Decoupling of MIMO Systems By Graph-Theoretic Approach

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Abstract: Decoupling techniques based on state variable concepts are appreciated by the control system theorists but up to now their application to practical cases is still under try. The point is that, the entries of the state, input and output matrices of the state space representation are regarded as exactly known numerical values while practicing engineers have to cope with varying parameters and more generally, with unavoidable uncertainties. Recently, in order to overcome these disadvantages of the space state theory, Reinschke proposed a graph-theoretic approach to the decoupling problem. The necessary and sufficient conditions for decoupling are directly interpreted in terms of properties of the digraph associated with the state space equations. Moreover using digraphs gives a good insight into the structural nature of the decoupling property, showing how the feedback coefficients offset the original coupling between the terminal variables of the plant. Unfortunately to apply the Reinschke design method it is necessary that the digraph associated with the plant equations presents a particular structure. This condition is severe and seldom satisfied in practice.
The aim of this paper is to overcome the limitations of the Reinschke approach. The main idea is to transform the plant equations into a properly chosen canonical form before associating the digraph with them. Any system which can be decoupled by state feedback controller can be reduced in such a canonical form which belongs to the class for which it is possible to apply the Reinschke design method.
The paper shows an application of the proposed approach to the synthesis of the controller of a synchronous machine supplying an infinite busbar. The example illustrates the effectiveness and easiness of the method.

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1. Introduction

The decoupling design methods of multivariable control systems have made remarkable progress in both theory and practice since the 6oies. Among these methods the following are considered to be successful and effective:

(a) the diagonal matrix approach developed by Tsien (1954), Mesarovic (1960), Schwarz (1967) and many others.

(b) The relative gain method introduced by Shinskey (1979), McAvoy (1979) and others.

(c) The characteristic locus method introduced by MacFarlane and Belletrutti (1973).

(d) The state variable method proposed by Falb and Wolowich (1967), Whonam and Morse (1971), Morse and Whonam (1971) and many others.

Usually the process control engineers do prefer the first two approaches since they are relatively simple to apply . Certainly inverse Nyquist array and the characteristic locus methods are actually effective but they need familiarity with advanced theory concepts and require a lot of computation work. As for decoupling techniques based on state variable concepts, they are appreciated by the control system theorists but up to now their application to practical cases is still under try. The point is that, the entries of the state, input and output matrices of the state space representation are regarded as exactly known numerical values while practicing engineers have to cope with varying parameters and more generally with unavoidable uncertainties. Recently, in order to overcome these disadvantages of the space state theory, Andrei (1985) and Reinschke (1988) proposed a graph-theoretic approach to the decoupling problem. The necessary and sufficient conditions for decoupling are directly interpreted in terms of properties of the digraph associated with the state space equations. Thus the decoupling property of state feedback controllers holds, independently of the numerical values of the entries of the matrices involved in the system model. Such entries may be considered free parameters providing degree-of -freedom to be used in subsequent design steps. Moreover using digraphs gives a good insight into the structural nature of the decoupling property, showing how the feedback coefficients offset the original coupling between the terminal variables of the plant. Unfortunately to apply the Reinschke design method it is necessary that the digraph associated with the plant equations presents a particular structure. This condition is severe and seldom satisfied in practice. Mainly the applications of the method reported in the technical literature refer to distillation columns which are modelled by sparse system equations. In this case the corresponding digraphs present a particular structure meeting the requirements necessary for applying the Reinschke decoupling method.

The aim of this paper is to overcome the limitations of Reinschke's approach. The main idea is to transform the plant equations into a properly chosen canonical form before associating the digraph with them. Any system which can be decoupled by state feedback controller can be reduced in such a canonical form. Moreover the digraph associated with system equations written in this canonical form belongs to the class for which it is possible to apply the Reinschke design method. It must also be mentioned that the main idea may be regarded as analogous to the Mesarovic concepts in the diagonal matrix approach to decoupling. Namely, following Mesarovic's ideas, better decoupling effects would be reached if a particular structure (the V canonical form) were adopted for transfer function matrix describing the plant. The method is applied to design a controller for synchronous generator connected to infinite power busbar. Starting with a linearized system model, the graph theoretic approach allows us to derive a state feedback control so that each synchronous generator output depends on its corresponding input only.

Section 2 introduces notation and precises the problem formulation. Section 3 deals with the main results of graph-theoretic approach to decoupling problem. Section 4 establishes a canonical form which is the key to apply Reinschke's method. Finally, Section 5 shows an example and Section 6 draws the conclusions.

2. Definitions and Problem Statement

Consider a linear time invariant system represented by the following equation:

$$\mathbf{x}^{(1)}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \qquad (1)$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) \qquad (2)$$

where $\mathbf{x}(t)\in\mathbf{R}^n$, $\mathbf{u}(t)\in\mathbf{R}^m$ and $\mathbf{y}(t)\in\mathbf{R}^m$ are the state, input and output vectors, respectively; the matrices $\mathbf{A}=\{a_{ij}\}$, $\mathbf{B}=\{b_{ij}\}$ and $\mathbf{C}=\{c_{ij}\}$ have proper dimensions and $\mathbf{x}^{(1)}(t)$ denotes the first derivative of $\mathbf{x}(t)$. For the sake of simplicity, the state space model (1) and (2) will also be referred as (**A**,**B**,**C**). Besides suppose we use the following feedback law:

$$\mathbf{u}(t) = \mathbf{G}\,\mathbf{v}(t) \qquad (3)$$

$$\mathbf{v}(t) = \mathbf{F}\,\mathbf{x}(t) + \mathbf{w}(t) \qquad (4)$$

where $\mathbf{v}(t)$, $\mathbf{w}(t) \in \mathbf{R}^m$, $\mathbf{F}$ is an $(m \times n)$ matrix and $\mathbf{G}$ is a nonsingular $(m \times m)$ matrix. Vector $\mathbf{w}(t)$ is the new input to the closed-loop system.

The decoupling problem lies in determining a pair (**G**,**F**) so that, for each $i\in\{1,2,\ldots,m\}$, the input $w_i(t)$ affects the output $y_i(t)$ only, and has no influence on the remaining output components $y_j(t)$ for $j\neq i$. In other words the closed loop transfer matrix

$$\mathbf{G}_C(s) = \mathbf{C}\,(s\mathbf{I} - \mathbf{A} - \mathbf{B}\,\mathbf{G}\,\mathbf{F})^{-1}\,\mathbf{B}\,\mathbf{G} \qquad (5)$$

must be diagonal, with non identically vanishing determinant.

The graph-theoretic approach to the decoupling problem is based on the preliminary notion of digraph associated with the system (**A**,**B**,**C**).

So let N be a set of vertices (or nodes) split into three disjoint subsets U, X and Y defined as follows. Labelling the vertices by the same symbols used for the system variables, let us associate a vertex with each input, state and output variable. Hence we get

$$N = U \cup X \cup Y \qquad (6)$$

where: $U = \{u_1,u_2,\ldots,u_m\}$ is the input node set, $X = \{x_1,x_2,\ldots,x_n\}$ is the state node set and $Y = \{y_1,y_2,\ldots,y_m\}$ is the output node set. Moreover let

$$E = E_U \cup E_X \cup E_Y \qquad (7)$$

be a set of directed edges between pairs of vertices from N, where: $E_U \subseteq$ in $(U \times X)$ is the set of input edges: the edge (u_j,x_i) is in E_U if and only if (iff for brevity) b_{ij} is non zero; $E_X \subseteq$ in $(X \times X)$ is the state edge set: $(x_j,x_i) \in E_X$ iff a_{ij} is non zero; $E_Y \subseteq$ in (XAY) is the output edge set: $(x_j,y_i) \in E_Y$ iff c_{ij} is non zero. The direction of each edge is graphically indicated by an arrowhead. The arrow heads for the second vertex (terminal vertex) of the node pair (v_i,v_j) describing the edge. The digraph

$$\mathfrak{g}(\mathbf{A},\mathbf{B},\mathbf{C}) = (N,E) \tag{8}$$

composed of the node set N and the edge set E, is defined as the digraph associated with the system **(A,B,C)**. Analogously, with an obvious meaning of the notation, the digraphs $\mathfrak{g}$**(A,BG,C)** and $\mathfrak{g}$**(A+BGF,BG,C)** can be defined.

Remark 1

Although, according to (3) and (4), the inputs to systems **(A,BG,C)** and **(A+BGF,BG,C)** are respectively **v** and **w**, the input vertex set for $\mathfrak{g}$**(A,BG,C)** and for $\mathfrak{g}$**(A+BGF,BG,C)** will still be denoted by U.

Now we introduce a few notions and properties that we use in the sequel. Let v_i and v_j be any two vertices from N. If $(v_i,v_j) \in E$, we say that v_i is adjacent to v_j and v_j is adjacent from v_i. Moreover if for the state vertex x_i there exists an input vertex (say u_j) such that $(u_j,x_i) \in E_U$, we say that x_i is input-adjacent. A directed path from v_i to v_j is an alternating sequence of distinct vertices from N and distinct edges from E, beginning with v_i and ending with v_j, such that each edge in the sequence is directed from the vertex preceding it to the vertex following it. Besides, if the digraph contains a path from v_i to v_j, then we say that v_j is reachable from v_i. In particular, if for a vertex $v_i \in U \cup X$ there exists an output vertex (say y_j) such that y_j is reachable from v_i we say that v_i is output-reachable.

Introducing the injection matrix **G** according to (3), changes $\mathfrak{g}$**(A,B,C)** into $\mathfrak{g}$**(A,BG,C)** only by modifying the edge set E_U. Now let us prove the following proposition.

Proposition 1

A state vertex x_i is input-adjacent in the digraph $\mathfrak{g}$**(A,BG,C)** iff it is input-adjacent in $\mathfrak{g}$**(A,B,C)**.

Proof

Let us assume x_i be input-adjacent in $\mathfrak{g}$**(A,B,C)**: this implies that the i-th row of **B** is non zero. Since matrix **G** is nonsingular, the same conclusion holds true for the i-th row of **BG**, proving sufficiency. The same pattern can be used to show necessity.

If the state feedback (4) is used, the digraph $\mathfrak{g}$**(A+BGF,BG,C)** is obtained from $\mathfrak{g}$**(A,BG,C)** by modifying only the subset E_X. In particular, if (x_j,x_i) is an element in the state edge set of $\mathfrak{g}$**(A,BG,C)** it is interesting to know under what conditions (x_j,x_i) does not belong to the state edge set of $\mathfrak{g}$**(A+BGF,BG,C)**. Indeed the analysis of these conditions suggests the way in which the feedback can eliminate the couplings between the variables of the plant **(A,B,C)**. Hence we introduce the following:

Proposition 2

Let (x_j,x_i) be an edge from E_X in the digraph $\mathfrak{g}$**(A,B,C)**. Such an edge can be removed by static state feedback iff its terminal node (i.e. x_i) is input-adjacent in $\mathfrak{g}$**(A,B,C)**.

Proof

Since (x_j,x_i) is in E_X, it holds:

$$a_{ij} \neq 0 \tag{9}$$

Therefore, in order to make zero the (i,j)-th entry of the closed loop state matrix (i.e. $(\mathbf{A}+\mathbf{BGF})_{ij}$), it must hold

$$(\mathbf{BGF})_{ij} \neq 0 \tag{10}$$

Condition (10) implies that both of the i-th rows of **BG** and **B** are non zero, i.e. x_i is input-adjacent. On the other hand, if x_i is input-adjacent, the i-th row of **BG** is non zero and it is always possible to choose the j-th column of **F** so that

$$(\mathbf{BGF})_{ij} = -a_{ij} \tag{11}$$ •

According to the previous proposition, the edges of $\mathfrak{g}$**(A,B,C)** whose terminal nodes are input-adjacent are named "eliminable edges". We remark that while a proper choice of the feedback matrix **F** always allows us to remove a single eliminable edge, the same is not true in general if we have to remove more than one eliminable edge simultaneously.

3. Graph-theoretic Approach To Decoupling By State Feedback

To develop the graph-theoretic approach to the decoupling, let us introduce some vertex subsets. Let V_i be the maximal subset of state vertices of $\mathfrak{g}(\mathbf{A},\mathbf{B},\mathbf{C})$ reaching the output vertex y_i with paths including no eliminable edge. Such a subset can be determined by the simple algorithm given in the Appendix.

Now let S_{V_i} indicate the subset of V_i containing all the input-adjacent vertices from V_i and put:

$$V=V_1 \cup V_2 \ldots \cup V_i \ldots \cup V_m \qquad (12)$$

and

$$S_V=S_{V_1} \cup S_{V_2} \ldots \cup S_{V_i} \ldots \cup S_{V_m} \qquad (13)$$

while W denotes the complementary subset of V with respect to X:

$$W = X - V \qquad (14)$$

To avoid trivialities let us assume that the system $(\mathbf{A},\mathbf{B},\mathbf{C})$ is output controllable. From the digraph point of view this implies that no subset S_{V_i} $(i=1,2,\ldots,m)$ is empty. In fact, output controllability implies the generic condition of structural output controllability. As proved by Franksen *et al* (1979), such a condition requires that every output vertex of $\mathfrak{g}(\mathbf{A},\mathbf{B},\mathbf{C})$ is reachable from at least one input vertex.

The graph-theoretic approach to the decoupling problem consists in finding a pair $(\mathbf{G},\mathbf{F})$ in such a way that the closed-loop system digraph $\mathfrak{g}(\mathbf{A}+\mathbf{BGF},\mathbf{BG},\mathbf{C})$ enjoys the following property: for each $i \in \{1,2,\ldots,m\}$ the output vertex y_i is reachable from the input vertex u_i, while no vertex y_j (for any $j \neq i$) is reachable from u_i.

Reinschke (1988) showed that a necessary condition for such a problem having a solution was given by the following theorem.

Theorem 1

For decoupling by static state feedback it is necessary that the state vertex subsets V_i $(i=1,2,\ldots,m)$ are disjoint.

As an immediate consequence of *Theorem 1* the non empty subsets S_{V_i} are disjoint too and

$$\text{cardinality of } S_V \geq m \qquad (15)$$

Now let $\mathbf{B}_{S_V}$ be the submatrix of $\mathbf{B}$ composed of the rows corresponding to the state vertices from S_V. A sufficient condition for the graph approach decoupling problem has a solution is given by the following result (Reinschke, 1988):

Theorem 2

Provided that the necessary condition of Theorem 1 is met, then the condition

$$\text{rank}(\mathbf{B}_{S_V}) = \text{cardinality of } S_V \qquad (16)$$

is sufficient for decoupling by static state feedback.

Let us assume that Theorems 1 and 2 hold true. In this case the computation of the resolving pair $(\mathbf{G},\mathbf{F})$ is straightforward, as we show next before concluding this section. To begin with, we first restrict ourselves to matrix G. Relations (15) and (16) imply

$$\text{rank}(\mathbf{B}_{S_V}) \geq m \qquad (17)$$

but, since $\mathbf{B}_{S_V}$ has m columns, only the equality sign must be considered in (17). Therefore we infer the following equations hold true:

$$\text{rank}(\mathbf{B}_{S_V}) = m \qquad (18)$$

$$\text{cardinality of } S_{V_i} = 1 \text{ for } i=1,2,\ldots,m \qquad (19)$$

and $\mathbf{B}_{S_V}$ must be a full rank square matrix of order m.

With (18) and (19) as background, let us re-arrange the state components according to the following state vertices sequence: V_1, V_2, $\ldots,V_m$, W, by considering in each subset V_i the vertex from S_{V_i} at first and, then, nodes from V_i-S_{V_i} $(i=1,2,\ldots,m)$. Moreover let us perform the corresponding row/column permutation on the state space equations (1) and (2). Abusing of the notation, the resulting matrices will be denoted by the same symbols as used for the original ones. It is clear that rearranging the state components does not affect at all the digraph associated with the system. On the contrary Eq. (3) introduces an input, $\mathbf{v}$, and new input vertices. However, according to Remark 1 the input vertex set and its members will further be indicated by U and u_i, respectively.

We are now in a position to determine $\mathbf{G}$ by imposing that in $\mathfrak{g}(\mathbf{A},\mathbf{BG},\mathbf{C})$ there be only one state vertex adjacent to u_i i.e. the only vertex in S_{V_i} $(i=1,2,\ldots,m)$. To this end it is necessary that the matrix

$$\mathbf{B}_{S_V}{}^* = \mathbf{B}_{S_V}\,\mathbf{G} \qquad (20)$$

be diagonal and non singular.

Before continuing, few remarks will provide the keys to understanding the structure of (**A**,**B**,**C**). Partitioning **B*** according to the partition of set X gives:

$$B^* = \begin{bmatrix} B^*_{V_1} & 0 & \dots & 0 \\ 0 & B^*_{V_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & B^*_{V_m} \\ B^*_{W_1} & B^*_{W_2} & \dots & B^*_{W_m} \end{bmatrix} \quad (21)$$

where $\mathbf{B}^*_{V_i}$ and $\mathbf{B}^*_{W_i}$ (i=1,2,...,m) are card(V_i)- and card(W)-column vectors, respectively. In particular each column $\mathbf{B}^*_{V_i}$ has the following form:

$$B^*_{V_i} = \begin{bmatrix} b^*_{S_{V_i}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (22)$$

with $b^*_{S_{V_i}} \neq 0$. The structures of (21) and (22) are easily explained by considering the following facts:

- all the vertices from V_i-S_{V_i} are not input-adjacent in $\mathfrak{g}$(**A**,**B**,**C**) by construction. They remain not input-adjacent in $\mathfrak{g}$(**A**,**BG**,**C**) by Proposition 1;
- the m x m submatrix $\mathbf{B}_{S_V}{}^*$ is diagonal.

Figure 1 shows the digraph $\mathfrak{g}$(**A**,**BG**,**C**).

Now it is to show how to determine the matrix **F**. Using the same partition as before for matrix **A** yields:

$$A = \begin{bmatrix} A_{V_1V_1} & \dots & A_{V_1V_m} & A_{V_1W} \\ \dots & \dots & \dots & \dots \\ A_{V_mV_1} & \dots & A_{V_mV_m} & A_{V_mW} \\ A_{WV_1} & \dots & A_{WV_m} & A_{WW} \end{bmatrix} \quad (23)$$

Blocks $\mathbf{A}_{V_iV_j}$, $\mathbf{A}_{V_iW}$, $\mathbf{A}_{WV_i}$ and $\mathbf{A}_{WW}$ have, respectively, the following dimensions: card(V_i) × card(V_j), card(V_i) × card(W), card(W) × card(V_i) and card(W) × card(W). Moreover the structures of $\mathbf{A}_{V_i V_j}$ (with i≠j) and $\mathbf{A}_{V_i W}$ are as follows:

$$A_{V_iV_j} = \begin{bmatrix} A_{S_{V_i}V_j} \\ 0 \end{bmatrix} \quad (24)$$

$$A_{V_iW} = \begin{bmatrix} A_{S_{V_i}W} \\ 0 \end{bmatrix} \quad (25)$$

where $\mathbf{A}_{S_{V_i}V_j}$ and $\mathbf{A}_{S_{V_i}W}$ are row vectors of proper sizes. Equations (24) and (25) are obtained by remarking that if (x_j,x_k) is an edge of $\mathfrak{g}$(**A**,**BG**,**C**) with $x_j \in V_i$, $x_k \in V_h$ and i≠h, then it must result: $x_k \in S_{V_h}$. Indeed, by definition of V_h, y_h is joined to x_k by a path containing no eliminable edge. Since $V_i \cap V_h$={0} by Theorem 1, the edge must be eliminable. As if this were not true x_j would belong to V_h. This makes us conclude that x_k must be input -adjacent, i.e. $x_k \in S_{V_h}$. The same conclusion is also drawn if $x_k \in V_h$ and $x_j \in W$. In fact in this case $(x_j,x_k) \in E_X$ requires $x_k \in S_{V_h}$.

The feedback matrix **F** can be partitioned as follows:

$$F = \begin{bmatrix} F_{1V_1} & \dots & F_{1V_m} & F_{1W} \\ \dots & \dots & \dots & \dots \\ F_{mV_1} & \dots & F_{mV_m} & F_{mW} \end{bmatrix} \quad (26)$$

where $\mathbf{F}_{i\,V_j}$ and $\mathbf{F}_{i\,W}$ (i=1,2,...m; j=1,2,...,m) are row vectors with card(V_i) and card(W) components, respectively. In order to obtain a decoupled system, **F** must be determined in such a way that each edge of $\mathfrak{g}$(**A**, **BG**, **C**) starting from any vertex in V_j or W and terminating in any vertex from S_{V_i} (for i,j=1,2,...,m and i≠j) should vanish in the new digraph $\mathfrak{g}$(**A**+**BGF**,**BG**,**C**). In other words, the edges of $\mathfrak{g}$(**A**,**BG**,**C**) to be removed are those corresponding to the entries of submatrices $\mathbf{A}_{S_{V_i}V_j}$ and $\mathbf{A}_{S_{V_i}W}$. This can immediately be accomplished by putting:

$$\mathbf{A}_{S_{V_i}V_j} + b^*_{S_{V_i}} \mathbf{F}_{iV_j} = \mathbf{0}$$

$$i,j=1,2,\dots,m \text{ and } i\neq j \quad (27a)$$

and

$$\mathbf{A}_{S_{V_i}W} + b^*_{S_{V_i}} \mathbf{F}_{iW} = \mathbf{0}$$

$$i=1,2,\dots,m \quad (27b)$$

The digraph for the closed-loop system is shown in Figure 2.

It must be remarked that the state components corresponding to vertices in W are made not output reachable by the feedback law: hence these components are unobservable in the system (**A**+**BGF**,**BG**,**C**). Another noteworthy fact is that Eq. (27a+b) imposes no constraint on submatrices F_{iV_i}: therefore these can be determined in order to satisfy further conditions, such as pole allocation.

4. Main Results

We point out that in order to apply the Reinschke design method, it is necessary that the digraph associated with the plant equations presents a particular structure. This condition is sever and seldom satisfied in practice. On the contrary, the Falb and Wolovich (1967) algebraic approach has a large applicability. The following theorem gives the necessary and sufficient condition for the existence of solving pair (**F**,**G**).

Theorem 3

Let $d_1, d_2, \ldots, d_m$ be m integers (decoupling indices) given by

$$d_i = \min \{j: \mathbf{c}_i \mathbf{A}^j \mathbf{B} \neq \mathbf{0}, j=0,1,\ldots,n-1\} \qquad (28)$$

or

$$d_i = n-1 \quad \text{if} \quad \mathbf{C}_i \mathbf{A}^j \mathbf{B} = \mathbf{0} \quad \text{for all } j \qquad (29)$$

with $\mathbf{c}_i$ being the i-th row of **C**. There exists a pair (**F**,**G**) solving the decoupling problem if and only if the following (m × m) matrix is non singular:

$$E = \begin{bmatrix} c_1 A^{d_1} B \\ c_2 A^{d_2} B \\ \cdots \\ c_m A^{d_m} B \end{bmatrix} \qquad (30)$$

The main result illustrated in this section will show that every time **E** is non singular it is possible to apply the decoupling graph-theoretic approach to a new state representation of the system (**A**,**B**,**C**)), equivalent to the original (**A**,**B**,**C**). So let us state the following theorem.

Theorem 4

If the system represented by (**A**,**B**,**C**) enjoys the necessary and sufficient condition of Theorem 3, then there exists a representation (**A**,**B**,**C**) equivalent to (**A**,**B**,**C**) such that $\mathfrak{g}$(**A**,**B**,**C**) satisfies the conditions stated by Theorems 1 and 2.

Proving Theorem 4 requires to introduce some notations and to state two lemmata.

Let us first group all the rows of **E** characterized by a decoupling index equal to zero. Let the resulting submatrix, indicated as ^{1}**CB**, be composed of m_1 rows. Obviously ^{1}**C** is a well defined ($m_1 \times n$) submatrix of **C**. Analogously, grouping together all the rows of **E** with unit decoupling index, the submatrices ^{2}**C A B** and ^{2}**C** are defined. Let m_2 denote the number of rows of each one of these blocks. Going on with this procedure, the m_i-rows submatrices $^i\mathbf{CA}^{i-1}\mathbf{B}$ and ^{i}**C** (for i=1,2,…,p) are defined, with dimensions ($m_i \times m$) and ($m_i \times n$), respectively. The integer $p \leq n$ is such that p-1 is the maximum decoupling index. Moreover it holds:

$$\sum_{i=1}^{p} m_i = m \qquad (31)$$

Now let us define the following matrices:

$$H = \begin{bmatrix} {}^1C \\ {}^2C \\ {}^2CA \\ {}^3CA^2 \\ \cdots \\ {}^iC \\ {}^iCA \\ {}^iCA^2 \\ \cdots \\ {}^iCA^{i-1} \\ \cdots \\ {}^pCA^{p-1} \end{bmatrix} \qquad (32)$$

$$H_1 = \begin{bmatrix} {}^1C \\ {}^2CA \\ \cdots \\ {}^iCA^{i-1} \\ \cdots \\ {}^pCA^{p-1} \end{bmatrix} \qquad (33)$$

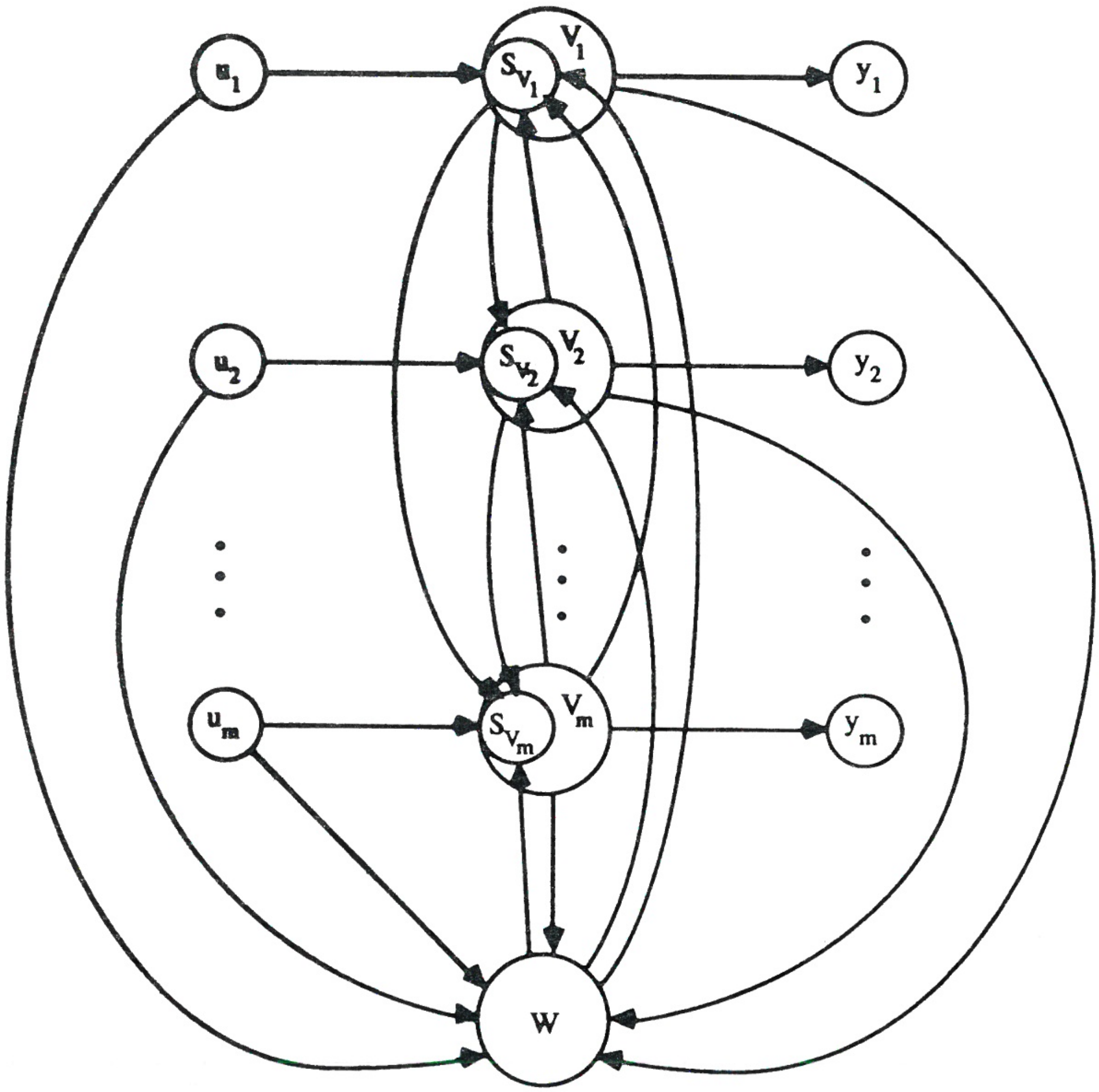

Figure 1. The Open Loop System Digraph g(A,BG,C).

$$H_2 = \begin{bmatrix} {}^2C \\ {}^3C \\ {}^3CA \\ {}^4C \\ \cdots \\ {}^iC \\ {}^iCA \\ {}^iCA^2 \\ \cdots \\ {}^iCA^{i-2} \\ \cdots \\ {}^pCA^{p-2} \end{bmatrix} \quad (34)$$

Putting

$$\mu = \sum_{i=1}^{p} i\, m_i \quad (35)$$

the dimensions of $\mathbf{H}$, $\mathbf{H_1}$ and $\mathbf{H_2}$ are respectively $\mu \times n$, $m \times n$ and $(\mu\text{-}m) \times n$.

Lemma 1

If matrix $\mathbf{E}$ given by (30) is non singular, then the following equation holds true:

$$\text{rank}(\mathbf{H}) = \text{rank}(\mathbf{H_2}) + m \quad (36)$$

Proof

By the construction of matrices $\mathbf{H_2}$ and $\mathbf{E}$ we infer:

$$\mathbf{H_2\, B} = \mathbf{0} \quad (37)$$

or, equivalently,

$$\text{Im}(\mathbf{H_2}^T) \subseteq \text{Ker}(\mathbf{B}^T) \quad (38)$$

where Im(•) and Ker(•) respectively stand for "the image of" and "the kernel of". The superscript T indicates transpose. Since the condition

$$\mathbf{H_1\, B} = \mathbf{E} \quad (39)$$

holds eventually but for a row permutation, it follows:

$$\text{rank}(\mathbf{H_1\, B}) = m \quad (40)$$

Considering the dimensions of $\mathbf{H_1}$ and $\mathbf{B}$, it is easy to realize that (39) gives

$$\text{Im}(\mathbf{H_1}^T) \cap (\text{Ker}(\mathbf{B}^T) = \{\mathbf{0}\} \quad (41)$$

By construction, matrices $\mathbf{H}$, $\mathbf{H_1}$ and $\mathbf{H_2}$ are in the following relation:

$$\text{Im}(\mathbf{H}^T) = \text{Im}(\mathbf{H_1}^T) + \text{Im}(\mathbf{H_2}^T) \quad (42)$$

therefore equations (38) and (41) give

$$\text{Im}(\mathbf{H}^T) = \text{Im}(\mathbf{H_1}^T) \oplus \text{Im}(\mathbf{H_2}^T) \quad (43)$$

and

$$\text{Rank}(\mathbf{H}) = \text{Rank}(\mathbf{H_1}) + \text{Rank}(\mathbf{H_2}) \quad (44)$$

But equation (40) implies

$$\text{Rank}(\mathbf{H_1}) = m \quad (45)$$

proving the lemma. •

Remark 1

Denoting by $(\mathbf{H_1})_i$ the i-th row of $\mathbf{H_1}$, a consequence of (43) and (45) is the following:

$$\text{Im}(\mathbf{H}^T) = \text{Im}[(\mathbf{H_1})_1]^T \oplus \text{Im}[(\mathbf{H_1})_2]^T \oplus \dots \oplus \text{Im}[(\mathbf{H_1})_m]^T \oplus \text{Im}(\mathbf{H_2}^T) \quad (46)$$

Lemma 2

If matrix $\mathbf{E}$ given by (30) is non singular, then it holds

$$\text{Rank}(\mathbf{H_2}) = \mu - m \quad (47)$$

Proof

Proceeding by contradiction, suppose the $(\mu - m)$ rows of $\mathbf{H_2}$ be linearly dependent. Hence there exists a non-zero $(\mu\text{-}m)$-row vector

$$\mathbf{z} = [z_1 \;\; z_2 \;\; \dots \;\; z_{\mu-m}] \quad (48)$$

such that

$$\mathbf{z\, H_2} = \mathbf{0} \quad (49)$$

Now let us consider all the submatrices ${}^i\mathbf{C\, A}^j$ of $\mathbf{H_2}$ ($1 < i \leq p$ and $j < i - 1$) containing at least one row weighted by a non zero entry of $\mathbf{z}$ in the product (49). Among these submatrices, let ${}^{i^*}\mathbf{C\, A}^{j^*}$ be one characterized by the minimum value of the index difference (i-j). So, multiplying on the right by $\mathbf{A}^{i^*-j^*-1}$ both sides of (49), gives

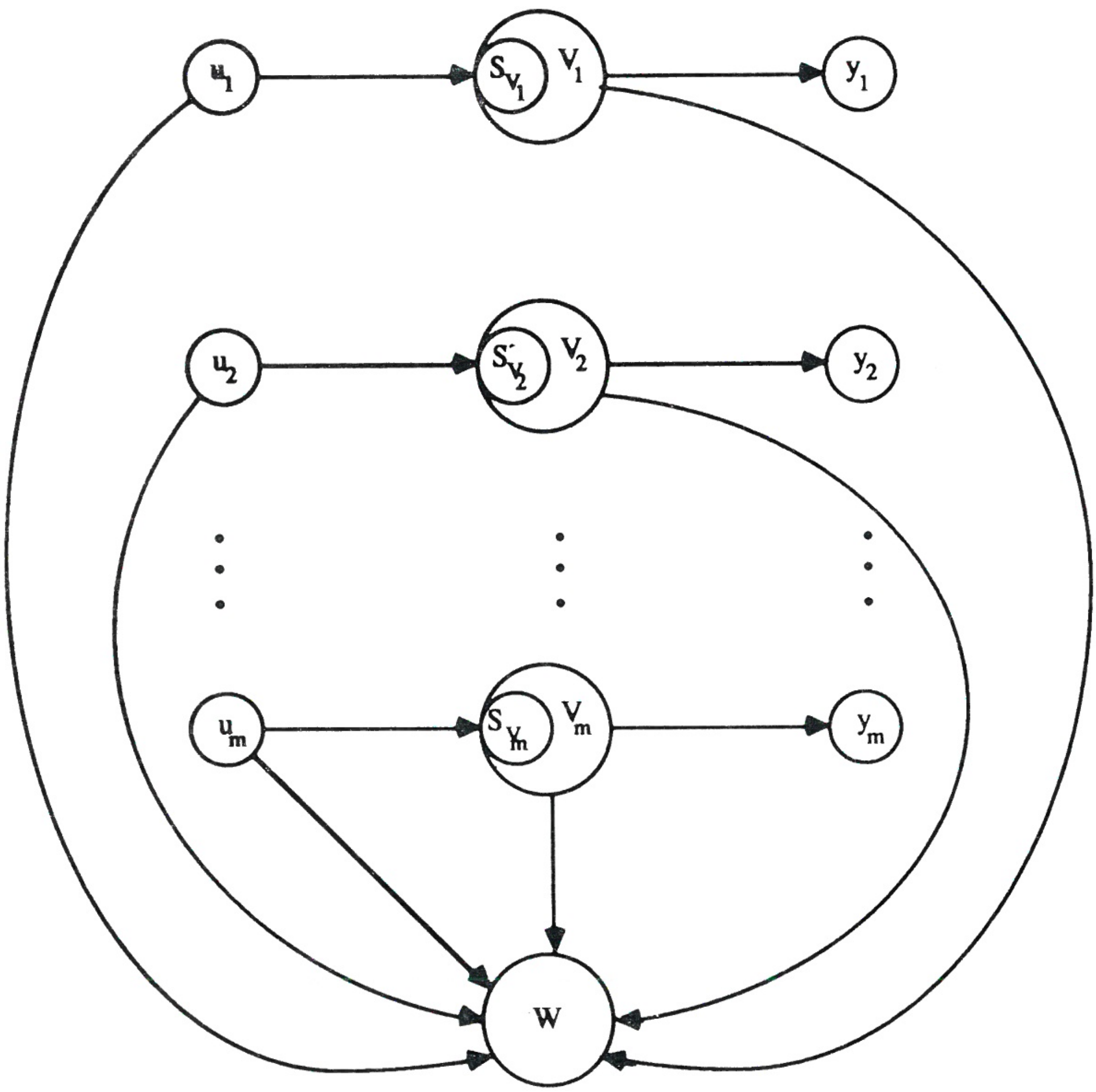

Figure 2. The Closed Loop System Digraph $g(A+BGF,BG,C)$.

$$z\,\mathbf{H}_2\,\mathbf{A}^{i^*-j^*-1} = \mathbf{0} \qquad (50)$$

Taking into account the meaning of indices i* and j*, it is easy to realize that the rows of $\mathbf{H}_2 \mathbf{A}^{i^*-j^*-1}$ weighted by non zero entries of **z**, are also rows of $\mathbf{H}_1$ or of $\mathbf{H}_2$. Therefore, equation (50) implies that at least one row of the submatrix ${}^{i^*}\mathbf{C}\,\mathbf{A}^{i^*-1}$ (i.e. a row of $\mathbf{H}_1$) can be expressed as a linear combination of the remaining (m-1) rows of $\mathbf{H}_1$ and of the rows of $\mathbf{H}_2$. But this contradicts (46), completing the proof. •

Remark 2

From the previous lemmata it follows that, if **E** is non singular, then the rank of matrix **H** is

$$\text{Rank}(\mathbf{H}) = \mu \qquad (51)$$

Obviously, being **H** a $(\mu \times n)$-matrix, it holds:

$$\mu \le n \qquad (52)$$

Proof of Theorem 4

Let **T** be a non-singular $(n \times n)$-matrix, given by

$$T = \begin{bmatrix} H \\ M \end{bmatrix} \qquad (53)$$

where **M** is a full rank $[(n-\mu)\times n]$-matrix, chosen such that each one of its rows is independent of the μ rows of **H**. Now consider the transformation

$$\underline{\mathbf{x}} = \mathbf{T}\,\mathbf{x} \qquad (54)$$

and partition $\underline{\mathbf{x}}$ as follows:

$$\underline{x} = \begin{bmatrix} \underline{x}_{1,0} \\ \underline{x}_{2,0} \\ \underline{x}_{2,1} \\ \underline{x}_{3,0} \\ \cdots \\ \underline{x}_{i,j} \\ \cdots \\ \underline{x}_{p,p-1} \\ \underline{x}_{M} \end{bmatrix} \qquad (55)$$

with:

$$\underline{\mathbf{x}}_{i,j} = {}^{i}\mathbf{C}\,\mathbf{A}^{j}\,\mathbf{x} \qquad i=1,2,\ldots,p; \quad j=0,1,\ldots,i-1 \qquad (56)$$

and

$$\underline{\mathbf{x}}_{M} = \mathbf{M}\,\mathbf{x} \qquad (57)$$

Consequently, each subvector $\underline{\mathbf{x}}_{i,j}$ (j=0,1,…,i-1) has m_i components, while $\underline{\mathbf{x}}_M$ has $(n-\mu)$ entries. It is to be remarked that m components of the state vector $\underline{\mathbf{x}}$ coincide with the system outputs; in fact, we get

$$\begin{bmatrix} \underline{x}_{1,0} \\ \underline{x}_{2,0} \\ \underline{x}_{3,0} \\ \cdots \\ \underline{x}_{p,0} \end{bmatrix} = \begin{bmatrix} {}^{1}C \\ {}^{2}C \\ {}^{3}C \\ \cdots \\ {}^{p}C \end{bmatrix} x = \underline{y} \qquad (58)$$

where $\underline{\mathbf{y}}$ is just the output vector **y**, but for an entry permutation. Let $\underline{\mathbf{y}}$ be partitioned as follows

$$\underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \cdots \\ \underline{y}_p \end{bmatrix} \qquad (59)$$

with

$$\underline{\mathbf{y}}_i = {}^{i}\mathbf{C}\,\mathbf{x} \qquad i=1,2,\ldots,p \qquad (60)$$

Let $\underline{\mathbf{y}}_i^{(j)}$ denote the j-th time derivative of $\underline{\mathbf{y}}_i$. Performing the first time derivative of (60) for i=2,…,p, gives:

$$\underline{\mathbf{y}}_i^{(1)} = {}^{i}\mathbf{C}\,\mathbf{A}\,\mathbf{x} + {}^{i}\mathbf{C}\,\mathbf{B}\,\mathbf{u} \qquad i=2,\ldots,p \qquad (61)$$

By the definition of matrix **E**, for i=2,…,p, it holds

$$ {}^{i}\mathbf{C}\,\mathbf{B} = \mathbf{0} \qquad (62)$$

therefore, considering (56), it follows:

$$\underline{\mathbf{y}}_i^{(1)} = {}^{i}\mathbf{C}\,\mathbf{A}\,\mathbf{x} = \underline{\mathbf{x}}_{i,1} \qquad i=2,\ldots,p; \qquad (63)$$

Computing the second derivative of $\underline{\mathbf{y}}_i$ and using the same arguments as above, we infer:

$$\underline{\mathbf{y}}_i^{(2)} = {}^{i}\mathbf{C}\,\mathbf{A}^2\,\mathbf{x} = \underline{\mathbf{x}}_{i,2} \qquad i=3,\ldots,p; \qquad (64)$$

Moreover, continuing on this way, the following general equation is obtained:

$$\underline{\mathbf{y}}_i^{(j)} = {}^{i}\mathbf{C}\,\mathbf{A}^{j}\,\mathbf{x} = \underline{\mathbf{x}}_{i,j} \qquad i=1,2,\ldots,p; \quad j=0,1,\ldots,i-1 \qquad (65)$$

Relations (65) also give

$$\underline{\mathbf{x}}_{i,j}^{(1)} = \underline{\mathbf{x}}_{i,j+1}$$

$$i=1,2,\ldots,p; \quad j=0,1,\ldots,i-2 \qquad (66)$$

Under the transformation (54), the original state model modifies to

$$\underline{\mathbf{x}}^{(1)} = \underline{\mathbf{A}}\,\underline{\mathbf{x}} + \underline{\mathbf{B}}\,\mathbf{u} \qquad (67)$$

$$\mathbf{y} = \underline{\mathbf{C}}\,\underline{\mathbf{x}} \qquad (68)$$

with:

$$\underline{\mathbf{A}} = \mathbf{T}\,\mathbf{A}\,\mathbf{T}^{-1} \qquad (69)$$

$$\underline{\mathbf{B}} = \mathbf{T}\,\mathbf{B} \qquad (70)$$

$$\underline{\mathbf{C}} = \mathbf{C}\,\mathbf{T}^{-1} \qquad (71)$$

Now let $\underline{\mathbf{x}}$ be partitioned as follows

$$\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \cdots \\ \underline{x}_i \\ \cdots \\ \underline{x}_p \\ \underline{x}_M \end{bmatrix} \qquad (72)$$

where $\underline{\mathbf{x}}_i$ denotes the following ($i\ m_i$)-subvector:

$$\underline{x}_i = \begin{bmatrix} \underline{x}_{i,0} \\ \underline{x}_{i,1} \\ \cdots \\ \underline{x}_{i,j} \\ \cdots \\ \underline{x}_{i,i-1} \end{bmatrix} \qquad i=1,2,\ldots,p \qquad (73)$$

If matrices $\underline{\mathbf{A}}$, $\underline{\mathbf{B}}$ and $\underline{\mathbf{C}}$ are partitioned according to the state vector partition (72), we get

$$\underline{x}_i^{(1)} = \sum_{j=1}^{p} \underline{A}_{ij}\,\underline{x}_j + \underline{A}_{iM}\,\underline{x}_M + \underline{B}_i u$$

$$i=1,2,\ldots,p \qquad (74)$$

and

$$y = \sum_{i=1}^{p} \underline{C}_i\,\underline{x}_i + \underline{C}_M\,\underline{x}_M \qquad (75)$$

where any symbol has an obvious meaning. Now let us perform a second level partition of $\underline{\mathbf{A}}_{ii}$, $\underline{\mathbf{A}}_{ij}$ ($i\neq j$), $\underline{\mathbf{A}}_{iM}$, $\underline{\mathbf{B}}_i$, according to (73). Equations (66) give the following structures for these matrices:

$$\underline{\mathbf{A}}_{ii} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} \\ * & * & * & \cdots & * \end{bmatrix} \begin{matrix} \leftarrow m_i \text{ rows} \\ \leftarrow m_i \text{ rows} \\ \\ \leftarrow m_i \text{ rows} \\ \leftarrow m_i \text{ rows} \end{matrix} \quad i\ m_i \text{ rows} \qquad (76)$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ m_i & m_i & m_i & & m_i \\ \text{columns} & \text{columns} & \text{columns} & & \text{columns} \end{matrix}$$

$i\ m_i$ columns

$$\underline{\mathbf{A}}_{ij} = \begin{bmatrix} \mathbf{0} \\ * \end{bmatrix} \begin{matrix} \leftarrow (i\text{-}1)\ m_i \text{ rows} \\ \leftarrow m_i \text{ rows} \end{matrix} \qquad (77)$$

$\uparrow$ $j\,m_j$ columns

$$\underline{\mathbf{A}}_{iM} = \begin{bmatrix} \mathbf{0} \\ * \end{bmatrix} \begin{matrix} \leftarrow (i\text{-}1)\ m_i \text{ rows} \\ \leftarrow m_i \text{ rows} \end{matrix} \qquad (78)$$

$\uparrow$ $n\text{-}\mu$ columns

$$\underline{\mathbf{B}}_i = \begin{bmatrix} \mathbf{0} \\ \underline{\mathbf{B}}_{i,i-1} \end{bmatrix} \begin{matrix} \leftarrow (i\text{-}1)\ m_i \text{ rows} \\ \leftarrow m_i \text{ rows} \end{matrix} \qquad (79)$$

$\uparrow$ m columns

Any symbol * in the previous matrices indicates a generic submatrix. Taking (70) into account, we get

$$\underline{\mathbf{B}}_{i,i-1} = {}^{i}\mathbf{C}\,\mathbf{A}^{i-1}\,\mathbf{B} \qquad (80)$$

Moreover, by (60) it follows

$$\underline{\mathbf{C}}_M = \mathbf{0} \qquad (81)$$

Let us observe that equations (60) are equivalent to

$$\underline{\mathbf{y}}_i = \underline{\mathbf{x}}_{i,0} \quad i=1,2,\ldots,p \qquad (82)$$

therefore, submatrices $\underline{\mathbf{C}}_i$ exhibit the following structure

$$\underline{\mathbf{C}}_i = \begin{bmatrix} \underline{\mathbf{C}}_{i,0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \end{bmatrix} \leftarrow m \text{ rows} \qquad (83)$$

$\uparrow$ m_i columns, $\uparrow$ m_i columns, $\uparrow$ m_i columns, …, $\uparrow$ m_i columns

$i\ m_i$ columns

where, according to (60), $\underline{\mathbf{C}}_{i,0}$ has $(m\text{-}m_i)$ zero rows, while the remaining m_i rows make up an unit matrix. Equations (76)-(83) allow to draw the digraph $\mathfrak{g}(\underline{A},\underline{B},\underline{C})$ as shown in Figure 3. In particular such a Figure displays all the edges directed to and from vertices associated with subvector $\underline{x}_i$ ($i \in \{1,2,\ldots,p\}$). Any edge incident with a vertex set signifies an aggregate of edges each one incident with a single vertex in the set. The structure exhibited by Figure 3 points out that the digraph $\mathfrak{g}(\underline{A},\underline{B},\underline{C})$ enjoys the condition required by Theorem 1.

In order to prove that the state model ($\underline{A}$, $\underline{B}$, $\underline{C}$) also satisfies the hypothesis of Theorem 2, let us consider the matrix $\underline{B}_{SV}$, which, minding Figure 3, is given by

$$\underline{B}_{SV} = \begin{bmatrix} \underline{B}_{1,0} \\ \underline{B}_{2,0} \\ \cdots \\ \underline{B}_{i,i-1} \\ \cdots \\ \underline{B}_{p,p-1} \end{bmatrix} \quad (84)$$

By (80), $\underline{B}_{SV}$ can be written as follows:

$$\underline{B}_{SV} = \begin{bmatrix} {}^{1}CB \\ {}^{2}CAB \\ \cdots \\ {}^{i}CA^{i-1}B \\ \cdots \\ {}^{p}CA^{p-1}B \end{bmatrix} \quad (85)$$

i.e. $\underline{B}_{SV}$ coincides with **E**, but for a row permutation. Therefore it can be inferred that:

$$\mathrm{rank}(\underline{B}_{SV}) = m = \text{cardinality of } S_V \quad (86)$$

completing the proof. •

We call the systems in the representation ($\underline{A}$, $\underline{B}$, $\underline{C}$) shown by Eqs. (74)-(75) in a "decoupling canonical form". If a system can be transformed into a similar canonical form, it is possible to apply the graph-theoretic approach and to derive the pair (**G**,**F**) solving linear equations (27a,b).

5. Example

In order to test the proposed method, let us consider the system consisting of a round rotor synchronous machine supplying an infinite busbar through a transformer and a transmission line (Figure 4). To derive the open-loop system representation, a third order model based on Park's equations is adopted for the machine (Schackshaft, 1963). The machine is assumed to be equipped by an exciter and a speed governing system. The state space representation of the systems, resulting from the cascade connection of the synchronous machine and the speed and voltage regulating systems is given by the following equations:

$$\mathbf{x}^{(1)}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t) \quad (87)$$

$$\mathbf{y}(t)=\mathbf{C}\mathbf{x}(t) \quad (88)$$

where $\mathbf{x}=[\Delta\delta\ \Delta\omega\ \Delta V_t\ \Delta E_{fd}\ \Delta P_m\ \Delta V_A\ \Delta g]^T$, the input vector $\mathbf{u}=[\Delta V_R\ \Delta\omega_R]^T$ and the output vector $\mathbf{y}=[\Delta V_t\ \Delta\delta]^T$. The matrices **A**, **B** and **C** are the following:

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -83.891 & -0.691 & -266.158 & 0 & 62.832 & 0 & 0 \\ -0.063 & 0.038 & -0.730 & 0.060 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 0 & -10 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -10 \end{bmatrix} \quad (89a)$$

$$B=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 400 & 0 \\ 0 & 25 \end{bmatrix} \quad C=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (89b)$$

Figure 5 points out that the digraph $\mathfrak{g}$(**A**, **B**, **C**) does not satisfy the conditions stated by Theorem 1. Moreover, the system (**A**, **B**, **C**) enjoys the necessary and sufficient condition of Theorem 3 with the decoupling indices $d_1=2$ and $d_2=3$. The graph-theoretic decoupling technique is used in Carnimeo *et al* (1992) to synthesise the controller. Moreover, we utilize the non singular transformation matrix **T** to carry the system (**A**, **B**, **C**) in its decoupling canonical form:

$$T = H = \begin{bmatrix} {}^{3}C \\ {}^{3}CA \\ {}^{3}CA^{2} \\ {}^{4}C_{3} \\ {}^{4}CA \\ {}^{4}CA^{2} \\ {}^{4}CA \end{bmatrix} = \quad (90a)$$

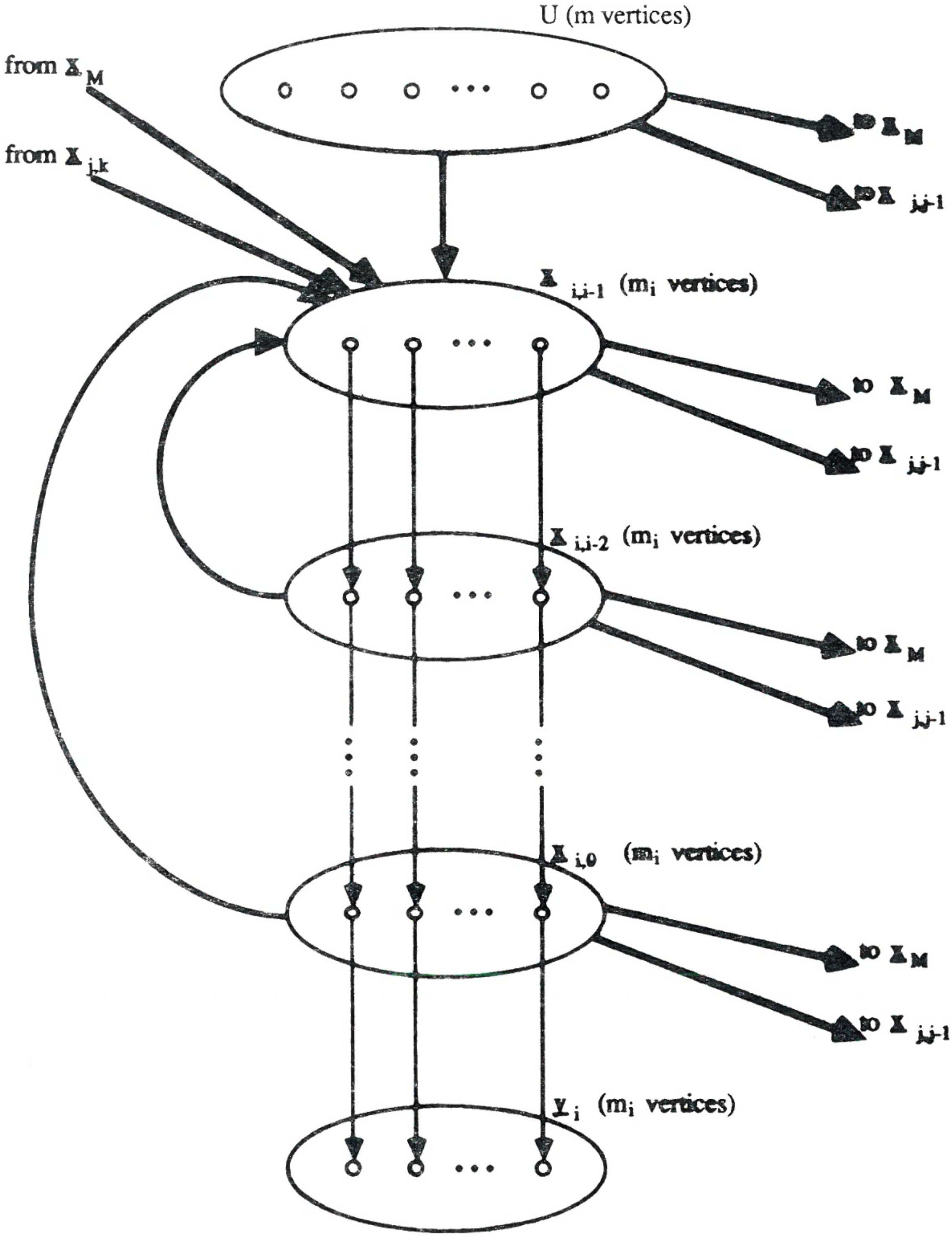

Figure 3. The Digraph g(A,BG,C) of the System (A,B,C) Equivalent to g(A,BG,C).

$$
= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.063 & 0.038 & -0.730 & 0.058 & 0 & 0 & 0 \\ -3.160 & -0.117 & -9.638 & -1.200 & 2.401 & 57.894 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -83.891 & -0.691 & -266.158 & 0 & 62.832 & 0 & 0 \\ 74.741 & -93.584 & 378.256 & -15.409 & -671.745 & 0 & 62.832 \end{bmatrix} \tag{90b}
$$

Under the transformation $\underline{\mathbf{x}} = \mathbf{T}\ \mathbf{x}$, vector $\underline{\mathbf{x}}$ is partitioned as follows:

$$
\underline{x} = \begin{bmatrix} \underline{x}_{3,0} \\ \underline{x}_{3,1} \\ \underline{x}_{3,2} \\ \underline{x}_{4,0} \\ \underline{x}_{4,1} \\ \underline{x}_{4,2} \\ \underline{x}_{4,3} \end{bmatrix} \tag{91}
$$

Since m_3=m_4=1, each subvector in right hand size of equation (91) is just a scalar; i.e.

$$
\underline{x} = \begin{bmatrix} \underline{x}_{3,0} \\ \underline{x}_{3,1} \\ \underline{x}_{3,2} \\ \underline{x}_{4,0} \\ \underline{x}_{4,1} \\ \underline{x}_{4,2} \\ \underline{x}_{4,3} \end{bmatrix} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \\ \underline{x}_4 \\ \underline{x}_5 \\ \underline{x}_6 \\ \underline{x}_7 \end{bmatrix} \tag{92}
$$

The transformation modifies the original triple (**A**, **B**, **C**) to the following triple ($\underline{\mathbf{A}}$,$\underline{\mathbf{B}}$,$\underline{\mathbf{C}}$):

$$
\underline{A} = \begin{bmatrix} \underline{A}_{33} & \underline{A}_{34} \\ \underline{A}_{43} & \underline{A}_{44} \end{bmatrix} = \tag{93}
$$

$$
= \left[\begin{array}{ccc|cccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -730 & -1051 & -71 & -63 & 34 & 3 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 26616 & -5323 & -266 & -8389 & -1747 & -198 & -21 \end{array}\right]
$$

$$
\underline{B} = \begin{bmatrix} \underline{B}_3 \\ \underline{B}_4 \end{bmatrix} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 23158 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1571 \end{array}\right] \tag{94}
$$

$$
\underline{C} = \begin{bmatrix} \underline{C}_3 & \underline{C}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{95}
$$

Considering the digraph g($\underline{\mathbf{A}}$, $\underline{\mathbf{B}}$, $\underline{\mathbf{C}}$) shown in Figure 6, the following vertex subset can be derived:

$V_1=\{\underline{x}_1, \underline{x}_2, \underline{x}_3\}$ $\qquad$ $V_2=\{\underline{x}_4, \underline{x}_5, \underline{x}_6, \underline{x}_7\}$

$S_{V1}= \{\underline{x}_3\}$ $\qquad$ $S_{V2} =\{\underline{x}_7\}$

We can verify that conditions required by Theorems 1 and 2 hold true. Hence, solving the corresponding equations (27a,b), matrix $\underline{\mathbf{F}}$ can be obtained:

$$
\underline{F} = \begin{bmatrix} 0 & 0 & 0 & .0027 & -.0015 & -.0001 & 0 \\ 16.9442 & 3.3888 & .1694 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{96}
$$

Finally, matrix $\mathbf{F} = \underline{\mathbf{F}}\ \mathbf{T}$ is given by:

$$
F = \begin{bmatrix} .0121 & -.0012 & .0294 & 0 & -.0060 & 0 & -.0001 \\ -.7488 & .1096 & 12.8371 & -.0072 & .4068 & 9.8097 & 0 \end{bmatrix} \tag{97}
$$

6. Conclusions

This paper proposes a method for overcoming the limitations of the graph-theoretic approach to the decoupling problem, introduced by Reinschke. We prove that any system which can be decoupled by state feedback controller, can be reduced in a canonical form, with a representative digraph that enjoys the required properties. So, a state feedback control law can easily be derived to decoupling the transformed system. A numerical example applies the approach to a system consisting of a round rotor synchronous machine. The example shows that the method is effective and simple in its application.

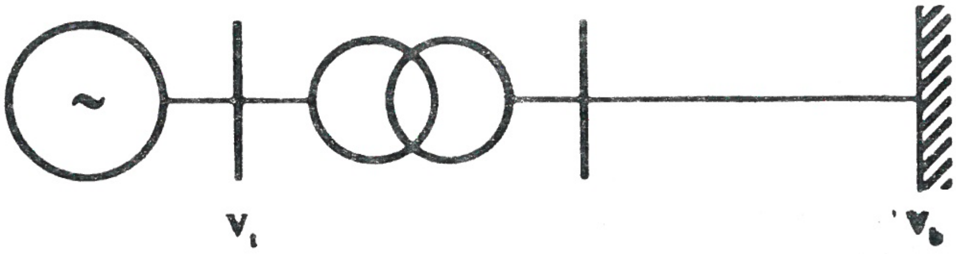

Figure 4. One-Line Diagram of the Studied System.

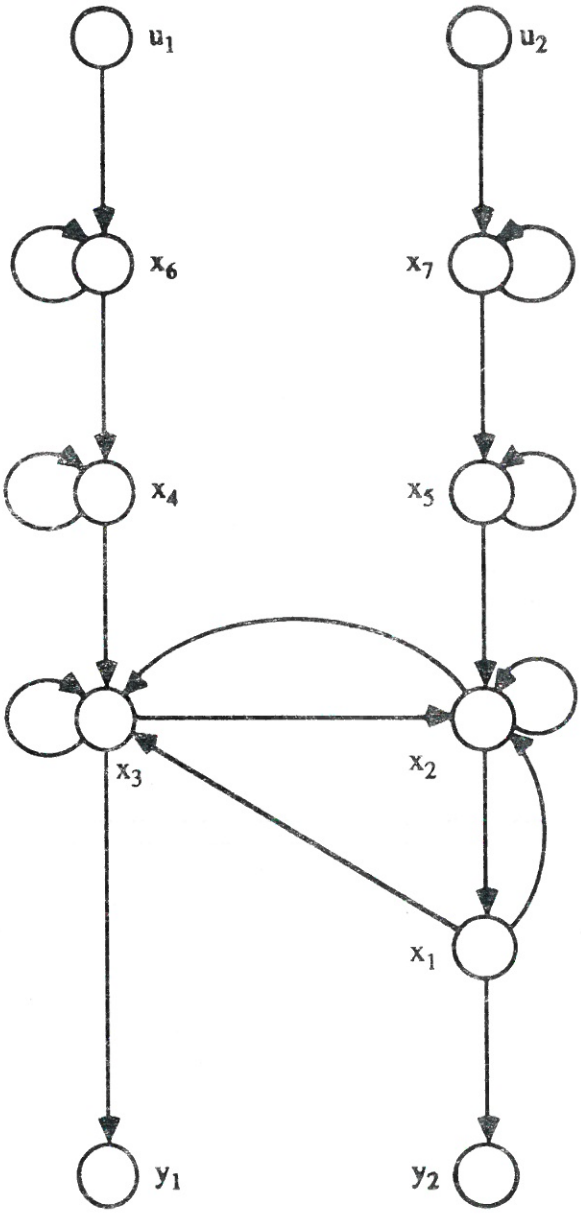

Figure 5. The Open Loop Digraph g(A,B,C).

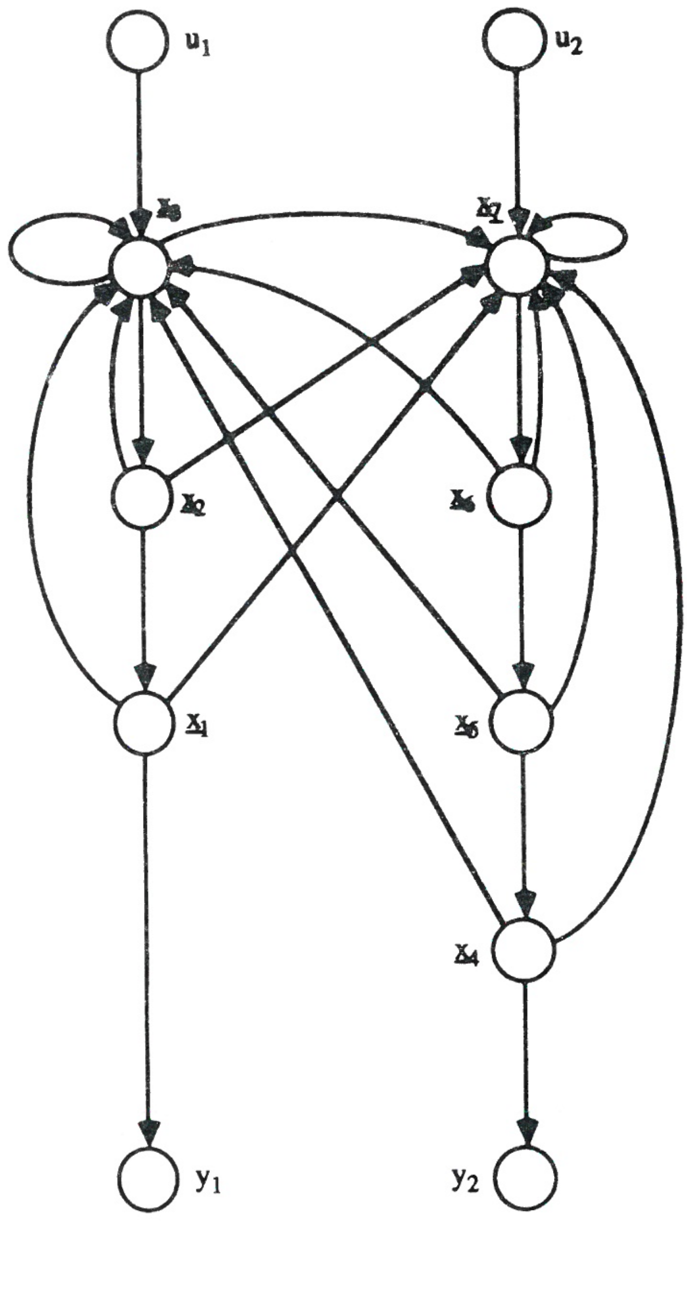

Figure 6. The Digraph g(A,B,C) of the Transformed System (A,B,C).

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Appendices

A1. Algorithm to determine V_i subsets

Step 1: Set k=1 and determine $V_i^{(k)}$ as the set of state vertices adjacent to vertex y_i.

Step 2: Determine $V_i^{(k+1)}$ as the set containing $V_i^{(k)}$ and all the state nodes adjacent to some non input-adjacent vertices from $V_i^{(k)}$.

Step 3: If $V_i^{(k+1)}=V_i^{(k)}$ put $V_i=V_i^{(k)}$ and STOP; otherwise put k=k+1 and go back to Step 2.

A2. List of symbols and system data

V_t	generator terminal voltage, p.u.
V_b	infinite busbar voltage, p.u.
δ	rotor angle with respect to infinite busbar, rad
f_0	rated frequency = 60 Hz
ω	rotor speed, rad/sec
H	inertia constant, s
P_m	mechanical power, p.u.
E_{fd}	generator field voltage, p.u.
x	reactance, p.u.
x_e	total reactance between generator terminals and infinite busbar, p.u.
P, Q	active and reactive power at generator terminals, p.u.
D	damping coefficient, p.u. MWs/rad
T'_{do}	open circuit time constant, s
g	steam-valve position, p.u.

T_G governor time constant, s

T_{ch} steam chest time constant, s

T_E, T_A exciter/regulator time constant, s

The system data are listed in the following (in p.u. unless other units are indicated):

$P=0.6$; $Q=0.3$; $V_b=1$;

$x_d=1.6$; $x'_d=0.32$; $x_q=1.55$; $x_{md}=1.5$;

$H=3s$; $D = 0.011$ p.u. MWs/rad ;

$T'_{do}=5s$; $x_e = 0.163$;

$K_E=50$; $T_E=0.05s$; $E_{fdmax} = 4.6$; $E_{fdmin}= -4.6$;

$K_A= 8$; $T_A = 0.02s$;

$F_{HP}=1$; $T_{ch}=0.1s$;

$K_G=2.5$; $T_G=0.1$.

All p.u. values have been referred to a 100 MVA base.