Theory and Algorithms for Linear Optimization An Interior Point Approach

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The subject of this book is linear optimization. The authors synthesize the latest aspects and results of the interior point methods for solving this class of problems, many of them quite new.They claim that the book is dedicated to linear optimization, but only the linear programming models are actually considered.

The linear programming area is old enough and the simplex method designed by Dantzig is still the most used algorithm, and it will possibly maintain its status in the future. Although efficient and elegant, the simplex method does not have a polynomial complexity. The question of polynomial complexity of the linear programming problems was solved in 1979 by Khachiyan. In 1984 Karmarkar published an algorithm with a polynomial complexity bound of $O(n^{3.5}L)$, lower than Khachiyan's, proving to be more efficient than the simplex method. A simplification of Karmarkar's algorithm is identical to the algorithm of Dikin published early 1967, which is now known as affine-scaling. One reason why Karmarkar's algorithm performs well in practice is that it avoids the boundary of the feasible set of the problem, keeping near the *central path*. This concept, which was introduced by Sonnevend in 1985, is a curve with some very important primal-dual properties. The properties of this concept and some other developments are the main subjects of this book. A comprehensive treatment of the Logarithmic Barrier Methods for Linear Programming and the Target-Following Approach to the Interior-Point Methods is presented. The algorithms based on these approaches use the Newton method in different ways, leading to a number of variants like: dual, primal, primal-dual full-step, adaptive-update, large-update, etc. All these algorithms are fully described and analysed with many new sharper results referring to the bounds of iterations, bounds of arithmetic operations, computational effort, etc. Finally, a new approach to sensitivity analysis is presented, and a good deal of crucial techniques for an efficient implementation of the interior point algorithms are discussed. Let us take a look inside it.

The book is organized in four Parts containing 20 Chapters. The *first Part:* ***Introduction. Theory and Complexity*** contains **Chapters 2, 3 and 4.** This Part makes an in-depth presentation of the main theoretical results for Linear Programming, as well as of a polynomial method for solving this class of problems. **Chapter 2** develops the theory of Linear Programming. This is viewed in a new manner, using the so-called *skew-symmetric model* of a Linear Programming problem, a concept introduced by Tucker in 1956. Given that the Interior Point methods, which the book centres on, are very close to the duality theory, this Chapter contains the main results on the duality theory for Linear Programming. This key concept, the *central path*, originated in the early 1968 from the nonlinear convex optimization papers written by Fiacco and McCormick, and was firstly interpreted by Nimrod Megiddo as an analytic curve in the interior of the admissibility domain of the problem that starts somewhere in the middle of the domain and ends somewhere in the middle of the optimal set of the problem. The duality theory results of linear programming are derived from the properties of the central path. The condition regarding the existence of a central path to a linear programming problem is established. This condition (called *interior-point condition*) implies the existence of the central path for the dual problem.The points on the dual central path are closely related to the points on the primal central path. All these results are firstly proved for the skew-symmetric model (*self-dual model*) of the problem which is a particular trivial LP problem for which the zero vector is both feasible and optimal. The results on the skew-symmetric model are extended to the LP problems in a canonical form in which the objective is to minimize a linear function subject to a set of inequality of greater-than-or-equal to type with nonnegative variables. **Chapter 3** is dedicated to presenting an algorithm for solving the skew-symmetric problems in polynomial time. This is based on the Dikin papers on his primal-affine-scaling method. The algorithm, based on a special affine-scaling direction, is presented in detail, as well as its convergence analysis, maximum number of iterations and arithmetic operations, the polynomial complexity results and a rounding procedure for determination of a sufficiently accurate solution are. Some numerical example illustrates the running of the algorithm. **Chapter 4** presents the general theory on the canonical problem solving. The approach considered here is based on a natural embedding of the primal and dual LP problems into a homogeneous skew-symmetric problem. The central path for the canonical problems is introduced. Some discussions about how an approximate solution for the canonical problem can be derived from an approximate solution of the embedding problem are related.

The algorithms for solving these types of linear programming problems are very simple, involving the determination of the search direction and the step-length along that direction. Their implementation is straight-forward. However, the theoretical iteration bound for these algorithms, also polynomial, is relatively poor in comparison with some other algorithms based on other search directions. Therefore, *Part II* of the book (**Chapters 5 through 8**), ***The Logarithmic Barrier Approach***, is dedicated to presenting some more efficient methods for solving LP problems based on the Logarithmic Barrier Methods. The algorithms corresponding to this method originate from Frisch's paper (published in 1955) and developed in some "classical papers" of Lootsma and Fiacco and McCormick (1968). After the presentation of the new polynomial-time algorithm for LP made by Karmarkar in 1984, the interest in logarithmic barrier approach greatly revived. This was the starting point for a wide class of polynomial-time algorithms for LP problems which has been implemented in a new generation of very efficient optimization packages like: CPLEX, HOPDM, OSL, OB1, ALPO, BPMPD, LIPSOL, Pcx, LOQO, etc. In **Chapter 5**, *Preliminaries*, the authors firstly review the classical duality theory of the LP problems in standard form (problems with nonnegative variables and equality constraints). Then, the logarithmic barrier function is introduced and some of its properties are considered. The main aspect here is *the interior-point condition* which stipulates that both the primal problem and the dual one have a positive solution. It is proved that this is equivalent to the fact that the primal logarthmic barrier function has a unique minimizer or that the KKT system has an unique solution. Based on these developments the central path is introduced and some equivalent formulations of the interior-point condition are articulated. **Chapter 6** presents *The Dual Logarithmic Barrier Method* for LP. A conceptual logarithmic barrier algorithm is presented. In this case the search direction is the Newton direction for minimizing the classical dual logarithmic barrier function with a barrier parameter. Some properties of the algorithm referring to the approximation of centers, definition and properties of the Newton step, local quadratic convergence and the duality gap are presented. Three variants of the algorithm are developed. The first variant is based on the idea of using a full Newton step and small updates of the barrier parameter. This gives the path-following methods which have the best possible iteration bound. The second variant considers adaptive updates of the barrier parameter illustrating good practical results. The third variant of the algorithm is based on the composite Newton algorithm idea using large updates of the barrier parameter and a bounded number of damped Newton steps between each pair of successive barrier updates. Some numerical examples (of small dimensions) illustrate the running of these algorithms and the fact that in practice large-update algorithms are much more efficient than the full Newton step method. **Chapter 7** is about the *Primal-Dual Logarithmic Barrier Method*. This is a natural extension of the ideas presented in Chapter 6. Thus, for the couple of primal-dual LP problems the same program as for the dual algorithms is presented. The Newton system associated with the primal and dual problems is considered, as well as the properties of the Newton step and the quadratic convergence results are proved. These theoretical results are used to develop the primal-dual logarithmic barrier algorithm with full Newton steps, the algorithm with adaptive updates of the barrier parameter, and a version of the algorithm with large updates of the barrier parameter. As a variant of the algorithm with adaptive updates of the barrier parameter the predictor-corrector algorithm is presented. In this case the Newton step is decomposed into two steps: an *affine-scaling step* and a *centering* one. Some small dimension numerical examples highlight the better behaviour of the primal-dual Newton methods compared with the dual or primal Newton method. The last Chapter of this part, **Chapter 8**, *Initialization*, is a presentation of a method for initializing the above described algorithms with a strictly feasible solution. The idea of this method is based on a transformation of the problem into an equivalent one for which a point on the central path is available. The disadvantage of this procedure lies in that it enlarges the size of the problem, but some techniques for an efficient implementation are explained in Chapter 20 where the implementation of the interior point methods is discussed.

Part III of the book ***The Target-Following Approach***, which consists of **Chapters 9 through 14**, makes a generalization of the methods considered in the previous Part. The idea is as follows: for a couple of dual linear programming problems, the KKT system can be associated. The right-hand-side of this system is a function of the barrier parameter. Assuming that the interior-point condition is satisfied, then the KKT system has a unique solution for every positive value of the barrier parameter. When the barrier parameter runs through the positive real line then the solutions of the KKT system run through the central path of the LP problems. All the methods presented in Part II approximately follow the central path to the optimal sets of the primal and dual LP problems. These are logarithmic barrier methods, or central-path-following methods because the points on the central path are minimizers of the logarithmic barrier functions for the considered primal and dual LP problems. The implementation of these methods is based on either the Newton method for solving the KKT system or the Newton method for minimizing the logarithmic barrier function of the primal and dual problems. Considering the first strategy we get the so-called *primal-dual method*. In the second case the *primal method* or the *dual method* is obtained depending on whether the logarithmic barrier function of the primal or of the dual is minimized by the Newton method or not. The generalization proposed in Part III of the book is based on the observation that if the barrier parameter in the RHS-term of the KKT system is replaced by any positive vector w, then the resulting system still has a unique solution. **Chapter 9**, *Preliminaries*, introduces the *target map* as the map associating any positive vector w with the pair of primal solution and the dual slack: (x(w),s(w)). The properties of this map and its inverse are thoroughly discussed. Based on this concept the *weighted-analytic centers* of the optimal sets of the primal and dual problems are defined. All these are further used to describe the main idea of the target-following approach and the corresponding algorithms. Firstly *the traceable target sequence* is defined as a sequence of positive vectors with the property that, given the primal dual pair for w, it is "easy" to compute the primal-dual pair for the next value of the w parameter. Based on this concept the Target-Following Algorithm is presented. The numerical method is again the Newton method, either for solving the KKT system which defines the primal-dual pair, or for minimizing a suitable barrier function. The same strategies as those considered in the second Part of the book are followed here. Thus, **Chapter 10** contains *The Pimal-Dual Newton Method* where the search directions are obtained by applying Newton's method to the weighted KKT system. This approach is called primal-dual because it uses search steps in both the x-space (primal space) and the s-space (dual space) at each iteration. The results of this Chapter are generalizations of those presented at Chapter 7. In this case the target is not the central path. Any positive vector w could be considered as a target. Two algorithms are presented: the primal-dual Newton step and the damped primal-dual Newton step. Some properties referring the feasibility of the primal-dual Newton step and the local quadratic convergence are proved. No numerical examples are provided. **Chapter 11**, *Applications*, gives some examples of traceable target sequences covering the most important primal-dual methods. Thus the central-path-following method, the weighted-path-following method, the centering method, the weighted-centering method, centering and optimizing method and the adaptive and large target-update methods, are presented. All these are attentively considered with a lot of very fine numerical estimations on the proximity function, number of iterations and computational effort. No numerical examples illustrate these theoretical results. In **Chapter 12**, *The Dual Newton Method* is considered. In this case assuming that we have a dual feasible solution, a dual method moves in the dual space until the optimal solution is reached. Firstly the weighted dual barrier function and the dual target-following algorithm are introduced. Then, the dual Newton step and the damped dual Newton step, as well as their properties on feasibility and quadratic convergence, are established. **Chapter 13** presents *The Primal Newton Method*. In this case moving is considered only in the primal space. The same methodology will apply. Starting with a positive primal feasible solution, a primal method moves in the primal space until an intermediate positive target vector is reached. The search direction is obtained by applying Newton's method in minimizing the weighted primal logarithmic barrier function. Taking full or damped Newton steps with respect to this function, we get the corresponding algorithms for the primal Newton approach. **Chapter 14**, *Applications to the Method of Centers*, the last one of this Part, considers a short discussion on the approaches to solving LP problems and the links among them: Karmarkar's algorithm, the barrier logarithmic approach, and Huard's methods of centers. The first two approaches have been amply analysed in the afore mentioned Chapters of the book. In this Chapter the authors concentrate on Renegar's method of centers. It is shown that this method can be described and analysed within the target-following framework.

Part IV of the book, ***Miscellaneous Topics*** (**Chapters 15 through 20**) considers a number of aspects of linear optimization very close to the material presented in the previous three Parts. In **Chapter 15** *Karmarkar's Projective Algorithm* is very deeply described and analysed. Special attention is given to the iteration bound and the explicit interpretation of the Karmarkar search direction. In **Chapter 16** some *More Properties of the Central Path* are presented. It is shown that the central path is differentiable with respect to the barrier parameter. Some bounds of these derivatives are established and the limits of the derivatives when the barrier parameter approaches to zero are given. Chapters 17 and 18 deal with the techniques for accelerating the interior point algorithms. Thus, **Chapter 17** considers the *Partial Updating*. In any interior point algorithm for solving the couple of LP problems each search direction is computed by solving a linear algebraic system involving a matrix of the form AD^2A, where the scaling matrix D is a positive diagonal matrix which in a primal-dual method $D^2 = XS^{-1}$, in a primal method $D = X$, and in a dual method $D = S^{-1}$. (X and S are diagonal matrices with primal variables x and dual slack variables on the main diagonal respectively). Although the AD^2A matrix varies from one iteration to another, it seems reasonable to expect that these variations are not too large. The idea of partial updating is to compute the search direction at the next iteration based on the calculations at the previous one. This idea can be applied to all classes of interior point methods. In this Chapter the so-called *rank-one updating* is considered in the context of the dual logarithmic barrier method with full Newton step. The barrier algorithm with rank-one updates is presented. It is shown that the complexity of this algorithm in terms of the number of iterations and of arithmetic operations is bounded by $O(\sqrt{n}L)$ and $O(n^3L)$, respectively. **Chapter 18** deals with the *Higher-Order Methods*. The motivation of these methods consists in the fact that the nonlinearities (of second order) of the KKT system have been so far neglected. So, the higher-order methods aim to approximate somehow the solution of the nonlinear weighted KKT system. Firstly the higher-order search directions are introduced. Then an estimation of the neglected quadratic term subject to the step-size is computed. These results are firstly applied to the primal-dual Dikin method and it is shown that for the higher-order variant the iteration bound can be reduced by the factor $\sqrt{n}$ without increasing complexity per iteration. Next, this idea applies to the primal-dual logarithmic barrier method thus obtaining its higher-order variant showing that in this case the higher-order search directions do not improve the iteration bound. **Chapter 19** considers the problems of *Parametric and Sensitivity Analysis of LP* from the perspective of interior point methods. This is a very important subject in the analysis of practical LP models and the authors show that the classical approach based on the use of optimal basic solution has some weaknesses. Firstly the parametric analysis is considered, then the sensitivity analysis. Some numerical examples are considered and a comparison with 5 packages (CPLEX, LINDO, PC-PROG, XMP, OSL) on a LP problem with 24 variables is made. The **last Chapter:** *Implementing Interior Point Methods*, gives some practical hints at and ideas about how the interior point methods presented in the previous Chapters could be implemented to work well in practice for solving large-scale LP problems. The essential elements of interior point algorithms are very well outlined. A prototype algorithm is presented and some very important practical problems concerning preprocessing of the problem before optimization, sparse linear algebra for solving the KKT system, starting point computation, determination of the step- size and stopping criteria, and optimal basis identification, are deeply analyzed. Finally, a list of 8 packages implementing interior point algorithms is given. The book has three Appendices with "Some Results from Analysis", "Pseudo-inverse of a Matrix", and "Some Technical Lemmas". A bibliography with 314 items, an author index and a subject index end up the book.

All in all the book is a very important contribution to the theory of Interior Point Methods for solving Linear Programming Models. Several polynomial interior point algorithms have been presented and discussed to prove that these provide the best theoretical complexity for this class of problems. This recommends the book as a reference one in the very active and extremely versatile area of optimization.

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