Graph Modelling Approach Application To A Distillation Column

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Abstract: In this paper, structured systems described by state space models are considered. For these systems, the entries of the state space model matrices are supposed to be either fixed zeros or free independent parameters. With such systems, one can associate a directed graph which is a useful tool to study some control properties of systems. In this context, we present here an illustrative application of disturbance rejection and input-output decoupling problems on a distillation column model.

Keywords: Structured systems, graph theory, input-output decoupling, disturbance rejection

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1 Introduction

In this paper[1], we consider linear systems represented by a quadruplet (A,B,C,E) where the entries of (A,B,C,E) are either fixed zeros or free parameters. With such systems, called structured systems, one can associate a directed graph in a natural way [11, 12, 13]. One can study structural properties, i.e. properties which are true for almost all values of the parameters. Most of these properties can be obtained from properties of the associated graph. Structural properties have been extensively studied during the last twenty years following [11].
For such systems, the generic infinite structure can be deduced from the associated graph [5, 13, 21] and corresponds to sets of vertex disjoint input-output paths. As an application, the structural solvability conditions of classical control problems can easily be checked on the associated graph. For instance, the disturbance rejection problem has been considered in [4, 5, 9, 21, 22]. The decoupling problem has been studied in [6, 12]. Efficient algorithms to determine this infinite structure and to solve control problems have been proposed in [1, 8, 17].
Differently, [13, 16] proposed some graphical structural studies for finding the feedback configuration of control problems like input-output decoupling and disturbance rejection. Structural numerical techniques have been developed for finding such feedbacks as those which have led to a MATLAB toolbox [1, 14].
The purpose of this paper is to combine the above two structural techniques in order to improve the existing procedures for input-output decoupling and disturbance rejection. The goal is to combine generic infinite structure conditions for such control problems, with graphical techniques for finding the feedback configuration.
These approaches are illustrated on a 13 tray binary distillation column model represented by classical state space equations.
The outline of this paper is as follows. We recall first some basic properties of applied graph theory and some results in structured system analysis. In Section 3, the distillation column model is presented and its associated graph is depicted. In Section 4, we discuss the input-output decoupling problem, show the existence of a feedback control law and compute it. Section 5 deals with the state feedback disturbance rejection problem. We show that this problem is generically solvable only if one of the disturbances is available for measurement, and calculate the feedback control law by both graphical and geometrical approaches. We end this paper with some concluding remarks in Section 6.

2 Structured Systems and Digraphs

In this Section, we recall the definition of linear structured systems, we introduce the associated graph of a structured system and give the graphical characterization of the generic infinite structure.
Consider the linear disturbed system Σ^d :

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t)+Bu(t)+Ed(t) \\ y(t) & = & Cx(t) \end{array} \right. \quad (1)$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $d(t) \in \Re^q$ and $y(t) \in \Re^p$ are respectively state, input, disturbance and output. A,B,C,E are real matrices of appropriate dimensions.
The system is said to be a structured system if the entries of (A,B,C,E) (resp. (A,B,C)) are either fixed zeros or free parameters. Denote Λ' (resp. Λ) the vector composed of the ρ' (resp. ρ) nonnull parameters λ_i $(i=1,\ldots,\rho')$ of the structured matrices (A,B,C,E) (resp. (A,B,C)). A structured system with a parameter set Λ' (resp. Λ) will be denoted $\Sigma^d_{\Lambda'}$ (resp. Σ_Λ).
With a structured system $\Sigma^d_{\Lambda'}$, one can associate a directed graph $G(\Sigma^d_{\Lambda'})=(Z,W)$ where the vertex set is $Z=U\cup D\cup X\cup Y$ where : $U=\{u_1,\ldots,u_m\}$, $X=\{x_1,\ldots,x_n\}$, $D=\{d_1,\ldots,d_q\}$, $Y=\{y_1,\ldots,y_p\}$
and the arc set is $W = \{(u_i,x_j),b_{ji}\neq 0\}\cup\{(d_i,x_j),e_{ji}\neq 0\}\cup\{(x_i,x_j),a_{ji}\neq 0\}\cup\{(x_i,y_j),c_{ji}\neq 0\}$ where b_{ji} (resp. a_{ji},e_{ji},c_{ji}) denotes the element (j,i) of the matrix B (resp. of A,E,C).
The associated graph of the structured system (A,B,C) is built the same way without considering neither the disturbance vertex set nor the arc set linked to the matrix E.
A directed path in the graph $G(\Sigma^d_{\Lambda'})=(Z,W)$ from a vertex i_{μ_0} to a vertex i_{μ_q} is a sequence of arcs :
$(i_{\mu_0},i_{\mu_1}),(i_{\mu_1},i_{\mu_2}),\ldots,(i_{\mu_{q-2}},i_{\mu_{q-1}}),(i_{\mu_{q-1}},i_{\mu_q})$ such that $i_{\mu_t}\in Z$ for $t=0,\ldots,q$ and $(i_{\mu_{t-1}},i_{\mu_t})\in W$ for $(t=1,\ldots,q)$. The length of a path is the number of its arcs, each arc being counted for as many times as it appears in the sequence. For the last sequence, the path has length q. Occasionally, we denote the path by the sequence of vertices it consists of, i.e. by $(i_{\mu_0},i_{\mu_1},\ldots,i_{\mu_{q-1}},i_{\mu_q})$. Moreover, if $i_{\mu_o}\in U$ and $i_{\mu_q}\in Y$, this path is called an input-output

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path. A set of k input-output paths with no common vertices is called a k vertex disjoint input-output paths set.
Let us recall the classical definition of the infinite zero structure :

Definition 1 *Let $\Sigma(A, B, C)$ be a linear system without disturbance ($E = 0$). Let $T(s)$ be the transfer matrix of Σ which is a $(p \times m)$ proper rational matrix. $T(s) = C(sI - A)^{-1}B$ can be factorized as follows :*

$$T(s) = B_1(s) \begin{bmatrix} \Delta(s) & 0 \\ 0 & 0 \end{bmatrix} B_2(s)$$

where :

$\Delta(s) = diag(s^{-n_1}, \ldots, s^{-n_r})$
n_i *integers with* $n_1 \leq n_2 \leq \ldots \leq n_r$
$r = rank(T(s))$
$B_1(s), B_2(s)$ *are bicausal matrices.*

The list $\{n_1, \ldots, n_r\}$ is uniquely defined and constitutes the infinite structure of $T(s)$. The n_i's are called the infinite zero orders of the system.

For a single input/ single output system, the infinite zero order of the system is simply the difference of degrees between denominator and numerator of the system transfer function. The infinite structure can be computed by calculation of minor degrees of the transfer matrix [5, 15]. The structure at infinity can be characterized for a structured system $\Sigma_\Lambda(A, B, C)$ on the associated graph $G(\Sigma_\Lambda)$ [5, 21] :

Theorem 1 *Let Σ_Λ be a linear structured system and $G(\Sigma_\Lambda)$ be its associated graph. One has the following :*

i) The structural rank of Σ_Λ which is the number of structural infinite zeros of Σ_Λ, is equal to the maximum number of input-output vertex disjoint paths in $G(\Sigma_\Lambda)$.

ii) The structural infinite zero orders of Σ_Λ are characterized on $G(\Sigma_\Lambda)$ as follows :

$$\begin{aligned} n_1 &= L_1 - 1 \\ n_k &= L_k - \sum_{j=1}^{k-1} n_j - k \\ &= L_k - L_{k-1} - 1 \qquad k = 2, \ldots, r \end{aligned}$$

where L_k is the minimal sum of k vertex disjoint input-output path lengths in $G(\Sigma_\Lambda)$.

This computation method will be useful to check the generic solvability of input-output decoupling and disturbance rejection on the distillation column model. Let us introduce the plant in the next Section.

3 Distillation Column Model

The test example is a 13 tray binary distillation column model [2, 10]. A diagram of the column with the model relevant variables is shown in Figure 1.
The objectives are to control the top composition X_D and the bottom composition X_B by means of the reflux flow rate L and steam flow V respectively and to prevent influence of the disturbances in feed flow L_F and feed composition X_F on the outputs. A non-linear process model has been linearized around a desired operating-point. The model has $[L, V]^t$ as the input vector, $[L_F, X_F]^t$ as the disturbance vector, and $[X_D, X_B]^t$ as the output vector. The state vector of the model, including the composition in the reboiler, each tray and the condenser, is $[X_R, X_1, ..., X_{13}, X_C]^t$.

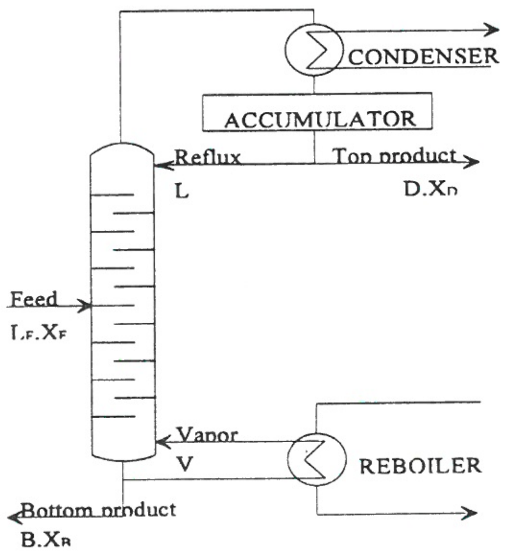

Figure 1: Distillation Column Plant

As described above, the linear model has 15 states, 2 inputs, 2 outputs and 2 disturbances. The different parameter values will be given in Appendix. In the following matrices, a "$\star$" corresponds to a free parameter.

$$A = \begin{bmatrix} \star & \star & 0 & \ldots & 0 \\ \star & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \star \\ 0 & \ldots & 0 & \star & \star \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & \dots & \dots & 0 & \star \\ \star & 0 & \dots & \dots & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ 0 & 0 \end{bmatrix} \quad , \quad E = \begin{bmatrix} \star & 0 \\ \star & 0 \\ \star & 0 \\ \star & 0 \\ \star & 0 \\ \star & 0 \\ \star & 0 \\ \star & \star \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Using this state space structure one gets the digraph presented in Figure 2.

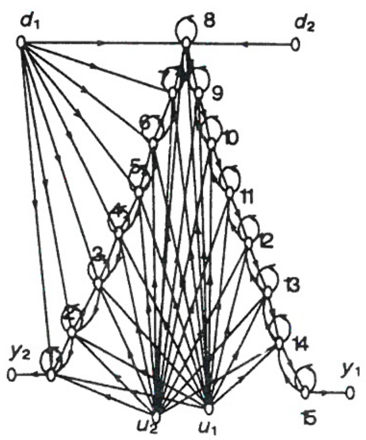

Figure 2: Associated Graph

4 Generic Input-Output Decoupling Problem

The decoupling problem has been studied using a transfer function approach, a state space approach, and a graph-theoretic approach (see references in [13], p 102). The basic idea in the approach proposed by Reinschke is to create new signal paths by means of state feedback in order to compensate unwanted coupling paths. The graph-theoretic approach demonstrates the principles in a particularly transparent manner. The generic infinite structure condition for decouplability [6] is general in the sense that it can be applied to all structured systems, but it gives no information about how to find the feedback configuration. Reinschke's method finds the feedback configuration, but it is not general in the sense that it only applies to compensation of one arc at a time (compensation of the first kind). Therefore a combination would improve the situation. We first explain the infinite structure condition, then give the Reinschke's computation. A more general computation method for finding the feedback based on the results in [7, 19] is then given.

4.1 Generic Condition

This condition is based on the infinite structure characterization. The following theorem is directly given for a structured system since the result is true in the general case.

Theorem 2 *Let Σ_Λ be a linear structured system defined by the triplet (A, B, C) whose transfer matrix $T(s) = C(sI - A)^{-1}B$ is a $(p \times m)$ full row rank proper rational matrix.*
Denote :
n_i, $i = 1, \dots, p$ the generic infinite zero orders of $\Sigma_\Lambda = (A, B, C)$
n'_i, $i = 1, \dots, p$ the generic infinite zero order of $\Sigma_\Lambda = (A, B, C_i)$, where C_i is the i^{th} row of C.
This system is generically decouplable by a feedback control law $u(t) = F_1x(t) + G_1v(t)$, G_1 regular, if and only if

$$\sum_{i=1}^{p} n_i = \sum_{i=1}^{p} n'_i \qquad (2)$$

So, by the computation of the infinite structure of the system, it will be easy to check the solvability of generic decoupling directly on the graph. Following [21], a computation method based on linear programming techniques is proposed in [8].

Consider now the distillation column model. Firstly, we will find the generic infinite structure of the model directly on the graph (Theorem 1):

1. a shortest input-output path is $\{u_1, x_1, y_2\}$. This path is of length $L_1 = 2$.

2. a shortest pair of vertex disjoint input/ output paths is $\{u_1, x_1, y_2\}$,$\{u_2, x_{14}, x_{15}, y_1\}$. This pair of paths is of length $L_2 = 5$.

So, the generic infinite zero orders of the structured system are according to Theorem 1 :

$$n_1 = 1 \quad , \quad n_2 = 2$$

The generic row-by-row infinite zero orders are calculated by computing the shortest length from the input set to each output :

1. the shortest path from the input set to y_1 is $\{u_2, x_{14}, x_{15}, y_1\}$. This path is of length $L'_1 = 3$.

2. the shortest path from the input set to y_2 is $\{u_2, x_1, y_2\}$. This path is of length $L'_2 = 2$.

So, the generic row-by-row infinite zero orders of the structured system are :

$$n'_1 = 2 \quad , \quad n'_2 = 1$$

Then according to Theorem 2, the decoupling problem for the distillation plant is generically solvable for almost all values of the parameters.

4.2 Feedback Computation

We will apply a control law of the form :

$$u(t) = G_2 v(t) \qquad v(t) = F_2 x(t) + w(t)$$

This control law can be shown to be equivalent to the previous law given in Theorem 2 when G_2 is regular. The resulting closed loop system is

$$\begin{cases} \dot{x}(t) &= (A + BG_2F_2)x(t) + BG_2 w(t) \\ y(t) &= Cx(t) \end{cases}$$

If successful the method results in a matrix pair (G_2, F_2) and information about additional feedbacks free for other purposes. The objective of decoupling by static state feedback is then to determine a matrix pair (G_2, F_2) such that, for each $i = 1, \ldots, r$, there is a reference input which can control the considered output without influencing the remaining outputs.

Let us now come back to the distillation column example, and in particular to the corresponding distillation column digraph (Figure 2). By adding a set of new inputs v_1 and v_2 and arcs from V to U, weighted with G_2, the paths $\{v_1, u_1, x_1\}$ and $\{v_2, u_2, x_{14}\}$ can be compensated by paths $\{v_1, u_2, x_1\}$ and $\{v_2, u_1, x_{14}\}$ respectively. But there are still two paths, $\{x_2, x_1\}$, and $\{x_{13}, x_{14}\}$, which have to be compensated in order to eliminate the influence of v_1 on y_2 and of v_2 on y_1. Thus two feedback arcs are needed from x_2 to the new input v_2, $f_{2,2}$ and from x_{13} to v_1, $f_{13,1}$. A toolbox for structural analysis of control systems based on digraphs is applied for the numerical calculations of the system decoupling [1]. The structural form of the state feedback matrix is as follows :

$$F_2 = \begin{bmatrix} \bullet & \bullet & \ldots & \bullet & \times & \times \\ \times & \bullet & \ldots & \bullet & \bullet & \bullet \end{bmatrix}$$

where a '$\bullet$' means an element fixed by equations (perhaps even at 0), a '$\times$' means that the element is free to be used for other purposes.

The numerical state feedback matrix and the compensating network are computed to :

$$F_2 = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & -15.44 & 0 & 0 \\ 0 & 34.59 & 0 & \ldots & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 5.50 & 4.95 \\ 5.00 & 5.50 \end{bmatrix}$$

After the input-output decoupling, the system can be decomposed into three sub-systems $\{x_1\}$, $\{x_2, \ldots, x_{13}\}$ and $\{x_{14}, x_{15}\}$ (see Figure 3).

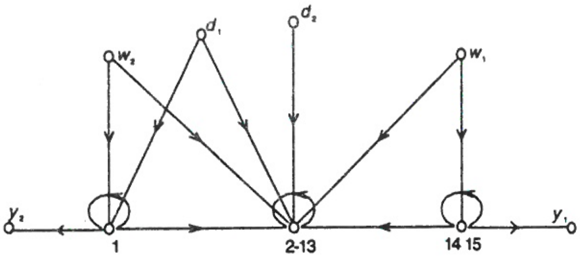

Figure 3: Digraph of The Condensed Closed Loop System

4.3 General Feedback Calculation

In this part, we use the well-known feedback calculation explained in [7, 19]. The simplest state feedback which solves the decoupling problem is given by $u(t) = F_w x(t) + G_w v(t)$ with :

$$\begin{cases} F_w &= -D^{*^{-1}} H^* \\ G_w &= D^{*^{-1}} \end{cases} \tag{3}$$

where

$$D^* = \begin{bmatrix} C_1 A^{n'_1 - 1} B \\ \vdots \\ C_m A^{n'_m - 1} B \end{bmatrix} \text{ and } H^* = \begin{bmatrix} C_1 A^{n'_1} \\ \vdots \\ C_m A^{n'_m} \end{bmatrix}$$

This solution leads to a decoupled integrator form where all the poles are located in 0.

For the distillation column model, the computation of the generic row-by-row infinite zero orders gives $n'_1 = 2$ and $n'_2 = 1$. Then

$$D^* = \begin{bmatrix} b_{14,1}a_{15,14}c_{1,15} & b_{14,2}a_{15,14}c_{1,15} \\ b_{1,1}c_{2,1} & b_{1,2}c_{2,1} \end{bmatrix}$$

and H^* has the following structure :

$$\begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & h^*_{1,13} & h^*_{1,14} & h^*_{1,15} \\ h^*_{2,1} & h^*_{2,2} & 0 & \ldots & 0 & 0 & 0 & 0 \end{bmatrix}$$

with :

$$\begin{aligned} h^*_{1,13} &= a_{14,13}a_{15,14}c_{1,15} \\ h^*_{1,14} &= a_{14,14}a_{15,14}c_{1,15} + a_{15,14}a_{15,15}c_{1,15} \\ h^*_{1,15} &= a_{14,15}a_{15,14}c_{1,15} + a_{15,15}a_{15,15}c_{1,15} \\ h^*_{2,1} &= a_{1,1}c_{2,1} \\ h^*_{2,2} &= a_{1,2}c_{2,1} \end{aligned}$$

By using (3), the feedback gain F_w has the following structure :

$$\begin{bmatrix} f_{1,1} & f_{1,2} & 0 & \dots & 0 & f_{1,13} & f_{1,14} & f_{1,15} \\ f_{2,1} & f_{2,2} & 0 & \dots & 0 & f_{2,13} & f_{2,14} & f_{2,15} \end{bmatrix}$$

with

$$\begin{aligned} f_{1,1} &= \frac{a_{1,1}b_{14,2}}{b_{14,1}b_{1,2} - b_{1,1}b_{14,2}} \\ f_{1,2} &= \frac{a_{1,2}b_{14,2}}{b_{14,1}b_{1,2} - b_{1,1}b_{14,2}} \\ f_{1,13} &= \frac{-a_{14,13}b_{1,2}}{b_{14,1}b_{1,2} - b_{1,1}b_{14,2}} \\ f_{1,14} &= \frac{-a_{14,14}b_{1,2} - a_{15,15}b_{1,2}}{b_{14,1}b_{1,2} - b_{1,1}b_{14,2}} \\ f_{1,15} &= \frac{-a_{14,15}b_{1,2}}{b_{14,1}b_{1,2} - b_{1,1}b_{14,2}} \\ &\quad - \frac{a_{15,15}a_{15,15}b_{1,2}}{a_{15,14}(b_{14,1}b_{1,2} - b_{1,1}b_{14,2})} \\ f_{2,1} &= \frac{a_{1,1}b_{14,1}}{b_{14,1}b_{1,2} - b_{1,1}b_{14,2}} \\ f_{2,2} &= \frac{-a_{1,2}b_{14,1}}{b_{14,1}b_{1,2} - b_{1,1}b_{14,2}} \\ f_{2,13} &= \frac{a_{14,13}b_{1,1}}{b_{14,1}b_{1,2} - b_{1,1}b_{14,2}} \\ f_{2,14} &= \frac{-a_{14,14}b_{1,1} - a_{15,15}b_{1,1}}{b_{14,1}b_{1,2} - b_{1,1}b_{14,2}} \\ f_{2,15} &= \frac{-a_{14,15}b_{1,1}}{b_{14,1}b_{1,2} - b_{1,1}b_{14,2}} \\ &\quad - \frac{a_{15,15}a_{15,15}b_{1,1}}{a_{15,14}(b_{14,1}b_{1,2} - b_{1,1}b_{14,2})} \end{aligned}$$

With the parameter values, one gets the state feedback matrices F_w and G_w :

$$F_w = \begin{bmatrix} -339.70 & 171.22 & 0 & \dots \\ -377.44 & 190.24 & 0 & \dots \end{bmatrix}$$
$$\begin{matrix} \dots & 0 & -84.92 & 411.66 & -515.81 \\ \dots & 0 & -77.20 & 374.24 & -468.91 \end{matrix}\Bigg]$$
$$G_w = \begin{bmatrix} 66.15 & -66.71 \\ 60.13 & -74.12 \end{bmatrix}$$

The two ways of feedback calculations do not necessarily give the same decoupling feedback matrices. In fact, the pairs of decoupling state feedback matrices belong to the family of decoupling state feedbacks which make the closed-loop system maximally unobservable. They can be parameterized as follows :

$$\left\{ \begin{array}{lcl} F & = & -D^{*^{-1}}(H^* + H^{**}) \\ G & = & D^{*^{-1}}K^{**} \end{array} \right. \qquad (4)$$

where K^{**} is an $(m \times m)$ diagonal real matrix and H^{**} represents an $(m \times n)$ real matrix allowing some pole placements while preserving decoupling. The i^{th} row of H^{**} is such that :

$$H_i^{**} = \sum_{j=0}^{n_i'-1} e_{ij}C_iA^j \qquad (i = 1, \dots, m)$$

with e_{ij} a free real parameter.
For the distillation column model, the computation of the generic row-by-row infinite zero orders gives $n_1' = 2$ and $n_2' = 1$. Then, H^{**} is as follows :

$$H^{**} = \begin{bmatrix} e_{1,0}C_1 + e_{1,1}C_1A \\ e_{2,0}C_2 \end{bmatrix}$$

that is :

$$H^{**} = \begin{bmatrix} 0 & 0 & \dots & 0 & h_{1,14}^{**} & h_{1,15}^{**} \\ h_{2,1}^{**} & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

with

$$\begin{aligned} h_{1,14}^{**} &= e_{1,1}a_{15,14}c_{1,15} \\ h_{1,15}^{**} &= e_{1,0}c_{1,15} + e_{1,1}a_{15,15}c_{1,15} \\ h_{2,1}^{**} &= e_{2,0}c_{2,1} \end{aligned}$$

Then , one gets the sum $(H^* + H^{**})$:

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & h_{1,13} & h_{1,14} & h_{1,15} \\ h_{2,1} & h_{2,2} & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}$$

Given that $e_{1,0}$, $e_{1,1}$ and $e_{2,0}$ are free real parameters, the sum $(H^* + H^{**})$ has the general following structure :

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & h_{1,13} & \times & \times \\ \times & h_{2,2} & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}$$

where '$\times$' denotes a free element.
Then by (4), the feedback matrix F has the form

$$\begin{bmatrix} \alpha_1 & f_{1,2} & 0 & \dots & 0 & f_{1,13} & \beta_1 & \gamma_1 \\ \alpha_2 & f_{2,2} & 0 & \dots & 0 & f_{2,13} & \beta_2 & \gamma_2 \end{bmatrix}$$

where α_1, β_1 and γ_1 are free elements and α_2 (resp. β_2, γ_2) is dependent on α_1 (resp. β_1, γ_1). It is worth noticing that this structure is similar to the structure of matrix F_2 in Section 4.2. Here, we precise the location of identical zero entries.

4.4 Remarks

Recall that Reinschke's method is applied with a control law of the form :

$$u(t) = G_2v(t) \qquad v(t) = F_2x(t) + w(t)$$

with G_2 regular, which gives :

$$u(t) = G_2F_2x(t) + G_2w(t) = F_rx(t) + G_rw(t)$$

By using the results of Subsection 4.2, the two matrices of the feedback control law are given by :

$$F_r = \begin{bmatrix} 0 & 171.22 & 0 & \dots & 0 & -84.92 & 0 & 0 \\ 0 & 190.24 & 0 & \dots & 0 & -77.20 & 0 & 0 \end{bmatrix}$$

$$G_r = \begin{bmatrix} 5.50 & 4.95 \\ 5.00 & 5.50 \end{bmatrix}$$

By manipulating Equations (4), it is easy to find the relations between the feedback pairs :

$$\begin{cases} K^{**} & = & G_w^{-1}G_r \\ H^{**} & = & D^*(F_w - F_r) \end{cases}$$

For the proposed feedback matrix pairs (F_r, G_r) and (F_w, G_w), we obtain :

$$K^{**} = \begin{bmatrix} 0.0831 & 0 \\ 0 & -0.0742 \end{bmatrix}$$

$$H^{**} = \begin{bmatrix} 0 & 0 & 0 & \dots \\ 5.0922 & 0 & 0 & \dots \end{bmatrix.$$
$$\begin{matrix} \dots & 0 & 6.2232 & -7.7975 \\ \dots & 0 & 0 & 0 \end{matrix}\Bigg]$$

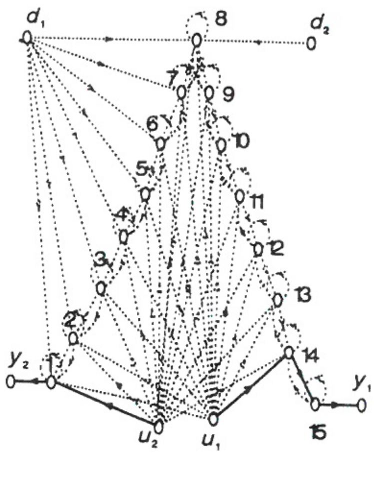

Figure 4: DRP When Disturbances Are Available for Measurement

5 Generic Disturbance Rejection Problem

In this part, we discuss the disturbance rejection problem (DRP) by a state feedback control law. In a first part, we present the generic infinite structure condition which has a nice graphical interpretation on the associated graph. A second part gives feedback computation. In the last part, we present the calculation of the feedback within a geometric framework.
In this Section, we consider the linear structured system with disturbances $\Sigma^d_{\Lambda'}(A, B, C, E)$ presented at Section 2.

5.1 Generic Condition

We will directly present here the generic infinite structure characterisation for the solvability of DRP. The conditions for DRP [5, 21] are summarised in the next proposition [9] :

Proposition 1 *The DRP for a structured system is generically solvable by a feedback control law $u(t) = Fx(t) + Hq(t)$ if and only if there exists a set of maximal number of vertex disjoint input-output paths of minimal total length in $G(\Sigma^d_{\Lambda'})$ which does not include vertices of $D = \{d_1, \dots, d_q\}$.*

When the disturbance $d(t)$ is not available for measurement ($H = 0$), we split each control input vertex u_i into two vertices u_i^- and u_i^+ connected by an arc (the state vertices come from u_i^+) and then we apply Proposition 1.

Consider the distillation column example when the disturbances are available for measurement. In that case, the maximal number of vertex disjoint input-output paths equals 2, and a pair of vertex disjoint input-output paths of minimal total length is (Figure 4) :

$$\{u_1, x_{14}, x_{15}, y_1\} \quad \text{and} \quad \{u_2, x_1, y_2\}$$

Then, the condition of Proposition 1 is satisfied. So the DRP with disturbance measurement is generically solvable.

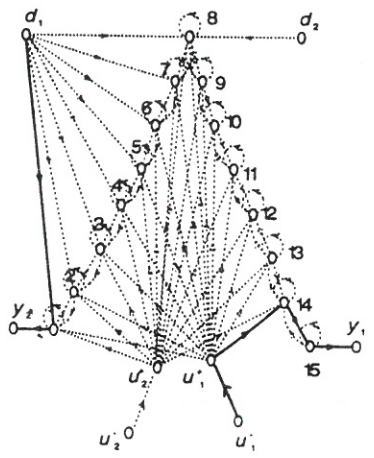

Figure 5: DRP When Disturbances Are Not Available for Measurement

When the disturbances are not available for measurement, i.e. $H = 0$, the maximal number of vertex disjoint input-output paths equals 2 but a pair of vertex disjoint input-output paths of minimal total length must include at least one vertex of $D = \{d_1, d_2\}$. For example (Figure 5):

$$\{u_1^-, u_1^+, x_{14}, x_{15}, y_1\} \quad \text{and} \quad \{d_1, x_1, y_2\}$$

Then, DRP is generically non-solvable if the disturbances are not available for measurement.

5.2 Graphical Feedback Calculation

The DRP amounts to finding a set of state feedbacks, $u(t) = Fx(t)$, to prevent an influence of the disturbances on the outputs. The paths from the disturbances to the output in the distillation column model pass through x_2, and through x_{13}. A complete disturbance rejection without the measurement of the disturbance feed flow is not possible because of direct influence of the disturbance on the output through x_{13}. By compensation of arc $\{x_2, x_1\}$, and $\{x_{13}, x_{14}\}$ the influence of the disturbances on the outputs can be eliminated by means of four state feedback gains from states x_2 and x_{13} to both inputs. The structural form of the state feedbacks is the following :

$$F = \begin{bmatrix} \times & \bullet & 0 & \dots & 0 & \bullet & \times & \times \\ \times & \bullet & 0 & \dots & 0 & \bullet & \times & \times \end{bmatrix}$$

where a '•' means an element fixed by equations (perhaps even at 0), an '×' means that the element is free to be used for other purposes, and a '0' means that the element is fixed to zero.
The numerical value of the state feedback is computed to

$$F = \begin{bmatrix} 0 & 171.22 & 0 & \dots & 0 & -84.93 & 0 & 0 \\ 0 & 190.25 & 0 & \dots & 0 & -77.21 & 0 & 0 \end{bmatrix}$$

Figure 6 shows the corresponding digraph after the disturbance rejection. By inspection, it is evident that the disturbances that enter the components x_2, x_{13} cannot continue to the outputs. The only problem is the d_1 entering x_1.

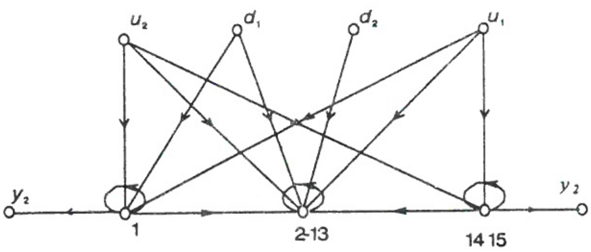

Figure 6: Digraph of The Closed Loop Distillation Column After Partial Disturbance Rejection

5.3 Geometrical Approach

Introducing the geometrical approach, we are able to parameterize the feedback control law for disturbance rejection in terms of the entries of the original system. Just recall the characterization of an (A, B)-invariant subspace :

Definition 2 *Consider the linear system Σ^d described in Equations (1) and denote by ImB the image of B. A subspace $\mathcal{V} \subset \mathcal{X}$ is said to be (A, B)-invariant if, whenever the initial state belongs to $\mathcal{V}$, the whole state trajectory can be held in it using appropriate inputs. This is geometrically equivalent to*

$$A\mathcal{V} \subset \mathcal{V} + \mathcal{B} \qquad (5)$$

Let $\mathcal{V}^*$ be the largest (A, B)-invariant subspace contained in Ker$C$, we obtain the well-known geometric condition of solvability of DRP (see [3, 20]).

Theorem 3 *DRP is solvable if and only if*

$$\mathcal{E} \subset \mathcal{V}^*$$

To illustrate this theorem, consider the distillation column model. A similar model was previously analysed by [18]. As shown in this paper, the largest (A, B)-invariant subspace contained in Ker$C$ is equal to :

$$\mathcal{V}^* = \{x \in \Re^{15} \quad | \quad x_1 = x_{14} = x_{15} = 0\} \quad (6)$$

Notice that $\mathcal{V}^*$ is independent of the precise values of the system parameters.
The condition of the above theorem is verified. The feedback gain has the following structure :

$$\begin{bmatrix} \times & f_{1,2} & 0 & \dots & 0 & f_{1,13} & \times & \times \\ \times & f_{2,2} & 0 & \dots & 0 & f_{2,13} & \times & \times \end{bmatrix} \quad (7)$$

where the "×" are free parameters chosen to be zero in the following. In this particular case, the four nonnull feedback gains can be computed as follows. Since $\mathcal{V}^*$ is (A,B)-invariant, one has (see [20]) :

$$(A + BF)\mathcal{V}^* \subset \mathcal{V}^* \qquad (8)$$

Using (6), (7) and (8), one gets :

$$(b_{1,1}f_{1,13} + b_{1,2}f_{2,13})x_{13} + (a_{1,2} + b_{1,1}f_{2,1} + b_{1,2}f_{2,2})x_2 = 0$$

$$(a_{14,13} + b_{14,1}f_{1,13} + b_{14,2}f_{2,13})x_{13} + (b_{14,1}f_{1,2} + b_{14,2}f_{2,2})x_2 = 0$$

Then, as these equations are true for all values of the vector x, one obtains :

$$f_{1,2} = \frac{a_{1,2}b_{14,2}}{b_{14,1}b_{1,2} - b_{1,1}b_{14,2}}$$

$$f_{2,2} = \frac{-a_{1,2}b_{14,1}}{b_{14,1}b_{1,2} - b_{1,1}b_{14,2}}$$

$$f_{1,13} = \frac{-a_{14,13}b_{1,2}}{b_{14,1}b_{1,2} - b_{1,1}b_{14,2}}$$

$$f_{2,13} = \frac{a_{14,13}b_{1,1}}{b_{14,1}b_{1,2} - b_{1,1}b_{14,2}}$$

On replacing the system's parameters by their value, one gets the above result calculated by Reinschke's method.

6 Concluding Remarks

The above results present a graphical approach to the generic input-output decoupling problem and the generic disturbance rejection problem for a distillation column plant.

Even if the parameters of a state space model were undecided, such that numerical computer routines cannot be applied, the structure of the model would still provide valuable information. The "digraph approach" focuses on the pattern of non-zero entries in the model, and it provides sufficient structural conditions for decouplability and disturbance rejection.

For the considered distillation column, the analysis shows that after input-output decoupling, the system can be decomposed into a number of sub-systems, making it easier to study such problems as disturbance rejection and pole placement. The structural form of the feedbacks for input-output decoupling or disturbance rejection is given. A parameterization of such feedbacks in terms of the entries of the original system is also provided.

The graph approach may also provide a way to cope with uncertainty and varying parameters. In practice the structure of a model may be known from professional insight into the physics of the system, but the parameters may vary according to the operating point or temperature variation. The digraph approach can nevertheless test necessary conditions for decoupling purely on structural grounds. The approach is typically applied in the following situations: the parameters of a model are undecided or vary; there is large sensitivity towards small variations; the parameters have been decided, but, the necessary matrix operations are numerically intensive.

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Appendix

$$A = \begin{bmatrix}
-5.09 & 2.57 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4.86 & -7.31 & 2.57 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4.75 & -7.12 & 2.57 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4.56 & -6.80 & 2.57 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4.24 & -6.33 & 2.57 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3.76 & -5.73 & 2.57 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3.17 & -5.13 & 2.57 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2.57 & -4.65 & 2.10 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.08 & -3.80 & 2.10 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.70 & -3.55 & 2.10 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.44 & -3.38 & 2.10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.28 & -3.29 & 2.10 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.19 & -3.25 & 2.10 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.15 & -3.22 & 2.10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.12 & -2.33
\end{bmatrix}$$

$$B = \begin{bmatrix}
0.07 & -0.07 \\
0.13 & -0.14 \\
0.23 & -0.25 \\
0.39 & -0.43 \\
0.61 & -0.67 \\
0.82 & -0.90 \\
0.91 & -1.00 \\
0.98 & -0.90 \\
0.87 & -0.79 \\
0.65 & -0.58 \\
0.42 & -0.38 \\
0.25 & -0.22 \\
0.14 & -0.12 \\
0.07 & -0.07 \\
0 & 0
\end{bmatrix}, \quad E = \begin{bmatrix}
1 & 0 \\
1 & 0 \\
1 & 0 \\
1 & 0 \\
1 & 0 \\
1 & 0 \\
1 & 0 \\
1 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}$$

$$C = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$