Conditions for Point Contact Between Robot Arm and An Object for Manipulation Modelling and Control

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Abstract: It has been demonstrated through past research that point contact is required between the robot arm and an object for effective modelling and control of manipulation. In this work we investigate sufficient conditions in order that the end effector of the robot and the object have point contact. Both the end effector and the object are rigid surfaces described by smooth equations in the 3-dimensional space. The analysis is built on the basic knowledge of Differential Geometry and particularly on the concept of the principal curvatures of a surface.

Keywords: Robots, Object Manipulation, Contact, Point Contact.

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1 Introduction

There is definitely a need for modelling of contact as well as for a description of the dynamics involved in problems of object manipulation by robotic arms. An appropriate model of these problems will result in more effective control schemes with many applications. There is a lot of research effort made towards this direction in a number of related fields like constrained dynamics, robotic hands and robotic control. One of the assumptions for the construction of a model of object manipulation is that the surfaces of the end effector of the manipulator and the object are rigid surfaces and point contact comes between the two surfaces (Figure 1).

In the present work we reduce the problem to that of point contact between two rigid surfaces. Thus we investigate for conditions which are sufficient that two smooth surfaces come into point contact. Our investigation is based on

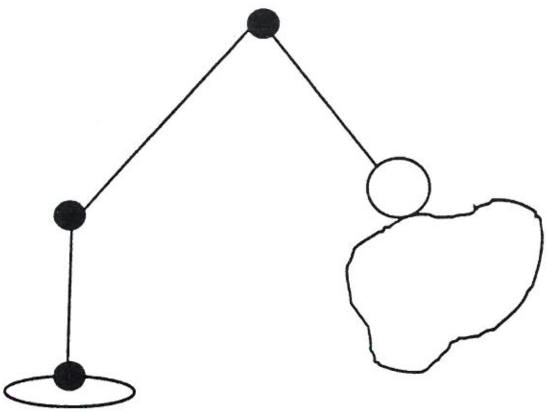


Figure 1: Object Manipulation by a Robot Arm with Point Contact

early results by H. W. Guggenheimer [3] which include conditions for point contact of two smooth plane curves using their curvature. We will attempt to extend those results using similar criteria for smooth surfaces. The key concept will be that of the principal curvatures of the surfaces. We extend our results to areas of points on the surface of the object which may come into point contact with areas on the surface of the end effector of the robot arm. This will allow us to get a clear view on the model of the constrained dynamics of the object and to construct an appropriate control algorithm in future work. In this work we require that the surfaces lie in $\Re^3$, and are defined by a function of the form $f(x, y, z) = 0$ at least locally.

2 Contact between Objects and Surfaces

Two surfaces are in point contact if they have one common point which is not a point of intersection. Throughout this work we will mean point contact whenever we talk about contact unless it is explicitly mentioned. It is noted that we will not investigate cases where we have multiple points of contact or parts of the surfaces in contact; we leave these cases for future research. An intersection point cannot be considered as a contact point. A simple way of excluding such points is to require that the two surfaces share a common tangent plane at each common point, Figure 2(a).

However, the condition of tangency does not exclude undesirable cases, see for example cases (b) and (c) in Figure 2. Even though the surfaces are tangent at the common point, because of the relative curvature of the surfaces they can intersect each other in a neighbourhood of the point of contact (b). In case (c) of Figure 2, contact occurs in the interior of the body (or surface) without violating the tangency condition. This is not physically meaningful since we are dealing with rigid bodies. These situations can be avoided by defining an orientation of the surfaces of the bodies, and by requiring, in addition to tangency, that each body (or surface) be in the exterior of each other. This added assumption is equivalent to requiring that the bodies and the surfaces are smooth orientable embedded manifolds in $\Re^3$.

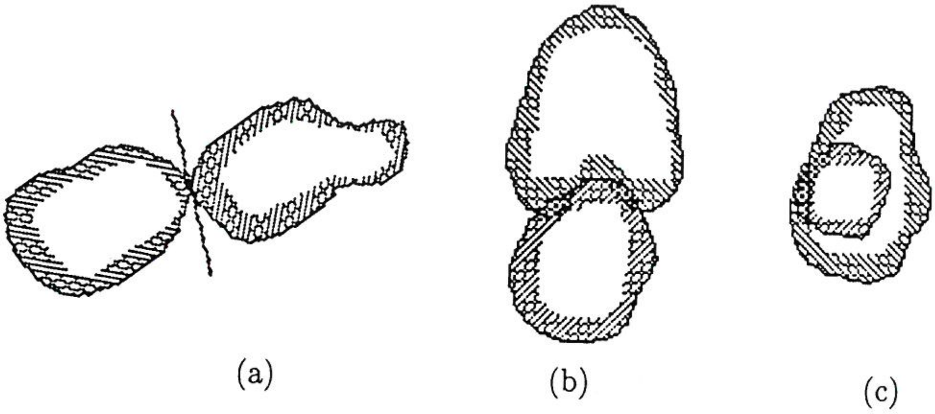


Figure 2: Condition for Tangency Does Not Guarantee Contact

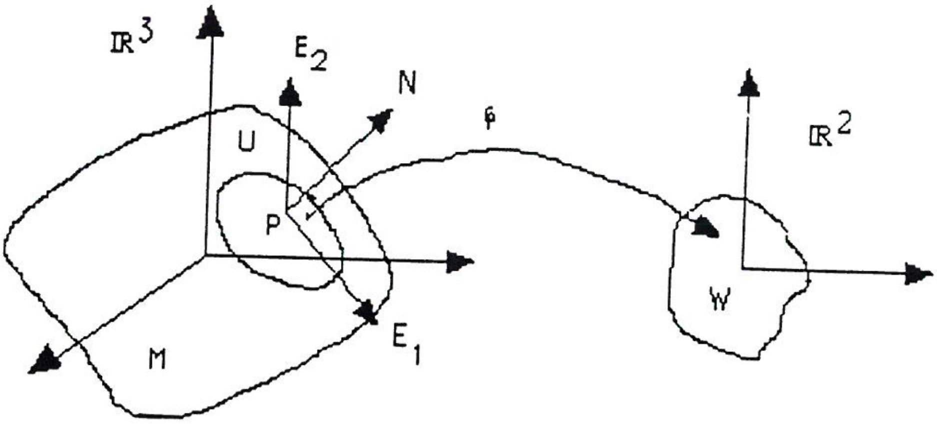


Figure 3: Embedded Surface in $\Re^3$

3 Curvature of Surfaces Embedded in $\Re^3$

The curvature of two surfaces, at two particular points (one on each surface), can be used to determine if the surfaces can come in contact at these points. We need define criteria which involve the curvature of surfaces in order to be able to decide in each case whether contact is possible. It is easy to determine curvature for surfaces which are embedded submanifolds in $\Re^3$.

Consider a surface M embedded in $\Re^3$, and a chart (U, ϕ) as shown in Figure 3, the map ϕ is continuous, and $W = \phi(U) \subset \Re^2$, where U is an open subset of M. Consider a point $p \in M$ then for every point p we can find a pair of vectors E_{1p}, E_{2p} such that E_{1p}, E_{2p} are tangent to the surface M and orthonormal. The vectors E_{1p}, E_{2p} also span the tangent space of M at the point p, and T_pM is considered as a subspace of $T_p\Re^3$. We assign to each point $p \in M$ of the surface the Gauss vector N_p which is the unit normal vector field of the surface M i.e. $N_p \perp T_pM$. The vectors E_{1p}, E_{2p} and N_p are chosen so that the triplet E_{1p}, E_{2p}, N_p is an orthonormal frame for $T_p\Re^3$ and N_p is consistent with the orientation of the surface M. The orientation of the surface coincides with the orientation of the solid body. This means that the internal part of the solid body is the same with the internal part of the surface. Since N_p is defined for all the points $p \in M$, we can consider the unit normal vector field N defined over M.

Let $c : I \longrightarrow M$ be a smooth curve, and assume that $p = c(0)$, where $p \in M$, with tangent vector $\dot{c}(0) = X_p \in T_pM$. We restrict the unit normal vector field N to the curve $c(t)$ such that $N(t) = N \mid_{c(t)}$, then $N(t)$ is a vector field defined on $c(t)$. Since $c(t)$ is also a curve in $\Re^3$, the derivative of the vector field $N(t)$ along $c(t)$ can be easily computed. We denote this by $\frac{dN}{dt}$ [2,p.298] , and the following theorem results for $\frac{dN}{dt}$ [2,p.367]:

Theorem: The vector $\frac{dN}{dt} \mid_{t=0}$ is independent of $c(t)$ and depends only on X_p. Let $S(X_p) = -\frac{dN}{dt} \mid_{t=0}$, then $X_p \mapsto S(X_p)$ is a linear map from T_pM to T_pM. $\triangle$

Using an analogy from linear algebra, we define a bilinear form:

$$\Psi(X, Y) = (S(X), Y)$$

where $(\cdot, \cdot)$ is the metrics on TM, considered as a subspace of $T\Re^3$, and Ψ is a co-variant tensor of order 2. The bilinear form Ψ has very useful properties which are apparent from the following two theorems, [2,p.368] and [2,p.370] :

Theorem: $S(X)$ is a symmetric operator on the tangent space TM and $\Psi(X, Y)$ is a symmetric covariant tensor of order 2. The components of S and Ψ are C^∞ if M is a C^∞ submanifold in $\Re^3$. $\triangle$

Theorem: At each point $p \in M$, the eigenvalues of the linear transformation S are real numbers k_1 and k_2, and we assume that $k_1 \geq k_2$. If $k_1 \neq k_2$, then the associated eigenvectors are orthogonal. If $k_1 = k_2 = k$ at p, then $S(X_p) = kX_p$ for every vector $X_p \in T_pM$. The numbers k_1 and k_2 are, respectively, the maximum and the minimum values of $\Psi(X_p, X_p) = (S(X_p), X_p)$, over all unit vectors $X_p \in T_pM$. $\triangle$

The two numbers k_1 and k_2 are the *principal curvatures* along the orthogonal directions F_{1p}, F_{2p} tangent to the surface, which are called *principal directions*, see Figure 4. The triad F_{1p}, F_{2p}, N_p is an orthonormal coordinate frame which conforms with the orientation of the surface. The meaning of the principal curvatures, at a point p, can be grasped by considering the plane F_{1p}, N_p, see Figure 4. This plane intersects the surface M along a curve $\gamma_1(t)$. The curvature of a plane curve is a well- defined quantity and k_1 is the curvature of the curve $\gamma_1(t)$ at p, considered as a planar curve on the plane F_{1p}, N_p. In a similar way, k_2 is the curvature of the curve $\gamma_2(t)$ at p on the plane F_{2p}, N_p. Any curve which is the intersection of an embedded surface M and a plane spanned by a tangent vector X_p and the normal vector N_p, at a point $p \in M$, is called a *normal section.* Thus γ_1 and γ_2 are normal sections.

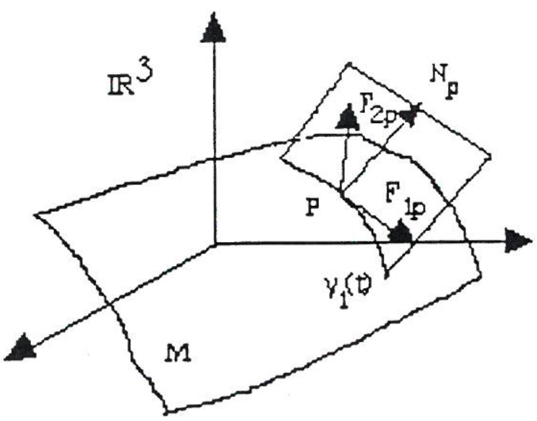


Figure 4: A normal Section on a Surface M in $\Re^3$

For a plane curve, the curvature at a point is defined as the rate of change of the tangent vector to the curve at the point. Any tangent vector at the point p can be written as a combination of F_{1p} and F_{2p}. Thus, the principal curvatures at a point p on a surface can be used to characterize the curvature of any normal section passing through this point. Let $X_p \in T_pM$ be a unit tangent vector, and assume that it is oriented to a direction which forms an angle θ with F_{1p}, see Figure 5.

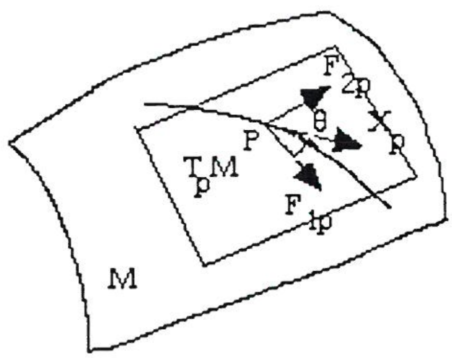


Figure 5: A Tangent Vector at Point p Forming an Angle θ with F_{1p}

Then, $X_p = F_{1p}cos(\theta) + F_{2p}sin(\theta)$, since F_{1p}, F_{2p} is an orthonormal frame which spans T_pM. Since k_1, k_2 are the eigenvalues, and F_{1p}, F_{2p} are the corresponding eigenvectors of the linear map S, then:

$$k_1(F_{1p}, F_{1p}) = (S(F_{1p}), F_{1p})$$

$$k_2(F_{2p}, F_{2p}) = (F_{2p}, S(F_{2p})).$$

Let $\gamma(t)$ be the normal section of the tangent vector X_p and the normal vector N_p, see Figure 5. Assume that $\gamma(0) = p$ and $\dot{\gamma}(0) = X_p$, then the curvature of the normal section $\gamma(t)$, as a plane curve at the point p, is given by the quantity $(N_p, \frac{d^2\gamma}{dt^2} \mid_p)$, where N_p is the unit normal vector at the point p. Since $(N_p, \frac{d\gamma}{dt} \mid_p) = 0$, then $(\frac{dN}{dt} \mid_p, \frac{d\gamma}{dt} \mid_p) = -(N_p, \frac{d^2\gamma}{dt^2} \mid_p)$, and at the point $p = \gamma(0)$ of the curve $\gamma(t)$ the curvature is $k = -(\frac{dN}{dt} \mid_p, X_p) = (S(X_p), X_p)$. The unit normal vector field N, is constrained to the curve $\gamma(t)$, $N(t) = N \mid_{\gamma(t)}$, such that $\frac{dN}{dt}$ is well- defined. The curvature at the point p of the normal section $\gamma(t)$, is determined by the tangent vector X_p, is given by $k = (S(X_p), X_p) = \Psi(X_p, X_p)$, and can be expressed as a function of θ if we use the relation $X_p = F_1 cos(\theta) + F_2 sin(\theta)$:

$$k(\theta) = k_1 cos^2(\theta) + k_2 sin^2(\theta).$$

This formula is also known as *Euler's* formula.

4 Curvature as a Contact Criterion

In the previous section we have analysed how the principal curvatures define the curvature of any normal section on a smooth surface. Next, we are going to see how the curvature of two plane curves can be used to define conditions in order that two surfaces (or bodies) are brought in point contact. Consider first two curves in $\Re^2$, namely $c_1(t)$ and $c_2(t)$, which can be used to represent the surfaces of two plane rigid objects. We give an orientation to the two curves, such that the unit normal vector points to the exterior of the object, see Figure 6.

We want to find out conditions in order that the two curves should be in contact at the points $c_1(0)$ and $c_2(0)$, without intersecting the interior of each other. Contact means that the two curves have a common point, and that the tangents to the curves at this point coincide. Let the curvatures of $c_1(t)$ and $c_2(t)$ be $k_{c_1}(0)$ and $k_{c_2}(0)$ at the points $c_1(0)$ and $c_2(0)$, respectively. Given the orientation of the curve $c_1(t)$ for example, at the point $c_1(0)$ it has negative curvature since it is bending away from the normal. If it were bending towards the normal it would have been positive. The following theorem [3,p. 28] gives a condition for the two plane curves $c_1(t)$ and $c_2(t)$ so as to be in contact at $c_1(0)$ and $c_2(0)$:

Theorem: Let $c_1(t)$ and $c_2(t)$ be two curves in $\Re^2$. Then $c_1(t)$ and $c_2(t)$ can be in (exterior) contact at $t = 0$, if one of the following conditions holds:

i) $-k_{c_2}(0) > k_{c_1}(0)$.

ii) $-k_{c_2}(0) = k_{c_1}(0)$,

and there is an interval $I \subset \Re$ with $I = (-\epsilon, \epsilon)$, ϵ is a small positive number, such that $-k_{c_2}(t) > k_{c_1}(t)$ for $t \in I$ and $t \neq 0$. $\triangle$

This theorem states that contact without intersection can occur if the sign of the difference $(-k_{c_2}(t) - k_{c_1}(t))$ does not change at the point $t = 0$. Using this theorem we can avoid cases like (a) in Figure 7, where there is a discontinuity of the curvatures at the point $t = 0$ in both curves and as a result the curves intersect or as in case (b) in Figure 7, where $k_{c_2}(0) = k_{c_1}(0)$, and the curves coincide for an interval $t \in I$ around $t = 0$, and they might intersect at a point $t \neq 0$.

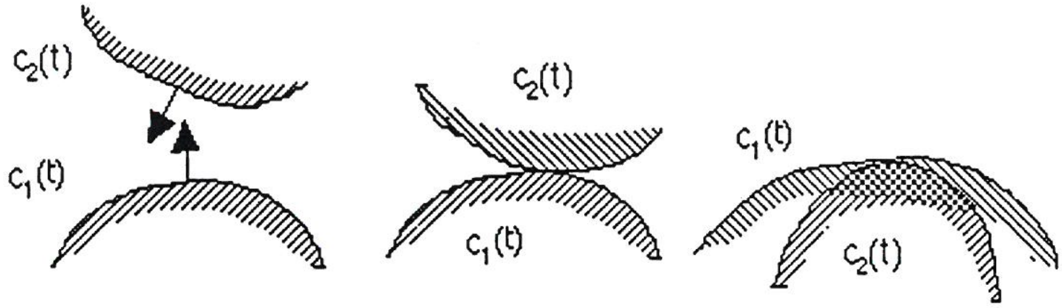
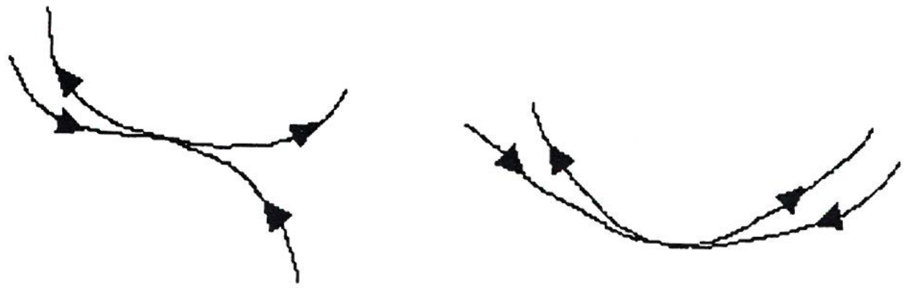


Figure 6: Two Plane Curves which Can Be in Contact



(a) (b)

Figure 7: Intersection between Curves which Are Tangent

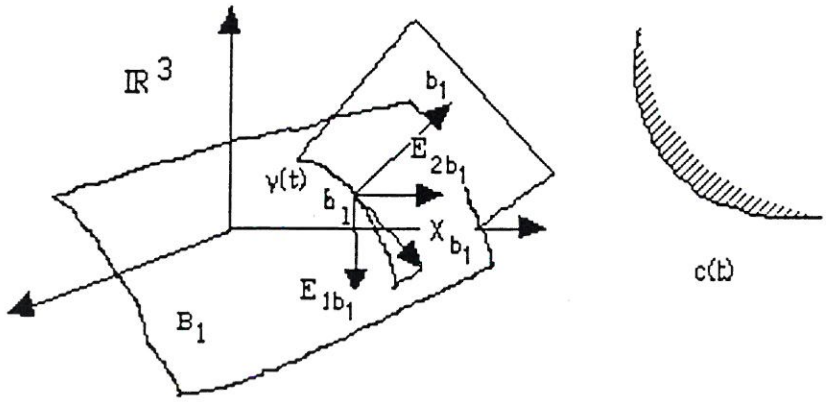


Figure 8: Surface B_1 and a Normal Section at a Point b_1

Let the surfaces B_1 and B_2 be smooth manifolds associated with two rigid bodies, and let b_1 and b_2 be two points on each surface, respectively. Sometimes, we use B_1 and B_2 to label the rigid bodies besides their surfaces. The principal curvatures at the points b_1 and b_2 can be used in checking if contact is possible between the surfaces at these points. Select one of the bodies as a reference and compare the curvature of the two bodies, let the reference surface be B_1. Initially, consider a normal section $\gamma(t)$ generated by a plane N_{b_1}, X_{b_1} at b_1, on the body B_1, such that $\gamma(0) = b_1$. We want to investigate if $\gamma(t)$ can be in contact with an arbitrary oriented planar curve $c(t)$ at the points $c_1 = c(0)$ and b_1, such that their interiors do not intersect. The interiors of the curves $\gamma(t)$ and $c(t)$ are defined by their orientation, for the curve $\gamma(t)$ it coincides with the interior of the body B_1. By contact in this case we require, of course, that the curve $c(t)$ lies on the same plane as $\gamma(t)$.

Let $k_\gamma(t)$ and $k_c(t)$ be the curvatures of $\gamma(t)$ and $c(t)$, respectively. Then, the curves can be in contact at the points b_1 and c_1 according to the theorem if $k_\gamma(0) < -k_c(0)$, or if $k_\gamma(0) = -k_c(0)$ and $k_\gamma(t) < -k_c(t)$ for $t \in I$ and $t \neq 0$. This illustrates how curvature can be used to construct criteria in order to investigate whether two smooth surfaces can be in contact at a given point. Consider an open neighbourhood $\mathcal{U} \subset B_1$, of a point b_1, such that all the normal sections $\gamma(t)$, satisfying $\gamma(0) = b_1$, also satisfy $\gamma(t) \in \mathcal{U}$ for $t \in I = (-\epsilon, \epsilon)$, for some small positive number ϵ. Let $k_1(u)$ and $k_2(u)$ denote the maximum and the minimum curvatures, respectively, at a point $u \in \mathcal{U}$. Notice that the maximum and the minimum curvatures at the point u, $k_1(u)$ and $k_2(u)$, are functions of the point u regardless of any passing curve through this point on the surface. Instead the curvature along a curve is a function of the parameter of the curve like $k_\gamma(t)$. Assume that $k_1(b_1) < -k_c(0)$ or if $k_1(b_1) = -k_c(0)$ then $\max_{u \neq b_1}(k_1(u)) < -k_c(t)$ for $t \in I$ and $t \neq 0$; then the curve $c(t)$ can be in contact with any normal section $\gamma(t)$ through the point b_1, since $k_\gamma(0) \leq (k_1(b_1)) < -k_c(0)$. If $k_1(b_1) = -k_c(0)$ then either $k_\gamma(0) < k_1(b_1)$ and thus we are covered by the first case or $k_\gamma(0) = k_1(b_1)$ and $k_\gamma(t) \leq \max_{u \neq b_1}(k_1(u)) < -k_c(t)$ for $t \in I$ and $t \neq 0$.

Next, consider the point b_2 on the surface B_2, and a normal section $\delta(t)$ satisfying $\delta(0) = b_2$. If we bring the surface B_2 into contact with the surface B_1 such that b_1 and b_2 coincide, then there can be an infinite number of common normal sections on both of the surfaces B_1 and B_2, see Figure 9. The contact of the two bodies is physically feasible if all pairs (γ, δ) of common normal sections on each body, have curvatures satisfying $k_\gamma(0) < -\lambda_\delta(0)$ or if $k_\gamma(0) = -\lambda_\delta(0)$ then $k_\gamma(t) < -\lambda_\delta(t)$ for $t \in I$ and $t \neq 0$. If this is possible for a certain position of the body B_2 relative to the body B_1, it means not that it applies to every relative configuration. In particular, if we rotate the body B_2 around the common normal at the point of the contact, it is possible to encounter a relative configuration where the interiors of certain common normal sections intersect.

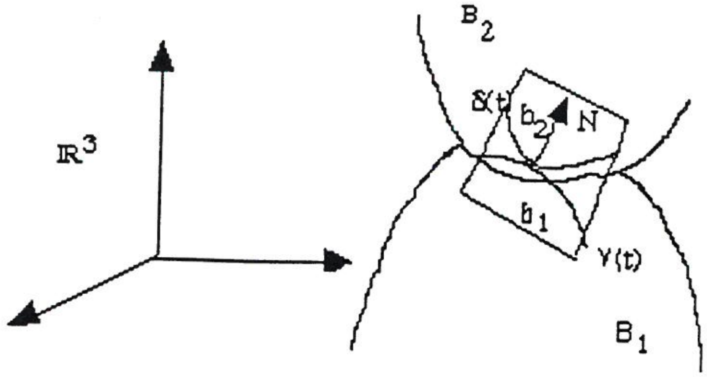


Figure 9: Corresponding Normal Sections on Two Bodies in Contact

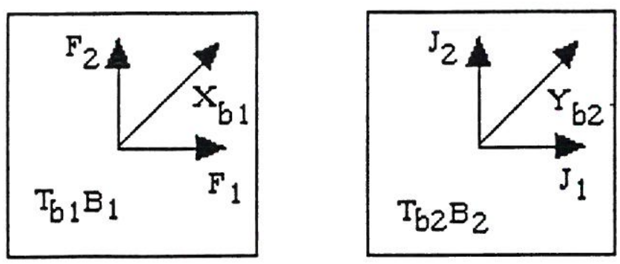


Figure 10: Position of the Tangent Vectors X_{b_1} and Y_{b_2}

From the analysis, it is apparent that in order to investigate if two bodies B_1 and B_2 can be in contact at the points $b_1 \in B_1$ and $b_2 \in B_2$, we have to initially check if the condition $k_\gamma(0) < -\lambda_\delta(0)$ holds. Let k_1, k_2 and λ_1, λ_2 denote the maximum and the minimum curvatures at the points $b_1 \in B_1$ and $b_2 \in B_2$, respectively. Consider a normal section along the direction of X_{b_1}, where $X_{b_1} = F_1 cos(\theta) + F_2 sin(\theta)$, then the curvature at the point b_1 is given by $k(\theta) = k_1 cos^2(\theta) + k_2 sin^2(\theta)$. Here F_1, F_2 [1] are the principal directions at the point b_1. In a similar way, let us consider a direction $Y_{b_2} = J_1 cos(\phi) + J_2 sin(\phi)$ at the point $b_2 \in B_2$, where J_1 and J_2 are the principal directions, see Figure 10. Then the curvature at the point b_2 of the normal section, along the direction Y_{b_2}, is given by $\lambda(\phi) = \lambda_1 cos^2(\phi) + \lambda_2 sin^2(\phi)$.

From the analysis, if $k_1 < -\lambda_1$, then it is possible that the two bodies are in contact at the points b_1 and b_2 in any relative configuration of the body B_2, with respect to B_1. If $k_1 < -\lambda_1$, then $k(\theta) < -\lambda(\phi)$ since $k(\theta) \leq k_1 < -\lambda_1 \leq -\lambda(\phi)$, and this happens for any $0 \leq \theta, \phi \leq \frac{\pi}{2}$. If $k_1 = -\lambda_1$, then $k(\theta) < k_1 = -\lambda_1 < -\lambda(\phi)$ if $\theta \neq 0$ and $\phi \neq 0$. When $\theta = 0$ and $\phi = 0$, the worst case happens when for a given relative configuration the direction of the vector F_1 in B_1 coincides with the direction of J_1 in B_2 and the pair of the corresponding normal sections in these directions, includes the curves with the maximum curvature at $b_1 \in B_1$ and $b_2 \in B_2$. We can denote these curves by $\gamma_1(t)$ and $\delta_1(t)$, respectively. Since $k_1 = k_{\gamma_1}(0)$ and $\lambda_1 = \lambda_{\delta_1}(0)$, then according to the theorem since $k_1 = -\lambda_1$ we have to check if the curvatures satisfy the condition $k_{\gamma_1}(t) < -\lambda_{\delta_1}(t)$, for $t \in I$ and $t \neq 0$; this determines point contact at the pair of points b_1, b_2. Further refinements might be necessary if the conditions $k_2 = k_1$ and $\lambda_2 = \lambda_1$ hold simultaneously.

[1]Here we dropped the subscript b_1 for salle of simplicity

5 Using the Principal Curvatures as A Priori Information

Next, we develop criteria based on the curvature condition to check if contact is possible. Given that $k_1, k_2, \lambda_1, \lambda_2$ are real numbers satisfying the constraints, $k_2 \leq k_1$ and $\lambda_2 \leq \lambda_1$, there exists a finite number of orderings. We can investigate the possibility of contact for each case, and we can organize the criteria for contact in terms of the particular relation among these numbers.

Before starting the investigation, we would like to mention the following facts as resulting from the geometry of the problem. First, on the tangent plane to the point b_1, the curvature of the normal sections as we move from the direction along F_1 (the maximum curvature direction) towards the direction of F_2 (the minimum curvature direction), decreases monotonically. This is obvious since the curvature of any normal section is given by $k(\theta) = k_1 cos^2(\theta) + k_2 sin^2(\theta)$. The derivative $\frac{dk}{d\theta} = (k_2 - k_1) sin(2\theta)$ is always negative for $0 \leq \theta \leq \frac{\pi}{2}$, since $k_2 - k_1 \leq 0$, and always positive for $-\frac{\pi}{2} \leq \theta \leq 0$. We can visualise the situation as illustrated in Figure 11, the arrows show the direction of motion such that the curvature of the corresponding normal sections, increases.

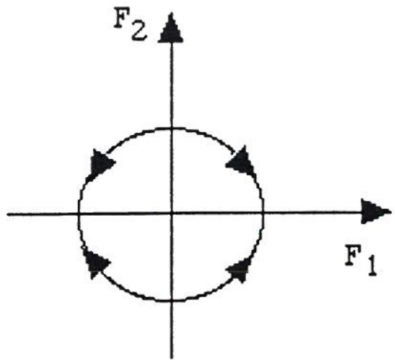


Figure 11: The Directions of Increase of Curvature

When the two bodies come into contact, then the tangent planes at the points b_1 and b_2 coincide. Recall that both pairs of vectors (F_1, F_2) and (J_1, J_2) are orthonormal, see Figure 12. Assume that when there is contact between the two bodies at the points $b_1 \in B_1$ and $b_2 \in B_2$, the vector J_2 is at an angle ψ from the vector F_1, see Figure 12.

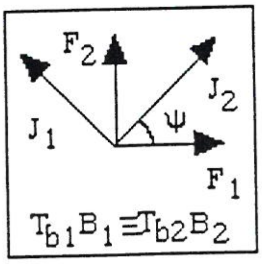


Figure 12: Relative Position of the Two Frames at Contact

The graph of the function $k(\theta)$ is illustrated in Figure 13.

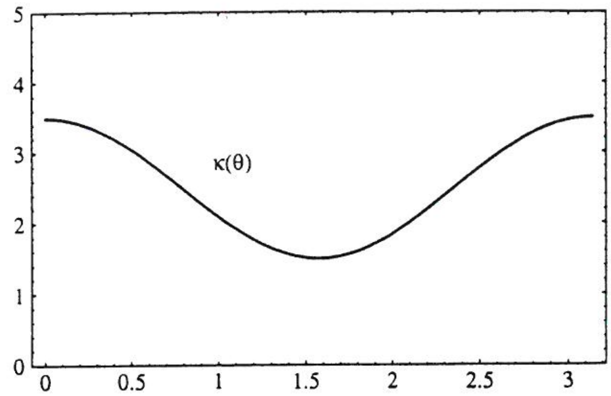


Figure 13: Curvature of a Normal Section as a Function of θ

In this graph the angle θ ranges between $0 \leq \theta \leq \pi$, and $k(\theta)$ is the curvature of the normal sections corresponding to each θ at the point $b_1 \in B_1$. A similar graph of the function $\lambda(\phi)$ gives the curvatures of the normal sections for $0 \leq \phi \leq \pi$, at the point $b_2 \in B_2$. When the two bodies are in contact at the points $b_1 \in B_1$ and $b_2 \in B_2$, and the angle between F_1 and J_2 is ψ, then we can superimpose the two graphs for comparing the curvatures of the corresponding normal sections along all the directions. The graph of $\lambda(\phi)$ is actually the graph of the relative curvature (with opposite sign) since we are comparing with reference to the surface B_1.

In Figure 14, the curvature function $\lambda(\phi)$ is given as a function of $\theta = \phi + \psi$. As we can see from the graph in Figure 14, the upper curve represents the graph of $\lambda(\phi)$ shifted by ψ. In the following graphs the intervals of θ of the curve $\lambda(\theta)$ where the curvature of the normal sections, at $b_2 \in B_2$, is less than the curvature of the corresponding normal sec-

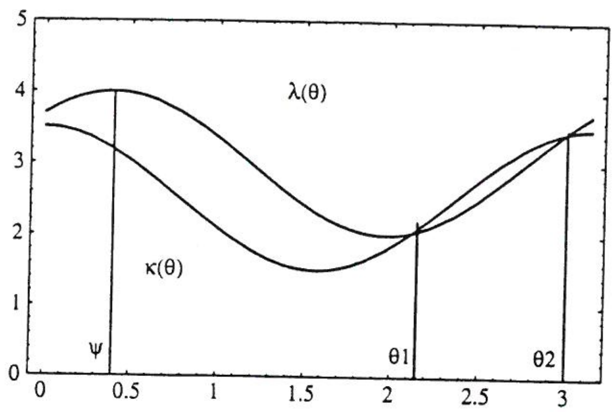


Figure 14: The Curvature Functions Superimposed

tions at $b_1 \in B_1$, correspond to situations where the two bodies B_1 and B_2 intersect and cannot physically be in contact. For example in Figure 14, in the interval θ_1 and θ_2, there cannot be contact. Next, we investigate all possible cases of ordering of $k_1, k_2, \lambda_1, \lambda_2$. We begin with the case where $-\lambda_1 = -\lambda_2$ and $k_1 = k_2$. This corresponds to the case where all the normal sections have the same curvature and the graphs of $k(\theta)$ and $\lambda(\theta)$ are straight lines. There are three possible cases :

i) If $k_2 = k_1 < -\lambda_1 = -\lambda_2$, we always have contact. We refer to this case as "completely contactable".

ii) If $-\lambda_1 = -\lambda_2 < k_2 = k_1$, then no contact is possible. We refer to this situation as "noncontactable".

iii) If $-\lambda_1 = -\lambda_2 = k_2 = k_1$, consider any pair of corresponding normal sections $\gamma(t) \in B_1$ and $\delta(t) \in B_2$ with $\gamma(0) = b_1$ and $\delta(0) = b_2$. Then $k_\gamma(0) = k_1 = k_2 = -\lambda_1 = -\lambda_2 = -\lambda_\delta(0)$ and we need investigate if $k_{\gamma(t)} < -\lambda_\delta(t)$ for t in some nonempty interval I, $t \neq 0$, then contact is possible. If we check on one pair of normal sections, it does not necessarily imply that the same happens to all other possible pairs. Thus, we need check on all the possible pairs of normal sections because in the neighbourhood of the points $k(0)$ and $\delta(0)$ the curvatures can change sufficiently. In general, the investigation of all possible pairs of normal sections not being possible, we refer to this situation as "undetermined".

For the case where $k_1 = k_2$ and $-\lambda_1 < -\lambda_2$, there are the following possibilities:

iv) If $k_2 = k_1 < -\lambda_1 < -\lambda_2$, in Figure 15, the points b_1 and b_2 are "completely contactable".

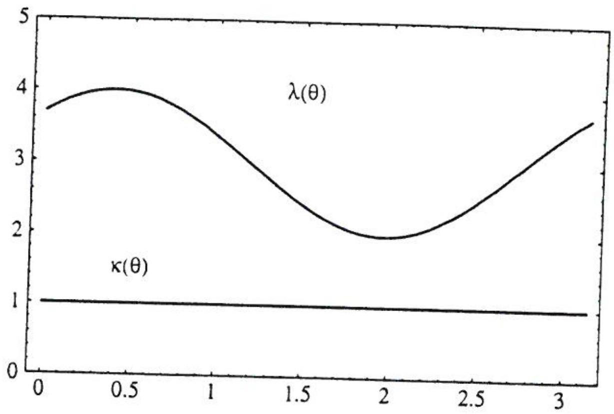


Figure 15: Comparative Graph of the Curvatures for the Case iv

v) If $k_2 = k_1 = -\lambda_1 < -\lambda_2$, generally the investigation of all possible cases being out of question, this is an "undetermined" situation.

vi) If $-\lambda_1 < k_1 = k_2 < -\lambda_2$, in Figure 16 the points are "noncontactable".

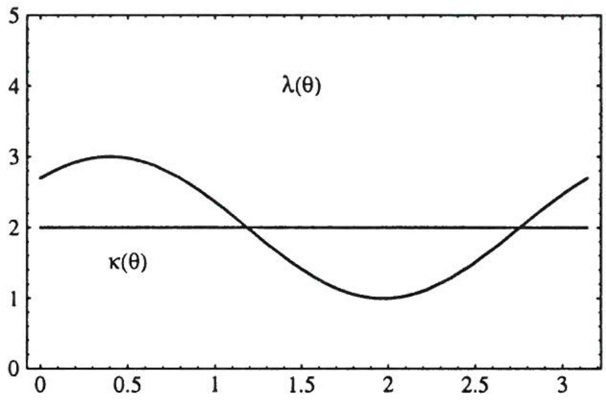


Figure 16: Comparative Graph of the Curvatures for the Case vi

vii) If $-\lambda_1 < -\lambda_2 < k_2 = k_1$, in Figure 17 the points are again "noncontactable".

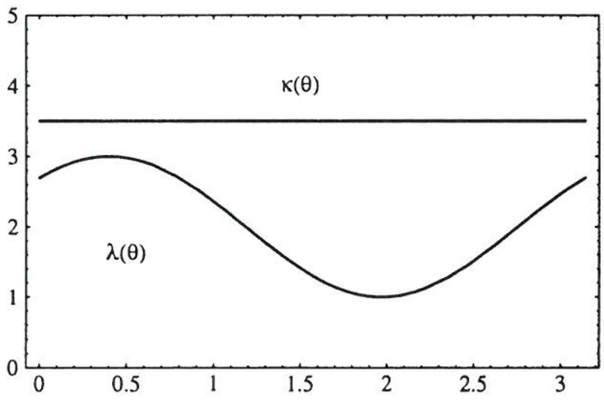


Figure 17: Comparative Graph of the Curvatures for the Case vii

For the cases where $k_2 < k_1$ and $-\lambda_1 = -\lambda_2$, we have the following combinations:

viii) If $k_2 < k_1 < -\lambda_1 = -\lambda_2$, then the points are "completely contactable", see Figure 18.

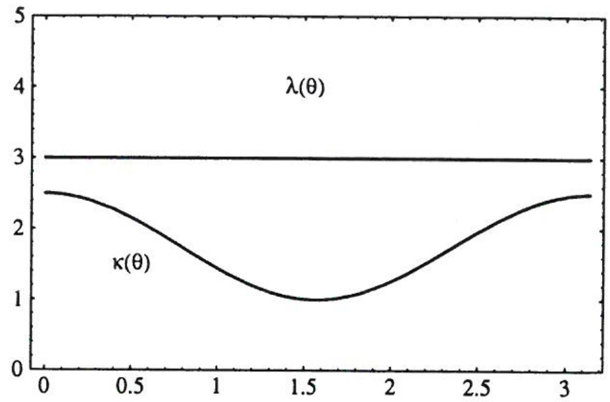


Figure 18: Comparative Graph of the Curvatures for the Case viii

ix) If $k_2 < k_1 = -\lambda_1 = -\lambda_2$, this case is "undetermined".

x) If $k_2 < -\lambda_1 = -\lambda_2 < k_1$, in Figure 19 this is a "noncontactable" case.

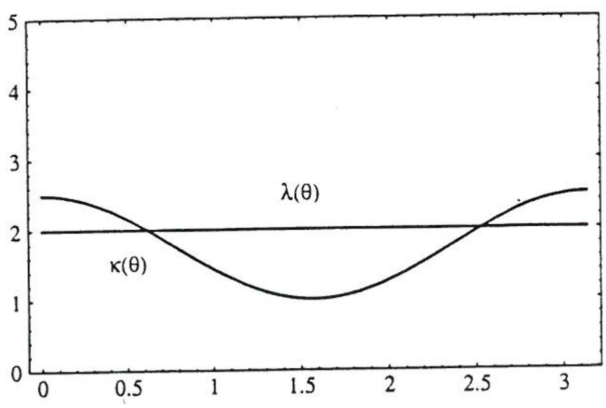


Figure 19: Comparative Graph of the Curvatures for the Case x

xi) If $k_2 = -\lambda_1 = -\lambda_2 < k_1$, or

xii) if $-\lambda_1 = -\lambda_2 < k_2 < k_1$, in Figure 20 such cases are "noncontactable".

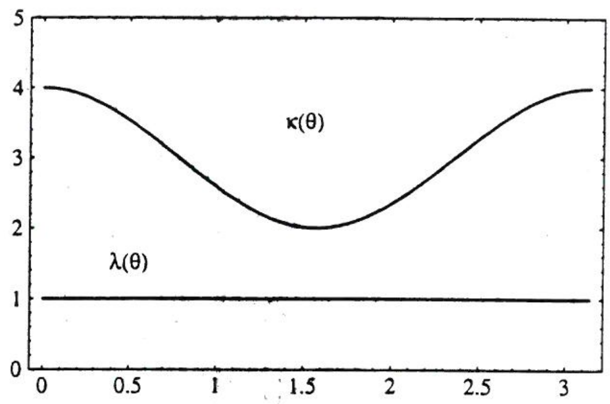


Figure 20: Comparative Graph of the Curvatures for the Cases xi and xii

When $k_2 < k_1$ and $-\lambda_1 < -\lambda_2$, the following combinations result:

xiii) If $k_2 < k_1 < -\lambda_1 < -\lambda_2$, in Figure 21 this is a "completely contactable" case.

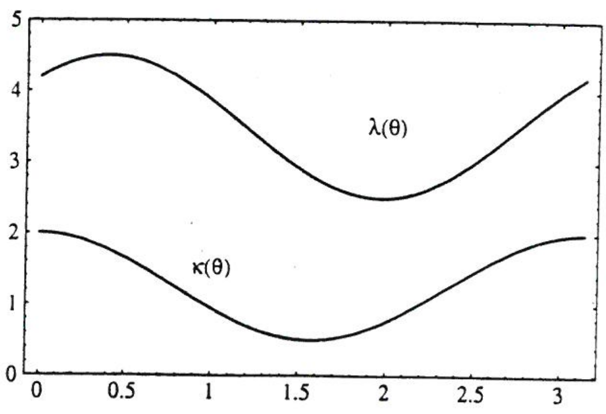


Figure 21: Comparative Graph of the Curvatures for the Case xiii

xiv) If $k_2 < k_1 = -\lambda_1 < -\lambda_2$, the case is "undetermined".

xv) If $k_2 < -\lambda_1 < -\lambda_2 < k_1$, see Figure 22, or

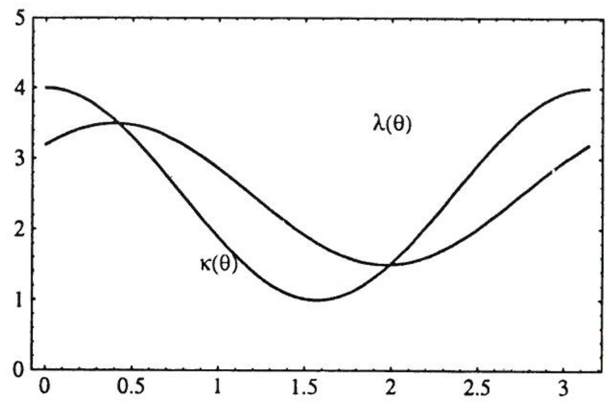


Figure 22: Comparative Graph of the Curvatures for the Case xv

xvi) if $-\lambda_1 < k_2 < -\lambda_2 < k_1$, see Figure 23, or

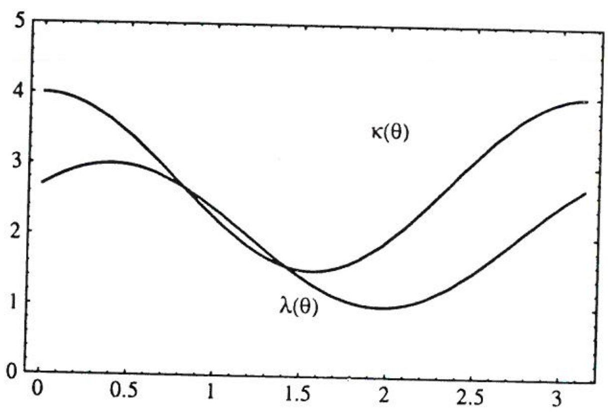


Figure 23: Comparative Graph of the Curvatures for the Case xvi

xvii) if $-\lambda_1 < k_2 < -\lambda_2 = k_1$, see Figure 24, or

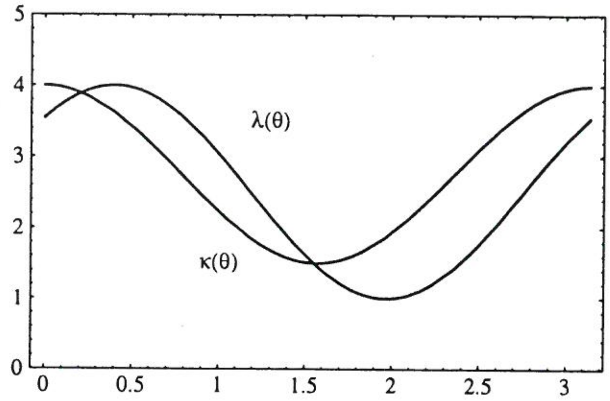


Figure 24: Comparative Graph of the Curvatures for the Case xvii

xviii) if $-\lambda_1 < k_2 = -\lambda_2 < k_1$, see Figure 25, or

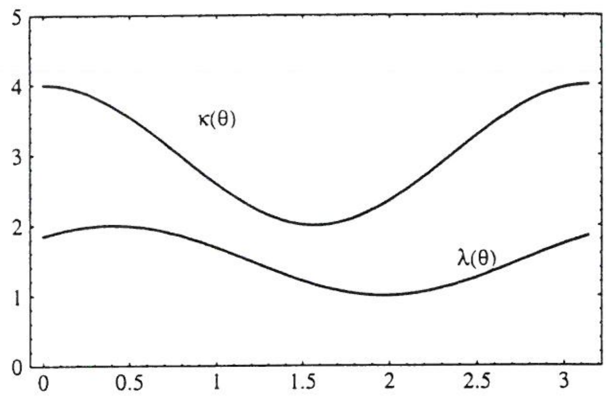


Figure 25: Comparative Graph of the Curvatures for the Case xviii

xix) if $-\lambda_1 < k_2 < -\lambda_2 < k_1$, see Figure 26, all these cases are "non-contactable".

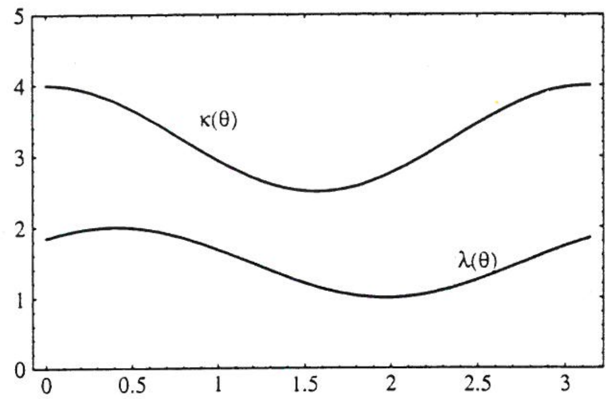


Figure 26: Comparative Graph of the Curvatures for the Case xix

xx) If $k_2 = -\lambda_1 < k_1 < -\lambda_2$, in Figure 27, then we have the "non-contactable" case except for the configuration in which the direction of the maximum curvature λ_1 is towards the minimum curvature k_2, which is "undetermined". We opt for classifying this case as "noncontactable" since the situation where contact is isolated to a single configuration is of no general interest.

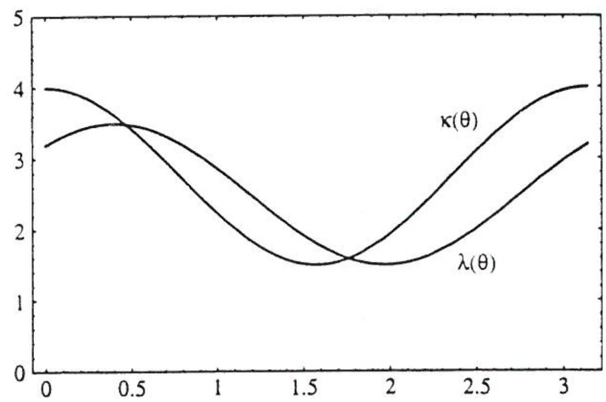


Figure 27: Comparative Graph of the Curvatures for the Case xx

xxi) If $k_2 = -\lambda_1 < k_1 = -\lambda_2$, the points are "noncontactable" except for one configuration, which is an "undetermined" case, see Figure 28.

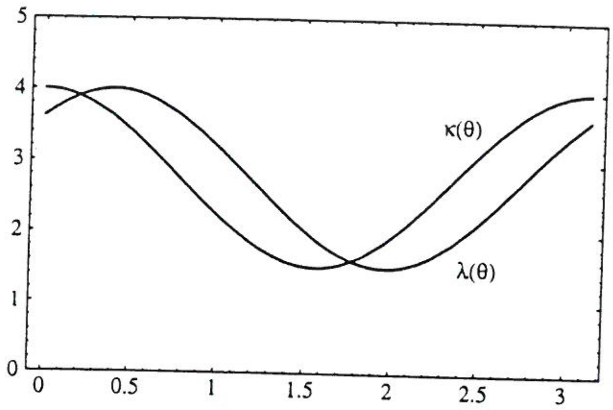


Figure 28: Comparative Graph of the Curvatures for the Case xxi

xxii) If $k_2 < -\lambda_1 < k_1 = -\lambda_2$, this case is, in general, "noncontactable", see Figure 29.

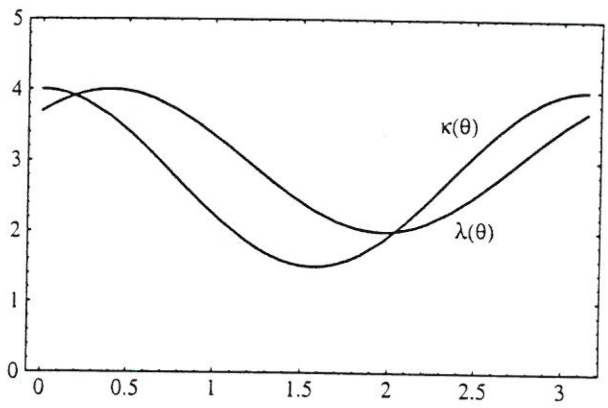


Figure 29: Comparative Graph of the Curvatures for the Case xxii

Finally,

xxiii) If $k_2 < -\lambda_1 < k_1 < -\lambda_2$, there is a certain angle ψ_0 such that if the angle ψ is in the interval $(-\psi_0, \psi_0)$, then every normal section at $b_2 \in B_2$ is greater than the curvature of the corresponding normal section at the point $b_1 \in B_1$, see Figure 30. This situation is referred to as "partially contactable" since contact depends on the orientation of the bodies according to the value of angle ψ.

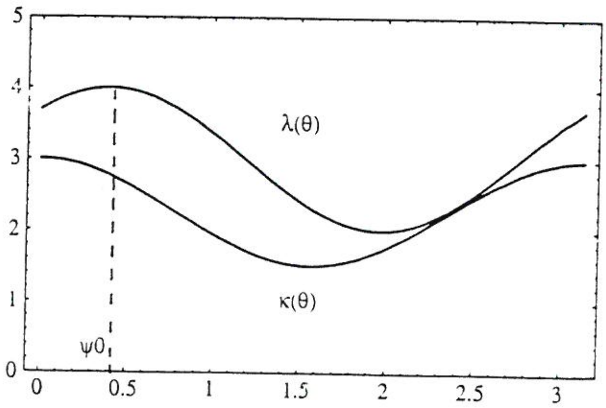


Figure 30: Comparative Graph of the Curvatures for the Case xxiii

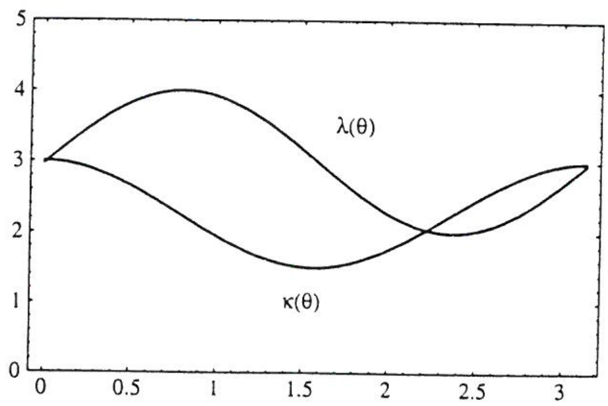


Figure 31: Comparative Graph for the Case xxiii when $\psi \geq \psi_0$

The following examples illustrate for simple surfaces embedded in $\Re^3$, the application of criteria for "contactability".

Example 1. Assume that the body B_1 is a hyperboloid with surface equation $-x_1^2 + x_2^2 + x_3^2 = 1$. The Gauss vector at a point $b_1 = (x_1, x_2, x_3)$ is given by $N(b_1) = (\frac{-x_1}{||b_1||}, \frac{x_2}{||b_1||}, \frac{x_3}{||b_1||})$. Let $b_1 = (0,0,1)$, then $N(b_1) = (0,0,1)$ and the principal directions are $F_1 = (1,0,0)$ and $F_2 = (0,1,0)$, and the principal curvatures are $k_1 = 1$ and $k_2 = -1$, see Figure 32.

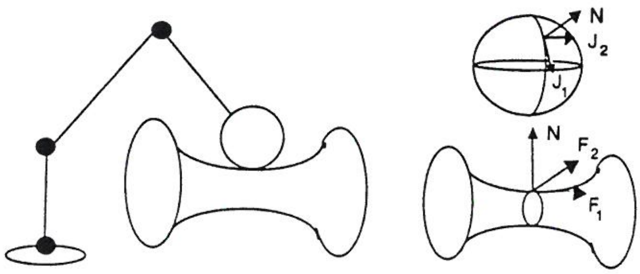


Figure 32: A Spherical End Effector and Hyperboloid Object Investigated

Let the end effector of the manipulator B_2 be a regular sphere, the surface equation is $x_1^2 + x_2^2 + x_3^2 = r^2$. The Gauss vector at a point $b_2 = (x_1, x_2, x_3)$ on the sphere is given by $N(b_2) = (\frac{x_1}{||b_2||}, \frac{x_2}{||b_2||}, \frac{x_3}{||b_2||})$, and the principal curvatures are the same at each point and along any direction, i.e. $\lambda_1 = \lambda_2 = -\frac{1}{r}$. If we choose a point b_2 on the sphere and $r < 1$, the point b_1 on B_1 and the point b_2 on B_2 are "completely contactable". Since $1 < \frac{1}{r}$, this corresponds to the case where $k_2 < k_1 < -\lambda_1 = -\lambda_2$. If $r = 1$, case $k_1 < k_2 = -\lambda_1 = -\lambda_2$ this is "undetermined" since we need investigate the curvature of all possible pairs of normal sections. In this particular case, since all the normal sections on the sphere are circles, it is easy to check if the condition $k_\gamma(t) < -\lambda_\delta(t)$ holds for t in some interval I and $t \neq 0$, and the points b_1 and b_2 are contactable for $r = 1$. In the case where $r > 1$, the points b_1 and b_2 are "noncontactable". In this case, $-\lambda_1 < k_2 < -\lambda_2 < k_1$ since $-1 < \frac{1}{r} < 1$, see Figure 33.

Example 2. Let the body B_1 be a hyperboloid as in Example 1, and the end effector B_2 be a cylinder, with surface equation $x_1^2 + x_2^2 = \frac{1}{4}$. At a point $b_2 \in B_2$, the Gauss vector is $N(b_2) = (\frac{x_1}{||b_2||}, \frac{x_2}{||b_2||}, 0)$. Let the point $b_2 = (0,1,0)$, then $N(b_2) = (0,1,0)$, and the principal curvatures are $\lambda_1 = 0$ and $\lambda_2 = -2$ along the principal directions $J_1 = (0,0,1)$ and $J_2 = (1,0,0)$, see Figure 34. If we consider the point $b_1 = (0,0,1)$ on B_1, $k_2 < -\lambda_1 < k_1 < -\lambda_2$ and the points b_1, b_2 are "partially contactable". In case (a), Figure 34, contact is possible because of the particular configuration of the body B_2 relative to the body B_1. This does not happen for the configuration of the end effector B_2 in case (b), as shown in Figure 34.

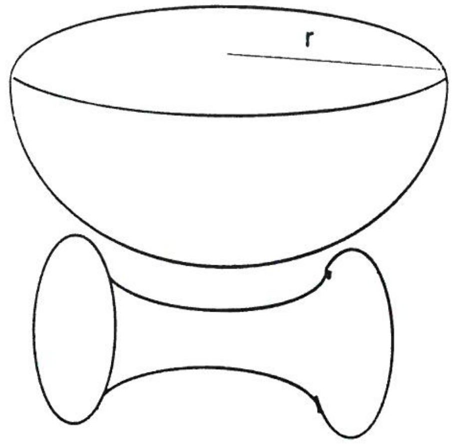
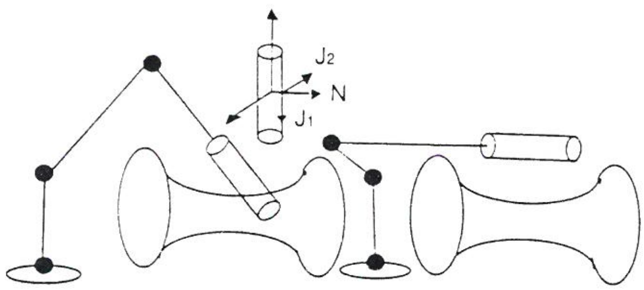


Figure 33: A Case where the Two Surfaces are Contactable at No Point



(a) (b)

Figure 34: A Case with Partially Contactable Points

6 A Graphical Representation of the Criteria for Contact

In order to systematically represent the results obtained in the paragraph above, we consider the plane with co-ordinates k_1, k_2 as shown in Figure 35. The line $k_1 = k_2$ separates the plane into two half planes, and we are interested in the half plane below the line $k_1 = k_2$, since $k_1 \geq k_2$. For a point $b_1 \in B_1$, consider the principal curvatures k_1, k_2; this corresponds to a point k_{b_1} on the plane k_1, k_2. Then consider the principal curvatures λ_1, λ_2 at a point $b_2 \in B_2$ and let λ_{b_2} denote the point $\lambda_{b_2} = (-\lambda_2, -\lambda_1)$ in the plane k_1, k_2. This point also belongs to the same half plane since $-\lambda_2 \geq -\lambda_1$.

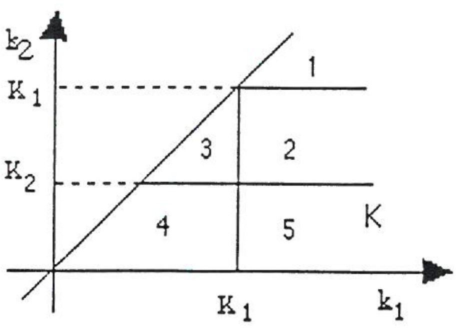


Figure 35: Graphical Representation of the Contact Criteria

We can decompose the half plane into five regions as shown in Figure 35, and these regions correspond to the following cases:

Region 1: $k_2 \leq k_1 \leq -\lambda_1 \leq -\lambda_2$

Region 2: $k_2 \leq -\lambda_1 \leq k_1 \leq -\lambda_2$

Region 3: $k_2 \leq -\lambda_1 \leq -\lambda_2 \leq k_1$

Region 4: $-\lambda_1 \leq -\lambda_2 \leq k_2 \leq k_1$
$-\lambda_1 \leq k_2 \leq -\lambda_2 \leq k_1$

Region 5: $-\lambda_1 \leq k_2 \leq k_1 \leq -\lambda_2$

If combining this decomposition of the plane (k_1, k_2) with the previous results we can summarize the results of the contactability analysis:

i) If $k_2 \leq k_1 < -\lambda_1 \leq -\lambda_2$ then the points are "completely contactable".

ii) If $k_2 \leq k_1 = -\lambda_1 \leq -\lambda_2$ then the points are "undetermined".

iii) If $k_2 < -\lambda_1 < k_1 < -\lambda_2$ then the points are "partially contactable".

iv) Otherwise the points are "noncontactable".

Let B_1 and B_2 be two rigid bodies, and choose a point $b_1 \in B_1$ with principal curvatures k_1, k_2, this separates the plane into five regions. Then a point b_2 on B_2 with principal curvatures λ_1, λ_2, is:

i) "completely contactable" if $\lambda_{b_1} = (-\lambda_2, -\lambda_1)$ belongs to the interior of region 1

ii) "undetermined" if it belongs to a boundary separating regions 1 and 2

iii) "partially contactable" if $\lambda_{b_1} = (-\lambda_2, -\lambda_1)$ belongs to the interior of region 2

iv) "noncontactable" if $\lambda_{b_1} = (-\lambda_2, -\lambda_1)$ belongs to region 3, region 4 or region 5.

We will only consider pairs of contact points which are "completely contactable". Actually isolated "completely contactable" points are not allowed either, and we need contact on an open neighbourhood of the surface of each body. In this situation, controlled motion is possible in which the bodies keep in contact.

7 Conclusions

The main task of this work was to investigate the conditions which are sufficient in order to have point contact between the surface of the end effector of a robot arm and the surface of an object under manipulation. Such an investigation was part of the on- going research in the area of constrained dynamics, robotic hands, robot control and object manipulation. Past discussions demonstrated that effective control schemes could result if the surfaces of robot arm and the object were considered rigid surfaces interacting through point contact. In the above areas there is a need for modelling of contact and its consequences on the modelling of the dynamics and control of a body. We also extended these results to areas of "contactable" points thus allowing the object to move. Related work has been published by D.J. Montana [4]. An appropriate model of the constrained configuration space will be the basis for a more accurate description of the dynamics of the constrained body. Within this context we used the principal curvatures of surfaces which were described by smooth equations. This is basic knowledge in Differential Geometry. We conducted our research on results related to sufficient conditions for point contact between two smooth plane curves where curvature was used as a criterion. We investigated how this work could be naturally extended to smooth surfaces using as criteria the principal curvatures of the surfaces. This has finally led to the development of a diagram that can be used for the graphic application of our results. In an applications environment the analysis done can result in a classification of objects which are "manipulatable" by the end effector of a robot.

Acknowledgment

I would like to thank Professor C.V. Massalas of the Department of Mathematics, University of Ioannina for his helpful remarks which had a substantial influence on this paper.

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