Modal and Multi-valued Logics

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Any sound judgement allows the transfer of the quality of truth from its premises to its conclusion. This is done by way of the "form" of the judgement and does not depend on the content of the sentences it contains. The role of logic is to establish the forms of sound reasoning.

In certain cases, the premises do not have the quality of truth in the full sense of the word (being only probable, possible, etc.). The conclusion will borrow from them these truth "deficiencies". In these situations logic - by adding the qualities of "probable", "possible", etc. to those of "true" and "false" - allows the distinction of sound and unsound forms of reasoning in two ways. The first is to give a (relatively) small number of such correct forms of reasoning from which one can then derive all the correct forms. This approach leads to the so-called *systems of modal logic* (or simply *modal logics*).The second way is to use "truth functions" and it leads to *multi-valued logics*.

The use of the plural "logics" might give rise to the (wrong) idea of a degree of arbitrariness in the choice of the correct form of reasoning. In this sense there is only one logic. There is multiplicity only in the number of interpretations that its terms can be given: possible, probable, even true and false, and from here follows the diversity of the systems of rules that operate on them.

The American logician C. S. Lewis enjoyed a certain amount of success in dealing with the relation of deducibility in bivalent logic.

The definition of the implication between two sentences as the relation that holds for all cases except when the hypothesis is true and the conclusion false gives birth to paradoxical consequences, for instance: from a false statement anything is deducible! This implication is also called "material" implication.

With the aid of the modality "possible", Lewis defines a – strict – implication between two assertions via the following condition: it is impossible that the hypothesis is true and the condition false.

Regarding the modalities thus introduced, first Lewis and then his followers proposed several systems of deductive rules representing as many systems of modal logic. Unfortunately, most of them are interesting especially from an algebraic point of view than from a logical perspective, and the reason for this is simply that we do not know how to interpret in intuitive terms all the (composite) modalities they operate with; it is to be noted that in some cases these are in infinite number! In fact, the only system that admits such a simple interpretation is the so-called S5 system. It contains, besides truth and falsehood, four other modalities: necessary and possible truth, as well as necessary and possible falsehood. From the point of view of the interpretation, an interesting result is the following: an assertion without modalities is necessary, in Lewis' logic, if and only if it is a theorem in the classical, bivalent logic. This shows that this logic meets the purpose for which it was created.

It must be said, however, that although the strict implication is exempted from the paradoxes of the material one, it falls under the incidence of other paradoxes that ruin its status of "deducibility relation". In fact, research into the properties of such a relation continues to our day.

Another line of research regarding the modalities was started by the Polish logician J. Lukasiewicz who used in his logistic analysis the method of truth tables. It is known that in classical, or bivalent logic, each logical operator is defined via a table. For instance, the conjunction "p and q" is a function of the truth of p and that of q, a function defined by the following Table:

p	q	p and q
T	T	T
T	F	F
F	T	F
F	F	F

Lukasiewicz extends the truth tables by introducing a third truth value: possible (P). Thus, the conjunction "p and q" has the truth value P if at least one of p and q have this truth value, and for the rest it takes the same values as in the bivalent case. Logical theorems in Lukasiewicz's three valued logic are those formulas that take the value T for any value T, F, P of their constituents (and called by Wittgenstein – in bivalent logic – tautologies).

From an algebraic point of view, the number of truth values is unlimited. One can build logics with four, five, etc. truth values. From the point of view of logical interpretation, such formalisms present serious deficiencies, even in the three valued case.

Like Lukasiewicz, who based his logical analysis on a critique of the law of the excluded middle, L. E. J. Brower questions – as a mathematician – the validity of this principle. As a consequence, he also rejects other logical laws such as the law of the double negation (a double negation is equivalent to an assertion) or the principle of reductio ad absurdum. In the new mathematics, called by Brouwer and his followers *intuitionistic*, several theorems in classical mathematics turn out to be false, and some of those that remain true require more difficult proofs. The Dutch logician A. Heyting was the one who developed the formalism for intuitionistic logic.

The Romanian mathematician G. Moisil managed to put together within the framework of a unitary formalism the common properties shared by the modal, multi-valued and intuitionistic systems.

The logic of probabilities developed by the German logician H. Reichenbach replaces the true-false pair by a continuous scale of values interpreted as probabilities.

Regarding the applications of the new logics we must distinguish between theoretical and pragmatic aspects. Theoretically, an attempt was made to overcome certain paradoxical situations (such as the ones that appear in set theory in relation to the set of all sets, or those in quantum mechanics in relation to certain empirical statements) by means of introducing extra truth values. However, there has not been much success in this direction. On the contrary, the practical use of multi -valued logics (for instance in modeling electrical circuits with contacts and relays) has shown good results. Does it follow that multi -valued logics are only tools having an operational character, while Logic remains, in essence, bivalent? We shall leave the reader to ponder on this problem, and to decide for himself, after further studies.

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