An Asymptotical Modification Of Two-Riccati Approach in Robust Stability Synthesis

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1 Introduction

Optimal robustness problems related to transfer function uncertainty have become prominent in the last decade. A basic approach to the problem can be found in [5],[2],[6], the most general treatment and current studies review are in [3]. The principal fact is that the robust controller design by the various uncertainties in the plant transfer function can be reduced to the synthesis on H_∞ criterion or H_∞-optimization [4].

There are two basic approaches to such a synthesis, namely, a factorizational approach and a state-space one. The first is based on the so-called J -factorization [6], [3] and does not require any additional restrictions of the internal system. Nevertheless it does not guarantee the designed controller to be proper, and numerical procedures (in a multi-input/ multi-output case especially) are quite complex. The direct application of a standard approach based on the two Riccati equations solutions ("two-Riccati approach") [1], is impossible, as it will be shown, because of the failure of regularity conditions.

An asymptotical modification of the 2-Riccati approach is proposed in the present paper that makes standard programs available. Recent results on asymptotical behaviour of H_∞ and H_2 norms [7] are used.

2 Formalization of the Optimal Stability Margin Problem

Let the transfer function of a plant be given as rational matrix $H(s)$, that can be presented as

$$H(s) = M^{-1}(s)N(s), \qquad (2.1)$$

where $M(s)$ and $N(s)$ are also rational (polynomial, in particular) matrices. The representation is known up to uncertainty terms $\Delta_M(s), \Delta_N(s)$:

$$M(s) = M_0(s)-\Delta_M(s), \quad N(s) = N_0(s)+\Delta_N(s), \qquad (2.2)$$

where $M_0(s)$ and $N_0(s)$ are determined and compose the basic plant model

$$H_0(s) = {M_0}^{-1}N_0(s),$$

and the uncertainties are bounded by

$$||\Delta_M(jw)|| \le |U(iw)|, \quad ||\Delta_N(jw)|| \le |V(iw)|. \qquad (2.3)$$

The boundary functions $U(s)$ and $V(s)$ are given, stable and rational.

The structure disturbances representation in the form of (2.1), (2.2) is sufficiently general and includes additive and multiplicative disturbances as particular cases.

With the help of the Nyquist criterion one can state the following sufficient condition. By any uncertainty bounded by (2.3), the plant (2.1) with the feedback $K(s)$ will be stable if the closed loop with basic plant is stable and condition

$$1 - \sup_w \|S_0(s)M_0{}^{-1}(s)U(s);$$
$$K(s)S_0(s)M_0{}^{-1}V(s)\|_{s=jw} > 0, \qquad (2.4)$$

holds (see [4]). Here $S_0(s)$ is the sensitivity function:

$$S_0(s) = [I - H_0(s)K(s)]^{-1}. \qquad (2.5)$$

It is natural to call the value of the left side of inequality (2.4) as a stability margin on structure disturbances. Design of the feedback $K(s)$ that maximizes the stability margin is called optimization on robustness criterion.

One can easily demonstrate that the problem is related to a H_∞-norm optimization class and equal to $K(s)$ design, minimizing the H_∞-norm of the following transfer function

$$P(s) = \{S(s)M_0{}^{-1}(s)U(s);\ K(s)S(s)M_0{}^{-1}(s)V(s)\} \qquad (2.6)$$

3 Transforming the Disturbance Attenuation Problem

Let us show the way of solving the robustness optimization problem via state-space representation and matrix Riccati equations analysis.

Let basic (undisturbed) plant description be given in the form

$$y = H_0(s)u.$$

Consider an auxiliary system

$$\begin{aligned} z_1 &= H_z(s)u + H_1(s)\xi \\ y &= H_y(s)u + H_2(s)\xi \\ u &= K(s)y. \end{aligned} \qquad (3.7)$$

Choose the transfer functions $H_z, H_y, H_1, H_2(s)$ for having

$$T_{z\xi}(s) = \begin{bmatrix} T_{z_1\xi} \\ T_{u\xi} \end{bmatrix}$$

equal to $P(s)$ in (2.6).

To do this it is necessary to make sure that the following equalities hold

$$H_zK[I - H_yK]^{-1}H_2 + H_1 = [I - H_0K]^{-1}M_0{}^{-1}U \qquad (3.8)$$

$$K[I - H_yK]^{-1}H_2 = K[I - H_0K]^{-1}M_0{}^{-1}V \qquad (3.9)$$

Let us choose $H_y(s) = H_0(s)$ and $H_2(s) = M_0{}^{-1}(s)V(s)$. Then the equality (3.8) holds and to satisfy (3.9) it is sufficient to set

$$\begin{aligned} H_1 &= M_0{}^{-1}U \\ H_z &= \tfrac{U}{V}H_0, \end{aligned}$$

that can easily be verified by a simple replacement. Therefore the following statement has been proved:

Theorem 1. *The stability margin maximization problem on structure disturbances is equal to H_∞-norm minimization problem of transfer function $T_{z\xi}(s)$ in the system*

$$z_1 = \frac{U(s)}{V(s)}H_0(s)u + M_0{}^{-1}(s)U(s)\xi; \quad u = K(s)y;$$

$$z^T = (z_1,\ u) \qquad (3.10)$$

$$y = H_0(s)u + M_0{}^{-1}V(s)\xi = \frac{V(s)}{U(s)}z_1. \qquad (3.11)$$

The operator description can be rewritten in a state-space form by standard procedure. However, it is more efficient to do it block -by - block. Provided that transfer functions $H_0(s)$ and $M_0{}^{-1}V(s)$ are strictly proper, the representation (3.11) could be written in the state-space form:

$$\dot{x}_0 = A_0x_0 + B_{20}u + B_{10}\xi; \qquad y = C_0x_0.$$

If $U(s)V^{-1}(s)$ is proper then the connection between y and z_1 can also have the state-space form

$$\dot{x}_\Delta = A_\Delta x_\Delta + B_\Delta y; \quad z_1 = C_\Delta x_\Delta + D_\Delta y,$$

where matrices $A_\Delta, B_\Delta, C_\Delta, D_\Delta$ can easily be derived in a view of $U(s), V(s)$ as scalar.

By joining these representations one gets

$$\begin{aligned} \dot{x} &= Ax + B_2u + B_1\xi \\ z_1 &= C_1x;\ \ y = C_2x, \end{aligned} \qquad (3.12)$$

where
$x^T = (x_0, x_\Delta); A = \{A_0 \ \ 0; B_\Delta C_0 \ \ A_\Delta\}; B_2^T = (B_{20}, 0); C_1 = \{D_\Delta C_2, C_\Delta\}; C_2 = \{C_0, 0\}$.

If $U(s) = V(s)$ then $y = z_1$ and the second subsystem becomes useless and

$$A = A_0;\ B_2 = B_{20};\ B_1 = B_{10};\ C_2 = C_1 = C_0.$$

Obviously, the system (3.12) does not satisfy the conditions of standard procedure of optimal feedback applying [1] because of the absence of noise disturbances in observations y. For the observability of the pair A, C_2 the output estimation controller can be designed, these estimations being calculated by the observation derivation. Generally speaking ,the feedback will turn out non-proper. The spectral approach leads to the same result [3].

Therefore, in order to design a proper feedback, one should regularize the problem.

4 Regularization Algorithm

Let us change the problem (3.12) by introducing "weak" noise η,

$$y = C_2x + \varepsilon\eta, \qquad (4.13)$$

where ε is a small positive value.

With this the problem takes the standard form, but the closed-loop transfer function from outer disturbances vector $w^T = (\xi, \eta)$ to output vector $z^T = (z_1, u)$ will be changed

$$T_{zw}(s,\varepsilon) = \{T_{z\xi}(s), T_{z\eta}(s)\} =$$

$$\begin{pmatrix} S_0 M_0{}^{-1} U & \varepsilon V U^{-1}(I - S_0) \\ K S_0 M_0{}^{-1} V & \varepsilon K S_0 \end{pmatrix} \qquad (4.14)$$

and the condition of Theorem 1 failed.

Nevertheless, by a sufficiently small ε

$$||T_{zw}(s,\varepsilon)|| = ||T_{z\xi}(s)|| + O(\varepsilon) \qquad (4.15)$$

and one can use the regularized problem solution which minimizes T_{zw} norm.

Following to [1] we introduce two matrix square equations

$$A^T P + PA - P(B_2 B_2{}^T - \gamma^{-2} B_1 B_1{}^T)P +$$
$$C_1{}^T C_1 = 0 \qquad (4.16)$$
$$AD + DA^T - D(\varepsilon^{-2} C_2{}^T C_2 - \gamma^{-2} C_1{}^T C_1)D +$$
$$B_1 B_1{}^T = 0. \qquad (4.17)$$

Let γ_P denote the lowest boundary of the variable γ thereby the solution of (4.16) exists, let $\gamma_D(\varepsilon)$ denote the corresponding boundary for (4.17) and let $\gamma_\rho(\varepsilon)$ denote the lowest boundary of γ thereby the inequality

$$\gamma^{-2}\rho\{P(\gamma)D(\gamma)\} < 1, \qquad (4.18)$$

holds, where $\rho\{a\}$ is the spectral radius of a matrix a.

Consequently, the minimal value of $T_{zw}(s,\varepsilon)$ norm is the following:

$$\gamma(\varepsilon) = max\{\gamma_P, \gamma_D(\varepsilon), \gamma_\rho(\varepsilon)\},$$

and the controller ensuring that is described by

$$u = F(\varepsilon)\hat{x} \qquad (4.19)$$

$$\dot{\hat{x}} = A\hat{x} + B_1\gamma^{-2}(\varepsilon)B_1{}^T P\hat{x} + B_2 u +$$
$$[I - \gamma^{-2}(\varepsilon)P(\gamma(\varepsilon))D(\gamma(\varepsilon),\varepsilon)]L(\varepsilon)(C_2\hat{x} - y), \qquad (4.20)$$

where

$$F(\varepsilon) = -B_2{}^T P(\gamma(\varepsilon))$$
$$L(\varepsilon) = -\varepsilon^{-2} D(\gamma(\varepsilon),\varepsilon)C_2{}^T.$$

A substantial reduction of the complexity of the procedure is possible if the plant is minimal-phase.

Theorem 2. *Let all zeros of the denominator of transfer function $V(s)M_0{}^{-1}(s)$ belong to the left half-plane. Then $\lim_{\varepsilon\to 0}\gamma_D(\varepsilon) = 0$ and $\lim_{\varepsilon\to 0}\gamma(\varepsilon) = \gamma_P$.*

The correctness of this theorem is directly derived from the results of [7], where the problem of "cheap control" has been considered, being a duplicate to the "weak noise" problem. Formal proof is given in the Annex.

Therefore, in order to find the limit stability margin by the structure disturbances it is sufficient to compute the constant γ_P from (4.16) corresponding to the full observation problem. To design the regularized feedback it is necessary to solve Equation (4.17) by fixed $\gamma = \gamma_P$ and some small ε.

Most of the characteristics can be given for the case when $U(s) = V(s)$ and consequently $C_1 = C_2$.

Theorem 3. *Let boundary functions of $U(s), V(s)$ coincide. Then*

$$\gamma_D(\varepsilon) = O(\varepsilon)$$

and stability margin is given by

$$\gamma(0) = max\{\gamma_P, \gamma_\rho(0)\}.$$

Moreover if conditions of Theorem 2 hold, then

$$\gamma(0) = \gamma_P.$$

$D(\gamma_P, \varepsilon)$ is the analytical function of $\varepsilon^{1/k}$, where k is the difference between the degree of nominator and denominator of the transfer function $M_0{}^{-1}(s)V(s)$, therefore it is not necessary to calculate it and the suboptimal feedback is given by

$$u = -P(\gamma_P)B_2{}^T\hat{x}$$

$$\dot{\hat{x}} = (A + \gamma_P{}^{-2}B_1B_1{}^T)\hat{x} + B_2u + \varepsilon^{-1}B_1(C_2\hat{x} - y).$$

The proof is based on the results of [7] and is given in the Appendix. Note that in this case the feedback turns out to be regular.

5 Computational Aspects

According to the described regularization algorithm one has to calculate the lower boundaries of $\gamma_P, \gamma_D(\varepsilon)$ thereby solutions of (4.16), (4.17) exist. A standard procedure from MATLAB Robust Toolbox is available.

Computational experiments show nevertheless that the solutions accuracy turns out to be doubtful when matrix entities are substantially different. That is exactly the case we have in (4.17) by small ε.

Therefore we have used a computationally stabler iterative procedure for Riccati equations solving, as that developed by V.Y.Katkovnik and M.A.Pasumansky. Some benchmark problems have been solved with it. It is worth noting that the case of unstable nominal plant needs special preliminary treatments. Otherwise formally applying MATLAB program on transferring the state-space description (tfm2ss.m) may yield incorrect results.

Numerical experiments confirmed the results of asymptotical analysis.

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APPENDIX

Proof of Theorem 2 Let us compare Equation (4.17) with

$$A^TX+XA-X(\varepsilon^{-2}B_2B_2{}^T-\gamma^{-2}B_1B_1{}^T)X+C_1{}^TC_1=0,$$

that was studied in [7]. Obviously, they coincide up to notification exchange

$$A\to A^T;\quad B_2\to C_2{}^T;\quad B_1\to C_1{}^T.$$

As shown in [7] ,

$$\lim_{\varepsilon\to 0}\gamma_X(\varepsilon)=0,$$

if

$$\det\{C_1(sI-A)^{-1}B_2\}$$

has its zeros in left half-plane only. Using the notification exchange, one reaches the conclusion that for the dual problem the mentioned condition is rewritten as

$$\det\{B_1{}^T(sI-A^T)^{-1}C_2{}^T\}=\det\{C_2(sI-A)^{-1}B_1\}=$$

$$det\{M_0{}^{-1}(s)V(s)\}.$$

One should take into account that according to (3.10), (3.12)

$$C_2(sI-A)^{-1}B_1=H_2(s)=M_0{}^{-1}(s)V(s).$$

The first part of Theorem derives from the mentioned duality.

The second part is true if

$$\rho\{P(\gamma_P)D(\gamma_P,\varepsilon)\}<\gamma_P{}^2,$$

but by $\varepsilon\to 0$ the left part vanishes, therefore it holds.

Proof of Theorem 3

By $C_1=C_2$ Equation (4.17) turns into

$$AD+DA^T-(\varepsilon^{-2}-\gamma^{-2})DC_1{}^TC_1D+B_1B_1{}^T=0.$$

Under conditions of controllability of A,B_1 and observability of A,C_1 the solution exists by

$$\varepsilon^{-2}>\gamma^{-2}\Rightarrow\gamma>\varepsilon.$$

From the latter $\gamma_D(\varepsilon)<\gamma_P$ is derived, and from this the first part of Theorem follows.

If conditions of Theorem 2 hold then one can use Theorem 3 from [7]. We see that $D(\gamma_P, \varepsilon)$ fits the equation

$$AD+DA^T-\varepsilon^{-2}(1-\varepsilon^2\gamma_P{}^{-2})DC_1{}^TC_1D+B_1B_1{}^T = 0.$$

Let us compare it with the dual equation

$$A^TX + XA - \varepsilon^{-2}XB_2B_2{}^TX + C_1{}^TC_1 = 0.$$

Obviously these equations coincide with notification exchange and an insignificant (by small ε) difference of the multiplier in round brackets from 1.

Therefore, according to the mentioned Theorem

$$D(\gamma_P, \varepsilon) = \mu\overline{D}(\mu), \quad \mu = \varepsilon^{1/k},$$

where $\overline{D}(\mu)$ is analytical along μ and k is the difference between the degree of nominator and denominator of the transfer function

$$C_2(sI - A)^{-1}B_1 = M_0{}^{-1}(s)V(s).$$

Substituting this asymptotical performance in Equation we get

$$\varepsilon^{1/k}[A\overline{D}+\overline{D}A^T]-\varepsilon^{-2(1-1/k)}[1-\varepsilon^2\gamma_P{}^{-2}]\overline{D}C_1{}^TC_1\overline{D}+ B_1B_1{}^T = 0.$$

The selection of main terms gives:

$$\overline{D}C_1{}^TC_1\overline{D} = \varepsilon^{2(1-1/k)}[B_1B_1{}^T + o(\mu)],$$

While

$$L = -\varepsilon^{-2}D(\gamma_P, \varepsilon)C_2{}^T = -\varepsilon^{-2}\varepsilon^{1/k}\overline{D}(\mu)C_1{}^T,$$

then

$$LL^T = \varepsilon^{-2(2-1/k)}\overline{D}C_1{}^TC_1\overline{D} = \varepsilon^{-2}B_1B_1{}^T + o(\mu),$$

that means

$$L \to B_1\varepsilon^{-1} \text{ by } \varepsilon \to 0.$$