $x \in I$ and $y \in A$;

(I3) If $x \in I$ and $y \in I$ then $x \oplus y \in I$.

3.4° Definition

A *strong filter* of **A** is a subset F of A such that the following conditions hold:

(F1) $F \ne \varnothing$;

(F2) $x \vee y \in F$ for each $x \in F$ and $y \in A$;

(F3) If $x \in F$ and $y \in F$ then $x \otimes y \in F$.

3.5 Lemma

Let I be a subset of A. The following conditions are equivalent:

(i) I is a *strong ideal*;

(ii) I satisfies the following conditions:

(I1') $0 \in I$;

(I2') $x \in I$ and $y - x \in I \Rightarrow y \in I$.

Proof

(i) $\Rightarrow$ (ii). Suppose that 3.5(i) holds, i.e. I verifies 3.4(I1)-(I3). From 3.4(I1, I2) it follows that 3.5(I1'), because $0 = x \wedge 0$, for each $x \in I$. If $x \in I$ and $y - x \in I$ then using 3.4(I3) here results $(x \vee y) \oplus 0 = x \oplus (y - x) \in I$, but

$$y \le (x \vee y) \oplus 0,$$

thus using 3.4(I2) one derives $y \in I$. This proves that 3.5(I2') also holds. Therefore 3.5(ii) is verified.

(ii) $\Rightarrow$ (i). Suppose that I verifies 3.5(ii). Relation 3.5(I1') implies Relation 3.4(I1). Also, I satisfies 3.4(I2), because using 3.5(I2') from $(x \wedge y) - x = 0 \in I$, $x \in I$ and $y \in A$ it follows that $x \wedge y \in I$. Here is the proof that I verifies 3.4(I3). Suppose that $x \in I$ and $y \in I$. Using 3.5(I2'), from Relation

$$[(x \oplus y) - y] - x = (x \oplus y) - (x \oplus y) = 0 \in I$$

it follows that $x \oplus y \in I$.
This completes the proof of lemma 3.5. □

3.5° Lemma

Let F be a subset of A. The following conditions are equivalent:

(i) F is a *strong filter*;

(ii) F satisfies the following conditions:

(F1') $1 \in F$;

(F2') $x \in F$ and $x \to y \in F \Rightarrow y \in F$.

Proof

Lemma 3.5° follows from Lemma 3.5 and the duality principle 2.6. □

3.6 Definition

An equivalence relation on A is called a *congruence relation* if from x R u and y R v it follows that

$(x \wedge y)\, R\, (u \wedge v)$; $(x \vee y)\, R\, (u \vee v)$;
$(x \oplus y)\, R\, (u \oplus v)$; $(x \otimes y)\, R\, (u \otimes v)$;
$(x - y)\, R\, (u - v)$; $(x \to y)\, R\, (u \to v)$,

for all $x, y, u, v \in A$.

3.7 Proposition

Let I be a *strong ideal* of **A**. Define a binary relation $\approx_I$ on A as follows:

$$x \approx_I y \text{ iff } d(x, y) \in I,$$

where $d : A \times A \to A \oplus 0$ is the distance function. Then the following conditions hold:

(i) Relation $\approx_I$ is a *congruence* of **A**.

(ii) *Quotient algebra* $\mathbf{A}/\approx_I$ is a *dual residuated algebra*.

Proof

(i) Suppose that $x \approx_I u$ and $y \approx_I v$, i.e.

(1) $d(x, u) \in I$;

(2) $d(y, v) \in I$.

Using Definition 3.4(I3), from (1) and (2) it follows that

(3) $d(x, u) \oplus d(y, v) \in I$.

From Definition 2.5, Lemma 2.7 and Proposition 3.3 it follows that the following conditions are satisfied:

(4) $[(x \wedge y) - (u \wedge v)] - [d(x, u) \oplus d(y, v)] = 0 \in I$;
(5) $[(x \vee y) - (u \vee v)] - [d(x, u) \oplus d(y, v)] = 0 \in I$;
(6) $d(x \oplus y, u \oplus v) - [d(x, u) \oplus d(y, v)] = 0 \in I$;
(7) $d(x \otimes y, u \otimes v) - [d(x, u) \oplus d(y, v)] = 0 \in I$;
(8) $[(x - y) - (u - v)] - [d(x, u) \oplus d(y, v)] = 0 \in I$;
(9) $[(x \to y) - (u \to v)] - [d(x, u) \oplus d(y, v)] = 0 \in I$

For example, (4) results from the following relations:

$[(x \wedge y) - (u \wedge v)] - [d(x, u) \oplus d(y, v)] =$
$= (x \wedge y) - (u \wedge v) \oplus [d(x, u) \oplus d(y, v)]$
$= (x \wedge y) - (p \wedge q)$;
$x \le u \oplus d(x, u) \le [u \oplus d(x, u)] \oplus d(y, v) = p$;
$y \le v \oplus d(y, v) \le [v \oplus d(y, v)] \oplus d(x, u) = q$;
$(x \wedge y) \oplus 0 \le (p \wedge q) \oplus 0$.

Using Lemma 3.5 (I2') and Definition 3.4 (I3), from relations (4)-(9) together with (3) it follows that

$(x \wedge y) \approx_I u \wedge v$; $(x \vee y) \approx_I (u \vee v)$;
$(x \oplus y) \approx_I (u \oplus v)$; $(x \otimes y) \approx_I (u \otimes v)$;
$(x - y) \approx_I (u - v)$; $(x \to y) \approx_I (u \to v)$.

Thus 3.7 (i) holds.

(ii) From Proposition 3.3(iv) it follows that

$$x \approx_I y \text{ iff } (x \oplus 0) \approx_I (y \oplus 0),$$

which shows that in the quotient algebra $\mathbf{A}/\approx_I$ the following equation holds:

$$[x] \oplus [0] = [x], \text{ for every } x \in A,$$

where [x] is the equivalence class of x with respect to $\approx_I$ i.e. 3.7(ii) holds. □

3.7° Proposition

Let F be a *strong filter* of **A**. Define a binary relation $\approx_F$ on A as follows:

$$x \approx_F y \text{ iff } e(x, y) \in F,$$

where $e : A \times A \to A \otimes 1$ is the equivalence function. Then the following conditions hold:

(i) Relation $\approx_F$ is a *congruence* of **A**.

(ii) *Quotient algebra* **A**/$\approx_F$ is a *residuated algebra*.

Proof

Proposition 3.7° follows from Proposition 3.7 and the duality principle 2.6. □

The next definition presents the notion of *deductive system* in order to characterize kernels of homomorphism. The manner how to construct *homomorphic images* is shown.

3.8 Definition

A *deductive system* is a couple (F, I), where F is a *strong filter* and I is a *strong ideal* such that the following conditions hold:

(i) $x \in F$ implies $\neg x \in I$.

(ii) $x \in I$ implies $\neg x \in F$.

3.9 Proposition

If R is a congruence on **A** then the couple

$$(F_R, I_R)$$

is a deductive system such that

$$R = \approx_{F_R} \cap \approx_{I_R},$$

where

$$F_R = \{x \in A \,/\, x\, R\, 1\};$$
$$I_R = \{x \in A \,/\, x\, R\, 0\}.$$

Therefore, the correspondence

$$(F, I) \mapsto \approx_F \cap \approx_I$$

is a bijection from the set of deductive systems of **A** and the set of congruences on **A**.

Proof

Proposition 3.9 is a consequence of the above definitions and Proposition 3.7. □

Remark:

If **B** is a biresiduated algebra and h : **A** → **B** is a homomorphism then the kernel of h is the congruence relation Ker(h) of **A** defined by

x Ker(h) y iff h(x) = h(y).

Conversely, each congruence relation of **A** is of the form Ker(h). We say that **B** is a homomorphic image of **A** if there is a surjective homomorphism h from **A** to **B**. From Proposition 3.9 it follows that homomorphic images of **A** are biresiduated algebras isomorphic to quotient algebras **A**/$\approx_F \cap \approx_I$, where (F, I) is a deductive system of **A**. □

The following definition presents the notions of *formula, valuation, valid formula* and *invalid formula* over each class C̲ of biresiduated algebras. Then we formulate *the word problem for free algebras over* C̲.

3.10 Definition

Let C̲ be a class of biresiduated algebras.

(i) Let V = {v, /, ∧, ∨, ⊕, −, ⊗, →, ¬} be a finite alphabet. By induction a denumerable set of variables $v_0, v_1, \ldots, v_n, \ldots$, is defined as follows:

$v_0 = v$; $v_{k+1} = v_k\,/$, for every natural number k.

Let V^* be the free monoid generated by V. The set Fml̲ of *formulas* is a subset of V^* inductively defined by the following clauses:

C1) each variable v_n is a formula;

C2) if p is a formula, then ¬p is a formula;

C3) if p and q are formulas, then ∧pq, ∨pq, ⊕pq, −pq, ⊗pq and →pq are formulas.

Fml̲ is associated with an algebraic structure

Fml̲ = (Fml̲, ∧, ∨, ⊕, −, ⊗, →, ¬)

of type (2, 2, 2, 2, 2, 2, 1) called *algebra of formulas* defined by

$p \wedge q = \wedge pq$; $p \vee q = \vee pq$;
$p \oplus q = \oplus pq$; $p \otimes q = \otimes pq$;
$p \to q = \to pq$; $p - q = -pq$;
$\neg p = \neg p$.

(ii) Let **A** be a biresiduated algebra. An **A**-*valuation* is a function v : Fml̲ → A such that v is a homomorphism from the algebra

Fml̲ = (Fml̲, ∧, ∨, ⊕, −, ⊗, →, ¬)

to the algebra

(A, ∧, ∨, ⊕, −, ⊗, →, ¬)

associated with **A**.

(iii) A formula p is called

- C̲-*valid*, if for every algebra **A** ∈ C̲ and for each **A**-*valuation* v, p is v-*valid*, i.e. $v(p) = 1$.
- C̲-*invalid*, if for every algebra **A** ∈ C̲ and for each **A**-*valuation* v, p is v-*invalid*, i.e. $v(p) = 0$.

The *word problem for free algebras over* C̲ consists in describing, by two effective procedures, both the set of formulas

T = {p ∈ Fml̲ / p is C̲-*valid*}

and the set of formulas

F = {p ∈ Fml̲ / p is C̲-*invalid*}.

A solution of the *word problem for free MV_algebras* ($\underline{C} = \underline{MV}$) is presented in [6] and a solution of the *word problem for free D_algebras* ($\underline{C} = \underline{D}$) is presented in [20].

Now the notion of *fuzzy set over a complete biresiduated algebra* **A** (called **A**-*set*) is introduced such that it includes some properties of distance and equivalence functions as presented in Propositions 3.3 and 3.3°.

3.11 Definition

(i) An *A-set* is a system

$$\mathbf{X[A]} = (X, d, e, \varphi),$$

where

(1) X is a set;

(2) **A** is a *complete biresiduated algebra*;

(3) $d : X \times X \to A \oplus 0$ is a *distance function* on X over **A**, i.e.

(D1) $d(x, x) = 0$;
(D2) $d(x, y) = d(y, x)$;
(D3) $d(x, z) \le d(x, y) \oplus d(y, z)$.

(4) $e : X \times X \to A \otimes 1$ is an *equivalence function* on X over **A**, i.e.

(E1) $e(x, x) = 1$;
(E2) $e(x, y) = e(y, x)$;
(E3) $e(x, y) \otimes e(y, z) \le e(x, z)$.

(5) $\varphi : X \to A$ is a *membership function* on X over **A**, i.e.

(M1) $\varphi(x) \otimes e(x, y) \le \varphi(y)$.
(M2) $\varphi(x) \le d(x, y) \oplus \varphi(y)$;

(6) $\oplus$-*complement of distance function is the closure of equivalence function*, i.e.

(C1) $\neg d(x, y) = e(x, y) \oplus 0$.

(7) $\otimes$-*complement of equivalence function is the interior of distance function*, i.e.

(C2) $\neg e(x, y) = d(x, y) \otimes 1$.

(ii) An **A**-set $\mathbf{X[A]} = (X, d, e, \varphi)$ will be called *separated* if satisfying the following *separation axiom*:

(SA) $d(x,y)=0$ and $e(x,y)=1 \Rightarrow x = y$.

Comment:
The notion of **A**-set includes the notion of *set over a complete Heyting algebra* as defined in Fourman and Scott [9]. Thus *sheaves represent a class of mathematical structures included in the class of A-sets, for A* $\in \underline{BR}$.

In order to develop a first-order logical system over the class of biresiduated algebras, one may consider the following *problems*:

- to produce a satisfactory solution of the *word problem for free biresiduated algebras*;
- to define corresponding *categories of A-sets and A-structures*.

3.12 Aggregating Mappings Associated With Multicriteria Decision Problems

This Section comments on the possibility of using generalized sets over biresiduated algebras defined as above, in solving multicriteria decision problems. The concept of aggregating mapping will be defined based on the notion of the Pareto optimal point.

A *multicriteria decision problem* is represented by a system (X, g, d), where :
- X is the set of alternatives;
- $g = (g_1, g_2, ..., g_n) : X \to A_1 \times A_2 \times ... \times A_n$ is a *membership function* from X into a direct product of biresiduated algebras $\mathbf{A_1}, \mathbf{A_2}, ..., \mathbf{A_n}$;
- $d = F(g_1, g_2, ..., g_n) : X \to A$ is a *membership function* from X into a biresiduated algebra **A** such that

$$(\forall x \in X)\ d(x) = F(g_1(x), g_2(x), ..., g_n(x))$$

and

$$F : A_1 \times A_2 \times ... A_n \to A$$

is an *aggregating mapping*, i.e. F satisfies the following condition of compatibility with respect to the Pareto solution of the multiattribute decision problem :

Aggregation axiom. If $x \in X$ and d(x) is a maximal element of the ordered set $d(X) \subseteq A$ then g(x) is a maximal element of the ordered set $g(X) \subseteq A_1 \times A_2 \times ... \times A_n$.

A multiattribute decision problem (X, g, d) as above is supposed to have n objectives expressed by n predicates $P_1(x), P_2(x), ..., P_n(x)$, namely, for all j = 1, 2, ..., n, $P_j(x)$ is defined by "$g_j(x)$ is a maximal element of the ordered subset $g_j(X) \subseteq A_j$". Suppose that

$$(\forall j \in \{1, 2, ..., n\})\ (\exists\ x \in X)\ P_j(x) \text{ is true.}$$

An alternative x is called an *ideal optimal decision* if x satisfies the conjunction $P_1(x) \wedge P_2(x) \wedge ... \wedge P_n(x)$. Often, the set of ideal optimal decisions is empty.

Therefore, it is necessary to introduce a notion of optimal decision including the notion of ideal optimal decision.

An alternative $x \in X$ is called a *Pareto optimal decision* if the vector $g(x)$ is a maximal element of the ordered subset $g(X) \subseteq A_1 \times A_2 \times \ldots \times A_n$. An *optimal decision* is any alternative x such that $d(x)$ is a maximal element of the ordered subset $d(X) \subseteq A$. Given that F satisfies the aggregation axiom, any optimal decision is a Pareto optimal decision. A fundamental *open problem* is *to work out a method for expressing aggregating mappings F in terms of basic biresiduated algebra operations.*

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