General Sliding Mode Systems Analysis and Design Via Flow-Invariance Method

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Abstract: This paper presents a flow invariance based approach to variable structure systems. Sliding mode in both single input and multi input systems is described via some special flow structure of the state space, pertaining to adjacent systems of variable structure. Quasi sliding motion is also treated using flow- invariance.A simulation procedure helps prove the advantages of this approach in designing quasi sliding mode with oscillation-free control.
Keywords: variable structure systems, sliding mode, flow-invariance.

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1. Introduction

The theory of variable structure systems (VSS) has been widely improved over the past 30 years, thus establishing a lot of connections with other different topics in system theory. Basic studies on VSS with sliding mode (VSSSM) are due to Emelyanov and Utkin, [1], [2]. A large and interesting work concerning VSS is due to Itkis, [3]. In the last 10 years one must remark the substantial contributions in the domain made by Sira-Ramirez [4], [5].

A larger interest in VSS (and especially in VSSSM) could be justified by their insensitivity to parameter variations and to exogenous disturbances caused by a feedback control strategy.

Studies on VSS have usually referred to continuous time systems, with sliding mode or quasi sliding mode (QSM). For this reason the term "VSS" is commonly used instead of "VSSSM".

Preoccupation has recently shifted from continuous to discrete VSS, due to the tremendous impact of computer technology on automatic control. The topics are still open to future investigation because, as one can see, every step forward proposes new facilities induced by VSS control strategy, in both theory and practice.

VSS are basically non-linear discontinuous robust controlled systems. But for the sake of surveying, let us notice that VSS are related, in many papers, with: adaptive model reference control, system identification, predictive control, linearization problems, fuzzy sets, neural networks.

This paper is concerned with some special characterizations of continuous time VSS, based on the flow- invariance method, [6], [7].

Some preliminaries, referring VSS and also a background of flow-invariance mathematical method, are given.

The main results related with scalar control VSS are presented in Section 2, [8], [9], [10]. Further, one extrapolates these results to multi input (MI) systems, [10]. Section 4 deals with some problems of quasi sliding mode (QSM). An illustrative example is given so that to highlight the special benefits of the proposed analytical methods.

1.1. Basic Definitions

Let us consider a non-empty family of dynamical systems, $\Im$, having the same time domain T, the same state space X and ξ being the domain of exogenous signals (reference inputs, disturbances).

Denote $\aleph$ the space $T*X*\xi$.

Also consider a function: α: $\aleph_0 \rightarrow \Im$, with $\aleph_0 \subseteq \aleph$(they might also be spaces), delineating, for every $x \in \aleph_0$, some system of $\Im$. Therefore α is a decision- like algorithm.

Definition 1.1. (Variable Structure System (VSS))

With the preceding statements, $\left\{A \in \Im; x \in \alpha^{-1}(A)\right\}$ defines the variable structure system (VSS) induced by α over $\Im$.

The membership systems, i.e. $\alpha(\aleph_0)$, are called the adjacent systems pertaining to VSS. ■

In other words, a VSS consists of a family of different systems and of a switching algorithm which , by means of states, time or environment conditions, defines the corresponding functioning structure. Note that α may be defined not only over $\aleph$, but also over a subset. Thus, even uncertain systems could be considered. This is, of course, a very general definition, but it addresses the larger class of possible VSSs, dealt with in the literature.

By choosing a domain S with zero measure in the state space X and by defining α on X\S the VSS structure will be changing when the state representative point (RP) passes over S. If the structure is changing with a (theoretically) infinite rate, the state being constrained on S, a sliding mode (SM) (or sliding motion) comes up.

The VSSSM distinct feature lies in the evolution of RP along a sliding domain (SD) (i.e. S) by means of indefinite high frequency switching control. In real plants such a motion meets bars because it is not possible to have only switching time finite and this time not arbitrarily small , thus chattering appears [11]. Therefore the RP moves rather in the neighbourhood of a sliding domain, namely quasi-sliding domain (QSD). This kind of a motion, referred to as quasi sliding mode (QSM) may be deliberately designed in VSS, in order to prevent or limit the oscillations of the actuating signal.

Sliding mode (i.e. "ideal" sliding mode - as a limit case of quasi sliding one) has to meet the following requirements:

1. From any initial state condition, RP hits the sliding domain. This is the reaching condition.
2. As soon as RP reaches SD, it remains indefinitely on it. This is the sliding condition.
3. The VSS behaviour in sliding mode has to be asymptotically stable, with the desired characteristics.

Consider the continuous-time dynamical scalar control system:

$$\dot{x} = f(t,x,u(t,x)) := F(t,x),\ t \in \Re_+,\ x \in \Re^n,\ u \in \Re^m, \tag{1.1}$$

where $x \equiv \left(x_1, x_2, \ldots, x_n\right)$, $u \equiv \left(u_1, u_2, \ldots, u_m\right)$, $F \equiv \left(F_1, F_2, \ldots, F_n\right)$.

Let us assume the continuous (possibly time-dependent) m- dimensional hypersurface in the state space:

$$S(t) := \left\{x \in \Re^{n};\ s(t,x) = 0\right\},\quad s \equiv \left[s_1 \ldots s_m\right]^T : \Re_+ * \Re^n \rightarrow \Re^m \tag{1.2}$$

Notice that the control u may be discontinuous on S. Consider that the state space originates from the equilibrium set point of (1.2) and belongs to S.

Let us also define the symmetrical neighbourhood of S:

$$R(t) := \left\{x \in \Re^n;\ \| s(t,x) \| \leq K(t,x)\right\},\quad K \equiv \left[K_1, \ldots, K_m\right]^T : \Re_+ * \Re^n_+ \rightarrow \Re^m_+ \tag{1.3}$$

where the inequality is componentwise determined, i.e. $|s_i| \le |K_i|$. By taking u as a discontinuous function along S, with the components:

$$u_i(t,x) = \begin{cases} u_i^-(t,x), & s_i(t,x)<0 \\ u_i^+(t,x), & s_i(t,x)>0 \end{cases}, i=\overline{1,m} \qquad (1.4)$$

where u $u_i^\pm$ are continuous and defined on the whole state space (by prolongation, for instance), the following family of dynamically smooth systems is set up:

$$\dot{x} = f\left(t,x,\left[u_1^\pm \dots u_m^\pm\right]\right) := \qquad (1.5)$$
$$:= F^p(t,x),\ p=\overline{1,2^m},\ x \in \Re^n.$$

The index p corresponds to all possible sign combinations. According to the previous statements, (1.5) with (1.2) and (1.4) define a VSS with 2^m adjacent systems.

The well-known basic concepts can be stated as follows.

Definition 1.2 (Ideal Sliding Domain (SD))

Switching hypersurface S (1.2) could be an ideal sliding domain (SD) for system (1.1) if it did not contain any trivial trajectory segments of adjacent systems (1.5), and if for each $\varepsilon > 0$ and $x^* \in S$, there existed a neighbourhood V* of X* such that for every $x_0 \in V^* \setminus S$, the trajectory of (1) starting from (t_0,x_0) evolved inside the domain:

$$S_\varepsilon := \left\{x \in \Re^n;\ \|s(t,x)\| \le \varepsilon\right\}, \qquad (1.6)$$
$$\text{for } t \ge t_0 \blacksquare$$

Definition 1.3 (Quasi Sliding Domain (QSD))

The set $R \le \Re^n$ could be a quasi sliding domain (QSD) for (1.1), (1.2) if any trajectory of (1.1),

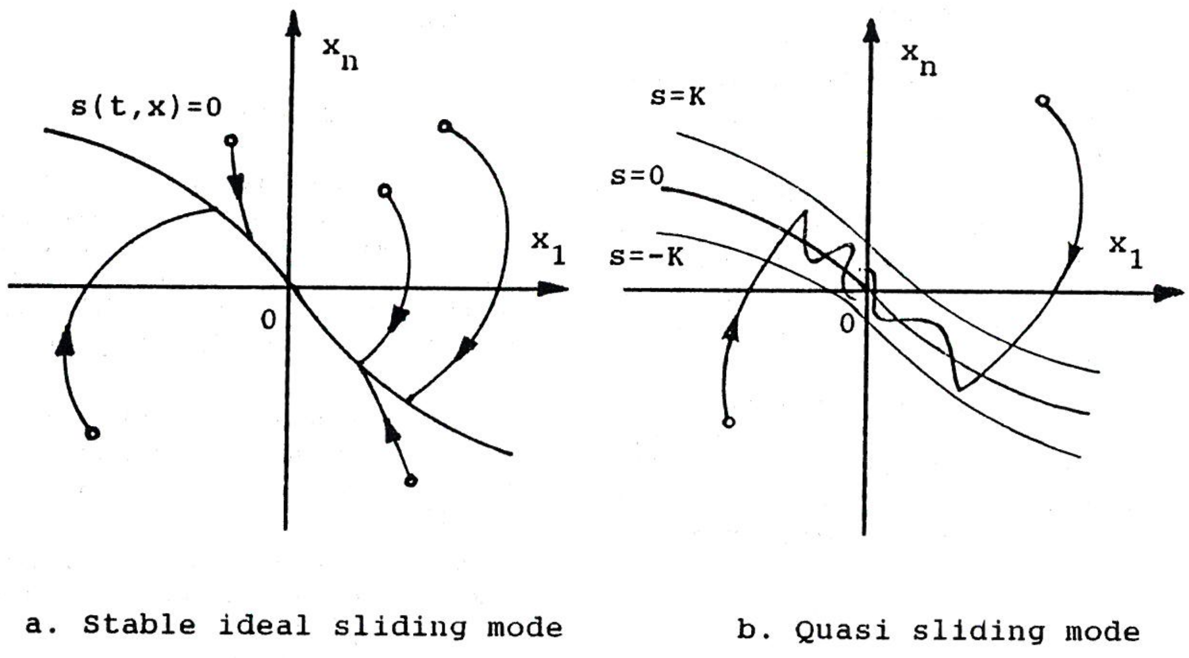


Figure 1.1 Sliding Mode and Quasi Sliding Mode

starting from (t_0, x_0), with $x_0 \in R$ evolved inside R, for every $t \geq t_0$. ■

The motion of state on SD is called (ideal) sliding mode (SM) and the motion in QSD is called quasi sliding mode (or pseudo sliding mode). The behaviours are depicted in Figure 1.1.

The function K(t,x), stated in (1.3), is called the reliable margin of QSM.

1.2. Sliding Mode Description

The most usual statement about sliding mode existence is the componentwise geometric condition:

$$\lim_{s_i \to +0} \dot{s}_i \leq 0, \ \lim_{s_i \to -0} \dot{s}_i \geq 0. \qquad (1.7)$$

It is well- known that (1.7) equation is only sufficient conditions, but, because of their simplicity, many authors look upon them as the definition of sliding motion.

Another possibility of designing SM is the Lyapunov function approach [1]. This is to choose a positive function (quadratic, for instance), depending on s: V(s,t), with $\dot{V}(s,t)$ negative in a certain vicinity Ω of the origin of $\Re^m$ space, without the origin.

Moreover, the preceding methods can also be used for the statement of reaching conditions. That is why, instead of (1.7),one says that $\dot{s}$ and s must have opposite signs in a larger vicinity of S. The domain Ω, related with the Lyapunov function, has to be the largest one, respectively.

Sliding mode means a special behaviour of VSS, which is different from any of the adjacent systems. The description of this mode is made by Utkin's equivalent method, [1], [2], leading to the sliding differential equation. The idea is to express u, as a smooth function, from:

$$\dot{s} = 0 \Leftrightarrow \nabla_x s \bullet f(t,x,u) + \frac{\partial s}{\partial t} = 0, \qquad (1.8)$$

where $\nabla_x := \left[\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \dots \frac{\partial}{\partial x_n}\right]$. The solution is called the equivalent control, u_{eq}. By replacing u_{eq} in (1.1) the differential equation with continuous right-hand side is obtained:

$$\dot{x} = f\left(t,x,u_{eq}\right), \qquad (1.9)$$

representing the model of the ideal sliding mode.

Because of $s = 0$ in SM, the m components of state vector x are to be retrieved as functions of the remaining n-m ones. Substituting them and removing the supplementary m equation, (1.9) yields :

$$\dot{x}^1 = F^1\left(x^1,t\right), \ x^1 \in \Re^{n-m}, \qquad (1.10)$$

which is called the sliding motion equation. The stability and transient characteristics of (1.1) in SM are equivalent to those of the system (1.10).

With SM not available in (1.1), but with QSM available in (1.1), and with (1.9) asymptotically stable, the state evolves by keeping a bounded distance from the state of the ideal sliding mode (1.10) [1]. So, in QSM the equivalent control method can also be used, but minding that the part x^1 of the state evolves "near" the solution of the equation (1.10), which starts from the same initial condition.

1.3. Flow Invariance

First, the flow invariance mathematical method has been developed apart from control system theory [6], but actually the fact that it represents a highly useful tool in approaching specific problems should be emphasized [7], [12]. It can be related with various aspects of stability and control design questions.

Let us more consider system (1.1), where **F** is assumed as being continuous and locally Lipschitzean function on some open set $T*X \leq \Re_+ * \Re^n$.

Therefore, for every $(t_0, x_0) \in T*X$ there exists a unique solution x(t), defined on $(a_0, b_0) \leq T$ such that $x(t_0) = x_0$. Functions

$$x^-(t)=x(t),\ t\in\left[a_0,t_0\right],\ x^+(t)=x(t),\ t\in\left[t_0,b_0\right),$$

are notified on the negative and positive solutions of (1.1) to (t_0, x_0), respectively.

Definition 1.4 (Flow Invariant Sets)

A time-dependent set $D(t) \subset X$, $t \in T$, is called negatively, and positively flow-invariant with respect to system (1.1) respectively, if for each $(t_0,x_0) \in T * D(t)$, the following conditions are met:

$$x^-\left(a_0,t_0\right] \le D(t),\ x^+\left[t_0,b_0\right) \le D(t). \blacksquare \qquad (1.11)$$

In other words, a set D is said to be positively flow invariant (or flow invariant), (PFI) for system (1.1), if every state trajectory which starts from D at some moment, remains in D for all subsequent time.

Theorem 1.1

D(t) is PFI with respect to system (1.1) if and only if $X \backslash D(t)$ is negatively flow-invariant (NFI) with respect to the same system. ■

Theorem 1.2

A closed set D(t) is PFI with respect to (1.1) if and only if

$$\liminf_{h \to 0} h^{-1} d\left(x+h\,F(t,x);\ D(t+h)\right)=0, \qquad (1.12)$$

for each $(t,x) \in T * D(t)$. ■

In (1.12) d denotes the distance in $\Re^n$.

Assume in the sequel that $T \equiv \Re_+$, $X \equiv \Re^n$ and D(t) is the time dependent n-dimensional parallelepiped:

$$D(t) := \prod_{i=1}^{n}\left[a_i(t), b_i(t)\right],\ a_i(t) \le b_i(t),\ t \in \Re_+, \qquad (1.13)$$

a_i, b_i being some differentiable functions.

Theorem 1.3

D (t) defined by (1.13) is PFI with respect to (1.1) if and only if

$$\begin{cases} F_i(t,x)\,\big|_{x=a_i} \ge \dot{a}_i(t) \\ F_i(t,x)\,\big|_{x=b_i} \le \dot{b}_i(t) \end{cases}, \qquad (1.14)$$

for each $t \in \Re_+$, $x \in D(t)$. ■

In (1.13) and, accordingly, in (1.14) it is possible to take $a_i = -\infty$ or $b_j = +\infty$ but, conversely, the corresponding items in (1.14) have to be removed.

Figure 1.2 suggests the significance of D (t) as a flow invariant set within the state space of some 2nd order dynamical system.

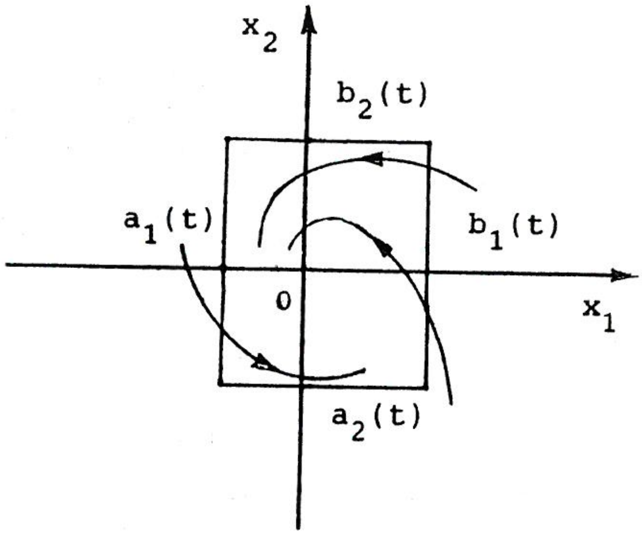


Figure 1.2 Positively Flow Invariant Parallelepiped

2. Single Input Sliding Mode System

This section characterizes the single input (SI) (or scalar) sliding mode systems, by means of flow invariance method.

The main results are only reviewed, with no proofs. For more detailed aspects, a list of references is available. Consider the system (1.1), with (1.2), for the case m = 1. For a simpler writing, (1.4) is replaced by:

$$u(t,x) = \begin{cases} u^-(t,x), & s<0 \\ u^+(t,x), & s>0 \end{cases}, \qquad (2.1)$$

and thus the adjacent systems are respectively:

$$\dot{x}=f\left(t,x,u^-(t,x)\right):=F^-(t,x), \qquad (2.2)$$

$$\dot{x}=f\left(t,x,u^+(t,x)\right):=F^+(t,x), \qquad (2.3)$$

both with continuous right - hand side.

If assuming, for the sake of convenience, that the switching function s, introduced in (1.2), with

$m = 1$, verifies the condition $\frac{\partial s}{\partial x_n} \neq 0$, then the non-singular state transform:

$$x \rightarrow \left[x_1\, x_2 \ldots x_{n-1} s\right]^T := \tilde{x} \qquad (2.4)$$

is allowed for (1.1). To avoid intricate expressions which could escape the basic idea, x_n is directly replaced by s in (2.1) - (2.3); accordingly, (2.2), for $m = 1$, is:

$$S := \left\{x \in \Re^n;\ x_n = 0\right\} \qquad (2.5)$$

Also define :

$$S^- := \left\{x \in \Re^n;\ x_n < 0\right\} \equiv \{s < 0\}, \qquad (2.6)$$

$$S^+ := \left\{x \in \Re^n;\ x_n > 0\right\} \equiv \{s > 0\}. \qquad (2.7)$$

Therefore in (1.6) x_n replaces s, and x^1 in (1.10) is defined by:

$$x^1 := \left[x_1\, x_2 \ldots x_{n-1}\right]^T. \qquad (2.8)$$

We are now in the position of formulating the main results.

2.1. Sliding Mode Existence via Flow Invariance Approach

On using the flow- invariance method, proper and equivalent conditions for the SM existence need be defined.

Theorem 2.1 (Sliding Mode Conditions)

Assume that the adjacent systems (2.2), (2.3) have no trajectory segments on the switching hypersurface S. Then, for system (1.5), the following statements are equivalent:

1^o S is the ideal sliding domain.

2^o Each functioning subset is negatively flow-invariant with respect to its own switching subsystem.

3^o The complementary element of each functioning subset is positively flow-invariant with respect to the switching subsystem of that functioning subset.

4^o $\lim\limits_{h \to 0} \inf h^{-1} d\left(x + h F^{\pm}(t,x); s \cup s^{\pm}\right) = 0,$

for each $(t,x) \in \Re_+ * \left(s \cup s^{\pm}\right)$.

5^o
$$\begin{cases} F_n^-\left(t, x_1, \ldots, x_{n-1}, 0\right) \geq 0 \\ F_n^+\left(t, x_1, \ldots, x_{n-1}, 0\right) \leq 0 \end{cases}, \qquad (2.9)$$

for each $t \in \Re_+$, $x^1 \equiv \left(x_1, \ldots, x_{n-1}\right) \in \Re^{n-1}$■

For proof, see [8], [9], [10].

The underlying idea is that of the adjacent system (2.2), which represents the VSS on S^+, having $s \cup s^-$ as PFI domain, and (2.3), which represents the VSS on S^-, having $s \cup s^+$ as PFI domain (see Figure 2.1. a,b). Given these features, one's (also intuitive) assertion is that the assembly behaves like a SM system.

Remark 2.1

Theorem 2.1 gives necessary and sufficient conditions so that SM exists. It is noticeable that only boundary restrictions are referred (i.e. (2.9)). Theorem 2.1 also depicts a special flow structure of the state space induced by VSSSM control.■

2.2. Reaching Condition

Reaching condition can also be characterized by means of some flow invariance issues, as follows.

Theorem 2.2 (Reaching Condition)

For each $t_0 \in \Re_+$, $x_0 \in \Re^n \setminus s$, the state of system (10) reaches the ideal sliding domain S if and only if there exists a differentiable function, depending on (t_0, x_0), $r: \Re_+ \rightarrow \Re$, satisfying the following conditions:

1^o There exists $\tau \in \left(t_0, +\infty\right)$ where $r(\tau) = 0$;

$$2^o \quad x_{on} \in \left(0, r\left(t_0\right)\right] \qquad (2.10)$$

3^o
$$\begin{cases} F_n^-\left(t, x_1, \ldots, x_{n-1}, r(t)\right) \geq \dot{r}(t), & \text{if } s\left(t_0\right) \equiv x_{on} < 0 \\ F_n^+\left(t, x_1, \ldots, x_{n-1}, r(t)\right) \leq \dot{r}(t), & \text{if } s\left(t_0\right) \equiv x_{on} > 0 \end{cases}, \qquad (2.11)$$

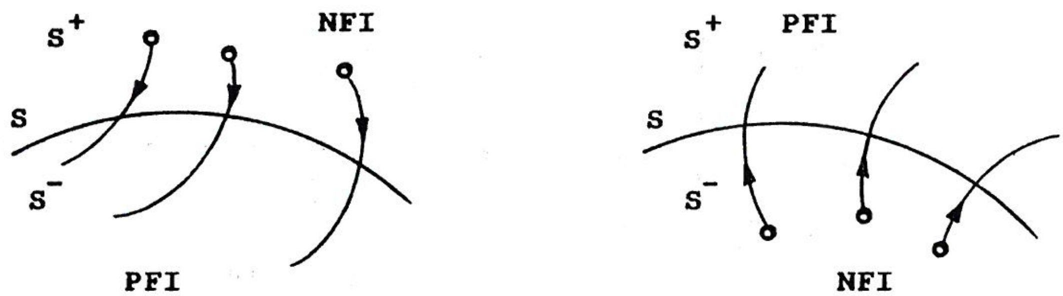


Figure 2.1 State Spaces for Adjacent Systems for a SI VSSSM

for each $t \in [t_0, \tau)$, $(x_1, \dots, x_{n-1}) \in \Re^{n-1}$. ■

For proof, see [9], [13].

According to (2.11), the state space of VSS is shared into the following flow structure: $\Re^{n-1} * (r(t), +\infty)$, is NFI with respect to (2.3) and $\Re^{n-1} * (-\infty, r(t)]$ is PFI with respect to the same system. Similarly, for (2.2), $\Re^{n-1} * (-\infty, -r(t)]$ is NFI and $\Re^{n-1} * [-r(t), +\infty)$ is PFI.

The reaching motion is suggested in Figure 2.2., where two different initial conditions for s were considered.

In order to use Theorem 2.2 in control problem-solving, an appropriate choice of r(t) must be made, allowing a desired velocity of the reaching process.

Remark 2.2

Theorems 2.1 and 2.2 are strongly related, both offering solutions to the different requirements made by VSSSM (section 1.1). Clearly, for $r(t) \equiv 0$ in (2.10), (2.11) one obtains (2.9), so Theorem 2.2 also refers the ideal SM as a subsequent phase of SD reaching. This very important aspect shows the net significance of the flow structure induced on the state space by VSS. ■

3. Multi-input Sliding Mode System

Consider the general case of multi input dynamical systems, (1.1) for which $1 < m < n$, together with the m-th order switching manifold (1.2). These elements involve the 2^m adjacent systems (1.5),

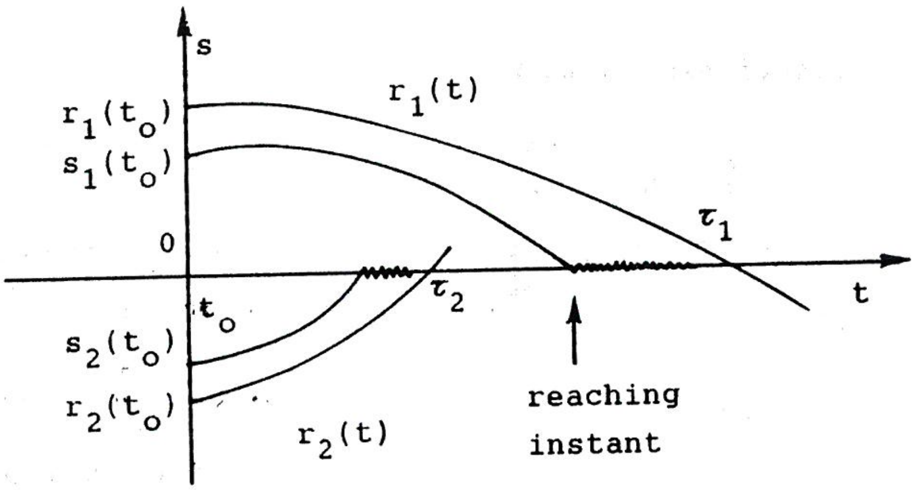


Figure 2.2 Reaching Phase via Flow Invariance Description

based on the discontinuous componentwise vector control (1.4) [1], [2], [4], [10].

The sliding mode statement is a little bit different in multi input (MI) case because of several possibilities being offered for its definition (Definition 1.2 or any equivalent definition).

To simplify any further analytical description, let us assume that:

$$\operatorname{rank}\left[\frac{\partial s_i}{\partial x_j}\right]_{i=\overline{1,m},\, j=\overline{1,n}} = m,$$

which allows the non-singular state co-ordinate transformation:

$$x := \left[x_1 \dots x_n\right] \rightarrow \tilde{x} := \left[x_1 \dots x_{n-m} s_1 \dots s_m\right]. \qquad (3.1)$$

As already done in the scalar case, similarly, we proceed on replacing the state by x, but also on denoting it with x, instead of x. Moreover, the following notations are made:

$$f(t,x,u) := \left[f_1 \dots f_{n-m} \varphi_1 \varphi_2 \dots \varphi_m\right](t,x,u) \equiv \qquad (3.2)$$
$$\equiv F(t,x) := \left[F_1 \dots F_{n-m} \phi_1 \phi_2 \dots \phi_m\right](t,x),\, t \in \Re_+, x \in \Re^n.$$

For the sake of simplicity of writing, the functions will often be expressed without their explicit variables.

By means of eqs. (3.1), (3.2), the VSS (1.1) turns into:

$$\begin{cases} \dot{x}_1 = f_1(t,x,u) \\ \dots\dots\dots \\ \dot{x}_{n-m} = f_{n-m}(t,x,u) \\ \dot{s}_1 = \varphi_1(t,x,u) \\ \dots\dots\dots \\ \dot{s}_m = \varphi_m(t,x,u) \end{cases} \Leftrightarrow \begin{cases} \dot{x}_1 = F_1(t,x) \\ \dots\dots\dots \\ \dot{x}_{n-m} = F_{n-m}(t,x) \\ \dot{s}_1 = \phi_1(t,x) \\ \dots\dots\dots \\ \dot{s}_m = \phi_m(t,x) \end{cases}. \qquad (3.3)$$

The individual sliding manifolds pertaining to S, (1.2), and the corresponding semi spaces are, respectively:

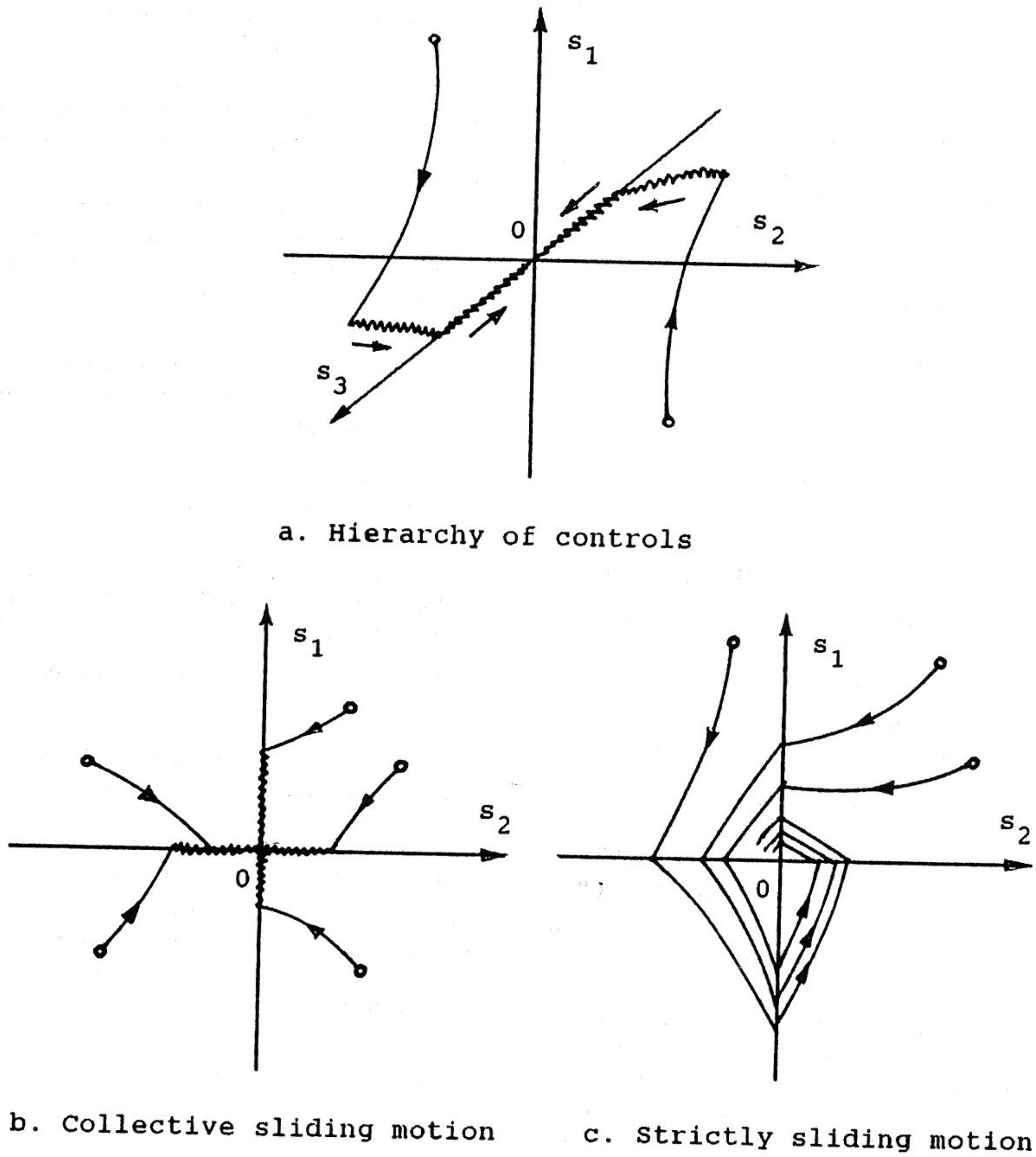


Figure 3.1 Multi Dimensional Sliding Mode, Projected on S-Space

$$\begin{cases} s^i := \{x \in \Re^n;\ s_i(x)=0\}; \\ s^{i+} := \{x \in \Re^n;\ s_i(x)>0\}; \quad i=\overline{1,m}. \\ s^{i-} := \{x \in \Re^n;\ s_i(x)<0\}, \end{cases} \tag{3.4}$$

In order to obtain a compact description of the VSS family (i.e. the adjacent systems), the multiple sign symbol will be introduced:

$$\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_m) \equiv \sigma_1\sigma_2\ldots\sigma_m,\ \sigma_i \in \{-, +\}, \tag{3.5}$$

σ_i being a sign symbol (but not a number). Obviously, there exist 2^m different σ symbols. According to eqs. (3.4) and (3.5) system (1.1) with (1.4) is a proper VSS, namely:

$$\dot{x}=f(t,x,u^\sigma(t,x)),\ u^\sigma \equiv \left[u_1^{\sigma_1},\ldots,u_m^{\sigma_m}\right]^T, \quad t\in\Re_+,\ x\in \bigcap_{i=1}^{m} s^{i\sigma_i}, \tag{3.6}$$

where the adjacent systems are, respectively:

$$\dot{x}=f(t,x,u^\sigma(t,x)) \Leftrightarrow \dot{x}=F(t,x)^\sigma,\ t\in\Re_+, x\in\Re^n, \tag{3.7}$$

and are further denoted by A^σ.

The operation of MI in SM should consider various kinds of motions; the most general one is, of course, the regime which corresponds to Definition 1.2, with no peculiarity. Nevertheless, by imposing some other restrictions, some useful and quite interesting modes could appear. Three main possibilities in multi - input sliding mode system (also remember that S is a vector function) are briefly presented in the sequel.

1. The method of hierarchy of controls, proposed by Utkin [1], disjoints the vector control problem in m successive scalar control problems. First, the 1st component of control, (i.e. u_1, (1.4)) leads the state to s^1, in a "partial" SM. The system turns into an n-1 order system in which the 2nd control component u_2 further sets a SM on $s_1 := s^1 \cap s^2$ and so on (Figure 3.1. a). The last phase lies in SM on $S \equiv S_m$.

2. The collective sliding mode refers the case in which every component of control, u_i, "partially" achieves SM on s^i, but simultaneously (Figure 3.1. b).

3. The strictly sliding mode is the regime under which the state hits on and is confined to S, but no SM "partially" arises in VSS (Figure 3.1. c). Despite intuition, such systems actually exist, as an original Utkin's example shows [1].

In any situation, some flow structures of the state space are attainable. A special difficulty is induced by the 3rd case, where a Lyapunov method would rather be used.

The following results, referring sliding and reaching problems, are just briefly outlined, without proofs or large remarks. For more details, also see [1], [2], [4], [6], [10]. The problems are subject to further investigation.

As Definition 1.2 suggests, let us assume in the sequel that none of the adjacent systems A^σ has trajectory segments on s^i, $i=\overline{1,m}$.

3.1. The Hierarchy of Controls

The method of control hierarchy [1], develops an artificial sequence of scalar SM fulfilments. Briefly saying, when the state of a system had hit the manifold $s_k := \bigcap_{i=1}^{k} s^i$, $k=\overline{1,m}$, the previous $u_1, \ldots, u_{k-1}$ components would have already been installing the sliding regime on $s_1, \ldots s_k$, successively. Thus, on s_k the order of system is n-k and only m-k components in control u are free to choose, the preceding ones being formally replaced by their own equivalent controls.

Since the vector control design problem is decomposed into m scalar problems, the analytical methods lend to be applied to the single input systems.

The following inequalities are manifest with the sliding mode existence on $s_1 \equiv s^1$:

$$\begin{cases} \Phi_1(t,x)^{+\sigma_2\ldots\sigma_m}\big|_{s_1=0} \le 0 \\ \Phi_1(t,x)^{-\sigma_2\ldots\sigma_m}\big|_{s_1=0} \ge 0 \end{cases},\ x\in\Re^n,\ s_1\equiv 0,\ t\in\Re_+, \tag{3.8}$$

the symbols σ_i, $i=\overline{2,m}$, being arbitrary. The equivalent control of this scalar sliding mode is:

$u^1_{eq}\left(t,x_1,\ldots,x_{n-m},s_2,\ldots,s_m\right)$ and results from the condition:

$$\varphi_1\left(t,x,u^1_{eq}\right)\Big|_{s_1=0}=0. \quad (3.9)$$

The n-1 order sliding differential equation is of the form:

$$\dot{x}^1=f^1\left(t,x^1,u^1\right):=F^1\left(t,x^1\right) \quad (3.10)$$

with

$$x^1:=\left[x_1\ldots x_{n-m}s_2\ldots s_m\right]^T, f^1:=\left[f^1_1\ldots f^1_{n-m}\varphi^1_2\ldots\varphi^1_m\right]^T,$$

$$F^1:=\left[F^1_1\ldots F^1_{n-m}\phi^1_2\ldots\phi^1_m\right]^T, u^1:=\left[u_2\, u_3\ldots\, u_m\right]^T.$$

Resuming this procedure, the switching variables $s_2,\ldots,s_m$, and also the controls $u_2,\ldots,u_m$ are successively removed.

As a result, the sliding mode equation, corresponding to the evolution of S state, is obtained. Moreover, one finds m restrictions on the components of control u, in terms of inequalities (3.8).

The reaching conditions can be formulated similarly as in the scalar case (Theorem 2.2), by confining the system to step k (evolving on s_{k-1}), to reach the surface s^k, i.e. the intersection s_k. Complicate relations would have to be written, with no special meaning, so one prefers not to enter into more details.

Remark 3.1

The flow structure employed by the control of hierarchy is given by (3.8), as follows. By sliding on s^{1-}, s^{1-} is laid down to be PFI for adjacent systems (3.7), corresponding to symbols $\sigma = +\sigma_2 \ldots \sigma_m$, and s^{1+} is NFI for the same adjacent systems.

s^{1+} is PFI for the adjacent systems (3.7) with $\sigma = -\sigma_2 \ldots \sigma_m$, and s^{1-} is NFI for the same family.

Sliding on $s_2\equiv s_1\cap s_2$ asks for $s^1\cap s^{2-}$ to be PFI systems of the form $\dot{x}^1=f^1\left(t,x,u^{1,\sigma}\right)$, with $\sigma=+\sigma_3\ldots\sigma_m$, and conversely $s^1\cap s^{2+}$ must be NFI for the same family, and so on.■

3.2. Collective Sliding Mode

If the state representative point (RP) reaches and slides on each individual n-1 dimensional manifolds s^i at least in the neighbourhood of s, this obviously yields the sliding mode on the m order domain s (i.e. the intersection of individual manifolds s^i).

Although it is quite restrictive, this collective SM is frequently used in design problems.

In the sequel, the condition for sliding mode existence and also other corresponding conditions are stipulated, by means of flow-invariance approach.

Theorem 3.2

System (3.3) satisfies the condition for collective sliding mode on s, if and only if for every $i=\overline{1,m}$ and $x\in \bigcap_{i=1}^{m} s^{i\sigma_i}$, respectively ,the following conditions hold:

$$\sigma_i \operatorname{sgn}\varphi_i(t,x)^\sigma\Big|_{s_i=0}\leq 0,\ \forall\sigma=\sigma_1\ldots\sigma_m\in\{-,+\} \blacksquare \quad (3.11)$$

The reaching requirement can be met by confining the RP to either moving directly to s, or hitting on a certain surface s^i, in any of the subsets split by s in the state space.

Note the availability of Theorem 2.2 in this frame, but for components. Accordingly, the RP necessarily reaches one of the individual manifolds, say s^j. This happens because the flow structure of the state space induced by SM control, forces the RP to $s_i(t)=0$, $i=\overline{1,m}$, after a finite time interval, anytime the initial state is resumed.

Theorem 3.3

One necessary and sufficient condition of system (3.3) for reaching the manifold S, is that every $t_0\in\Re_+\cdot x_0\in\Re^n \setminus S$, should have a differentiable function, depending on $\left(t_0,x_0\right)$, $r\equiv\left[r_1\ldots r_m\right]^T:\Re_+\to\Re^n$, which satisfies the statements:

1° There exists $\tau>t_0$, such as $r_i(\tau)\leq 0$, $i=\overline{1,m}$;

2^o $s_i(t_0) \in (0, r_i(t_0)]$, $i=\overline{1,m}$;

3^o For each $i=\overline{1,m}$ and $\sigma=\sigma_1 \dots \sigma_m \in \{-,+\}^m$:

$$\sigma_i \text{sgn}\left(\varphi_i^{\sigma}(t,x) - \dot{r}(t)\right) \Big|_{s_i=r_i(t)} \le 0,$$
$$x \in \bigcap_{i=1}^{m} S^{i\sigma_i},\ t \in (t_0, \tau). \blacksquare \tag{3.12}$$

3.3 Strictly Sliding Mode

VSS (3.3) can evolve in sliding mode on s without any individual sliding on s^i. This behaviour is addressed as "strictly" sliding mode [10], rather being an unusual regime, if compared to the previous ones (Figure 3.1. b).

It seems to be fairly difficult to generally deal with strict SM. Most frequently, this problem is approached by the Lyapunov method. But many authors express the Lyapunov function by imposing the well- known geometrical restriction of $s_i \dot{s}_i < 0$.

Unfortunately, this does not apply here because such conditions determine a collective sliding mode, not a properly strict one.

A characterization of strict SM can be produced in terms of flow invariance, by assuming a new variable:

$$\xi := ||s|| \equiv \left(s_1^2 + s_2^2 + \dots + s_m^2\right)^{1/2}, \tag{3.13}$$

representing the usual norm in $\Re^m$, $m>1$. In this space, the prospective sliding domain is written as $\{\xi=0\}$. In any of the functioning domains $s^{i\pm}$, (3.4), the components s_i can respectively be expressed as some functions of ξ, namely:

$$s_i = \pm\left(\xi^2 - s_1^2 - s_2^2 - \dots - s_m^2\right)^{1/2},\ x \in S^{i\pm}. \tag{3.14}$$

In $\Re^n \backslash S$, system (3.3) can be defined by means of the new state vector:

$$x := \left[x_1 \dots x_{n-m} s_1 \dots s_{m-1} \xi\right],$$

as the following discontinuous right side differential equation:

$$\begin{cases} \dot{x}_1 = F_1(t,x)^{\sigma} \\ \dots\dots \\ \dot{x}_{n-m} = F_{n-m}(t,x)^{\sigma} \\ \dot{s}_1 = \varphi(t,x)^{\sigma} \\ \qquad\qquad\qquad , \quad x \in \bigcap_{i=1}^{m} S^{i\sigma_i}. \\ \dots\dots \\ \dot{s}_{m-1} = \varphi_{m-1}(t,x)^{\sigma} \\ \dot{\xi} = \varphi(t,x)^{\sigma} \end{cases} \tag{3.15}$$

Assume now that the 2^m adjacent systems pertaining to (3.15) are continuously prolongated over the whole state space $\Re^n$.

Theorem 3.4

One sufficient condition for strictly sliding motion of (3.15) on the manifold S, is that for every multi symbol σ, the following inequality holds:

$$\varphi(t,x)^{\sigma}\Big|_{\xi=0} \le 0,\ t \in \Re_+,\ x \in \Re^{n-1} * \{0\}. \blacksquare \tag{3.16}$$

See [10] for a proof. In terms of flow invariance, the reaching condition can also be stated for system (3.15). Notice that here only a sufficient reaching restriction is made on (3.3).

Theorem 3.5

One necessary and sufficient condition for (3.15) so as to reach the surface $\{\xi=0\}$ is that for each $t_0 \in \Re_+, x_0 \in \Re^n \backslash S$, there exists a differentiable function depending on (t_0, x_0), $r: \Re_+ \to \Re$, for which the following conditions are met:

1^o There exists $\tau > t_0$, such as $r(\tau)=0$;

2^o For ξ_0, (i.e. the n-th co-ordinate of x_0): $\xi_0 \le r(t_0)$;

3^o For every $\sigma=\sigma_1 \dots \sigma_m \in \{-,+\}^m$:

$$\varphi(t,x)^{\sigma}\Big|_{\xi=r(t)} \le \dot{r}(t),\ x \in \bigcap_{i=1}^{m} S^{i\sigma_i},\ t \in (t_0, \tau). \blacksquare \tag{3.17}$$

Remark 3.2

The previous approach emphasizes some flow-invariances , but it is merely a Lyapunov like approach, because $||s||$ is involved.

Nevertheless, Theorem 3.5 is a little more eloquent since it asserts not only reaching condition, but also the dynamics of reaching motion. These characteristics are strongly related to the expression of r. ■

4. Quasi Sliding Motion in Single Input System

This section addresses single input VSS in which not sliding but quasi sliding motion (QSM) manifests. Although control u in an ideal sliding mode can be designed using high (theoretically infinite) switching frequency, the real actuator devices use non-zero switching time. On the other hand, chattering appearing in control could be destructive for the equipment. For this very reason, it is necessary to introduce, for both analysis and control synthesis purposes, a quasi-sliding (or real sliding) motion [1], [3], [10], [14]-[17].

This special sort of behaviour has already been stated in Introduction, where quasi sliding domain (QSD) was defined (Definition 1.3). Remember that domain R (t), (1.3), is a QSD for (1.1) if every state trajectory originating in R (t) cannot leave it. Let us consider only the case of single input VSS, as described by (1.1) with m = 1. Let us also consider a scalar variable s (x,t) which defines the manifold S, (2.5). By proceeding like in Section 2, the new state variable (2.4) will directly replace the old variable x.

Thus, x_n is used instead of s. Accordingly, the considered neighbourhood of S is:

$$R(t):=\left\{x\in\Re^n;|s(t,x)|\leq K(t,x);K:\Re_+*\Re^n\to\Re_+\right\}. \quad (4.1)$$

We also consider the special variant where K only depends on t :

$$R(t):=\left\{x\in\Re^n;|s(t,x)|\leq K(t);K:\Re_+\to\Re_+\right\}. \quad (4.2)$$

As Definition 1.4 suggests that a special link between QSM and flow- invariance can be established, namely the set R(t) must be PFI with respect to VSS. However this assessment may not be directly expressed in terms of Theorem 1.3, because of the right hand side of (1.10) having discontinuities on S. A rigorous approach deals with some flow invariant sets pertaining to adjacent systems.

4.1 Flow Structure of Quasi Sliding Motion

Based on the adjacent systems (2.2), (2.3), let us build the following VSS :

$$\dot{x}=\begin{cases} F^-(t,x),\ x\in\left\{x\in\Re^n;\ x_n\leq -K(t)\right\}:=R^-(t) \\ H(t,x),\ x\in\left\{x\in\Re^n;|x_n|<-K(t)\right\}:=R^0(t), \\ \qquad\qquad t\in\Re_+, \quad (4.3) \\ F^+(t,x),\ x\in\left\{x\in\Re^n;\ x_n\geq +K(t)\right\}:=R^+(t) \end{cases}$$

For the time being, the switching algorithm of the VSS can establish the structure (i.e. the functioning adjacent systems) but only outside $R^0(t)$. This is the reason why H (t,x) may represent some multivalent function. Obviously, (4.2) and system (1.1), with control (2.1) have the same structure outside R (t).

We are prepared to give the main result concerning the quasi- sliding motion. If K only depends on t, i.e. R(t) is given by (4.2.), the following holds.

Theorem 4.1

Each trajectory of system (1.1), with the scalar control (2.1), starting from R(t) cannot leave this domain, i.e. R(t) is a QSD, if and only if:

$$\begin{cases} F_n^-\left(t,x_1,\dots,x_{n-1},-K(t)\right)\geq -\dot{K}(t) \\ F_n^+\left(t,x_1,\dots,x_{n-1},\ K(t)\right)\leq\ \dot{K}(t) \end{cases}, \quad (4.4)$$

for every $t\in\Re_+, x^1:=\left(x_1,\dots,x_{n-1}\right)\in\Re^{n-1}$. ■

A proof runs in [10], [15].

The QSM existence, if K depends on t and x, namely R(t) expressed as (4.1), will be properly characterized by the following theorem.

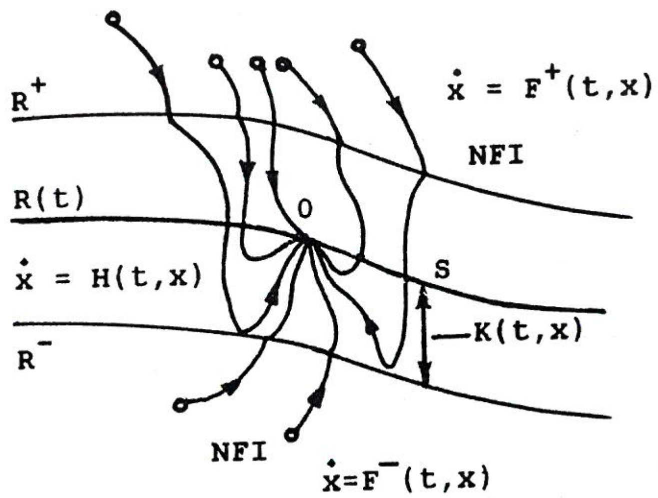


Figure 4.1 Flow Structure of Quasi Sliding Motion

Theorem 4.2

The set R defined in (4.1) is a QSD for system (1.1), with the scalar control (2.1), if and only if:

$$\begin{cases} F_n^- + \frac{\partial K}{\partial t} \nabla K F^- \geq 0, \text{ on } \left\{ x \in \Re^n, x_n = -K(t,x) \right\} \\ F_n^+ + \frac{\partial K}{\partial t} \nabla K F^+ \leq 0, \text{ on } \left\{ x \in \Re^n, x_n = +K(t,x) \right\} \end{cases}, \tag{4.5}$$

for every $t \in \Re_+, x^1 := \left(x_1, \ldots, x_{n-1}\right) \in \Re^{n-1}$. ■

Remark 4.1

Theorems 4.1 and 4.2 produce boundary conditions, which represent, despite their simplicity, a useful tool in quasi-sliding mode analysis and synthesis. This quasi sliding motion can be deliberately generated in VSS. By appropriately choosing K(t) as a decreasing function, the quasi-sliding motion can attain the ideal sliding. Moreover if K(t) depends on the initial state and so $x(t_0) \in R(t)$ does, the reaching condition is possible. In fact, if the state point belonged to R(t) at every moment, the reaching problem would make no sense. ■

It is worth noting that, according to Utkin's well-known findings, [1], the system behaviour in quasi-sliding mode is stable if the corresponding ideal sliding across S is asymptotically stable.

The flow structure of system (4.3), working in QSM, lies in that R^+ is NFI with respect to adjacent system (2.3), and $R^- \cup R^0$ is PFI with respect to the same adjacent system. The same goes with (2.2). For the assembly, i.e. VSS defined in (4.3), it is clear that R^0 is PFI, so R(t) is a QSD.

4.2. Quasi Sliding Control Synthesis for Linear Plant

Theorem 4.1 can be directly applied to control design, due to its simple statements (4.4). Consider the linear plant with scalar control.

$$\dot{x} = A(t)x + B(t)u, \; t \in \Re_+, \; x \in \Re^n, \; u \in \Re, \tag{4.6}$$

A(t) and B(t) matrices are adequately dimensioned and include continuous and locally Lipschitzean time-dependent elements. Control u is taken as:

$$u(t,x)=\begin{cases}u^-(t,x), & x\in R^-(t)\\ u^+(t,x), & x\in R^+(t)\end{cases},\qquad (4.7)$$

R^- and R^+ being stated in (4.3); meanwhile, u is not defined on $R^0(t)$. Consider the linear hyperplane defined in (1.2) by a linear function:

$$s(t,x):=cx, c:=\left[c_1 c_2 \dots c_{n-1} 1\right], c^T B(t)\neq 0, t\in \Re_+. \qquad (4.8)$$

The transformation (2.4). i.e. $\left(x_1,\dots,x_{n-1},x_n\right)\rightarrow$ $\rightarrow\left(x_1,\dots,x_{n-1},s\right)$, yields the equivalent form of (4.6):

$$\begin{cases}\dot{x}^1=E^1x^1+\tilde{a}^n s+\tilde{B}n\\ \dot{s}=c\left[A^1(t)-a^n(t)c^1\right]x^1+ca^n(t)s+cB(t)u\end{cases}\qquad (4.9)$$

where A^1 is the matrix A with the last column dropped, a^i is the i-th column of A, c^1 is c with the last element missing, and $x^1:=\left[x_1\dots x_{n-1}\right]^T$.

The forms of E^1, $\tilde{a}^n$ and $\tilde{B}$ simply result from calculus. Assume the control:

$$u(t,x)=-\left[\psi_1\psi_2\dots\psi_{n-1}\right]x^1+v\left(t,x^1\right)w(s), \qquad (4.10)$$

with:

$w(s) = \text{sign}(s)$, if $|s(t)| \geq K(t)$,

and where w might be a multivalent function (for instance, a hysteresis law). Therefore, if the state x lies in $R^0(t)$, there will no longer be restrictions on w.

Let the structure for v be [1]:

$$v\left(t,x_1\right)=-\alpha_0-\sum_{i=1}^{n-1}\alpha_i\left|x_i\right|, \qquad (4.11)$$

where the following inequalities hold:

$$\inf_t\left\{cB(t)\left(\alpha_i+\psi_i\right)-\left[ca^i(t)-ca^n(t)c_i\right]\right\}\geq 0, i=\overline{1,n-1} \qquad (4.12)$$

$$\sup_t\left\{cB(t)\left(-\alpha_i+\psi_i\right)-\left[ca^i(t)-ca^n(t)c_i\right]\right\}\leq 0, i=\overline{1,n-1} \qquad (4.13)$$

They are equally described by:

$$cB(t)\alpha_i\geq\left|ca^i(t)-ca^n(t)c_i-cB(t)\psi_i\right|, t\in\Re_+, i=\overline{1,n-1}$$

The next result comes directly from Theorem 4.1 [15].

Theorem 4.3

The set R(t) is a QSD for system (4.6), with (4.10)-(4.13), if and only if:

$$c\left[a^n(t)K(t)-\alpha_0 B(t)\right]\leq\dot{K}(t), t\in\Re_+. \blacksquare \qquad (4.14)$$

The conditions (4.10)-(4.13), coupled with the corresponding reliable margin of QSM, K(t), make a complete tool for control synthesis. The reaching of S could be solved by choosing K(t) as a function depending on t_0, x_{0n}.

Notice that the suggested QSM control must cope with some uncertainties, for $|s|<K(t)$. The art of control design is to choose w, inside R^0, in order to diminish the chattering of control u, as well as the state oscillations. Heuristics are not only allowed here, but also recommended. Valuable ideas are presented in several papers [16]-[20].

Attempts at testing some rules for w, involving good characteristics regarding oscillation alleviation, are made in the following example.

4.3. Simulation Example

To illustrate the previous method of designing deliberate quasi- sliding motion in linear VSS, please consider the linear unstable system:

$$\dot{x}=\begin{bmatrix}0.5 & 7.5 & -9\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}x+\begin{bmatrix}1\\0\\0\end{bmatrix}u, \qquad (4.15)$$

$$x=\left[x_1\, x_2\, x_3\right]^T\in\Re^3, u\in\Re.$$

We must find the control law u in order to stabilize (4.15) in quasi-sliding mode, within a QSD alike (4.2). Let us choose, according to (4.8):

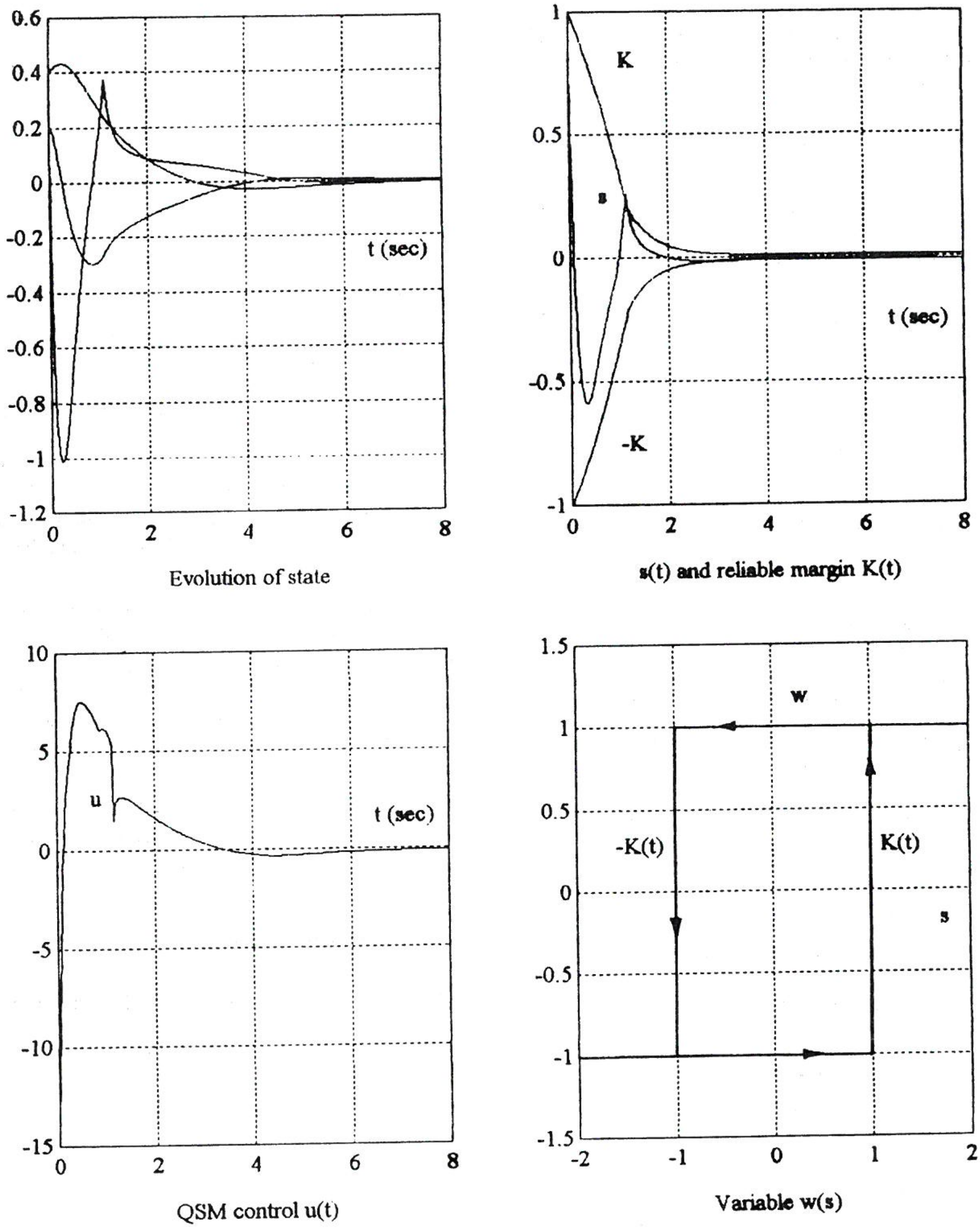


Figure 4.2 QSM for Example 4.3, with Sharp Hysteresis w

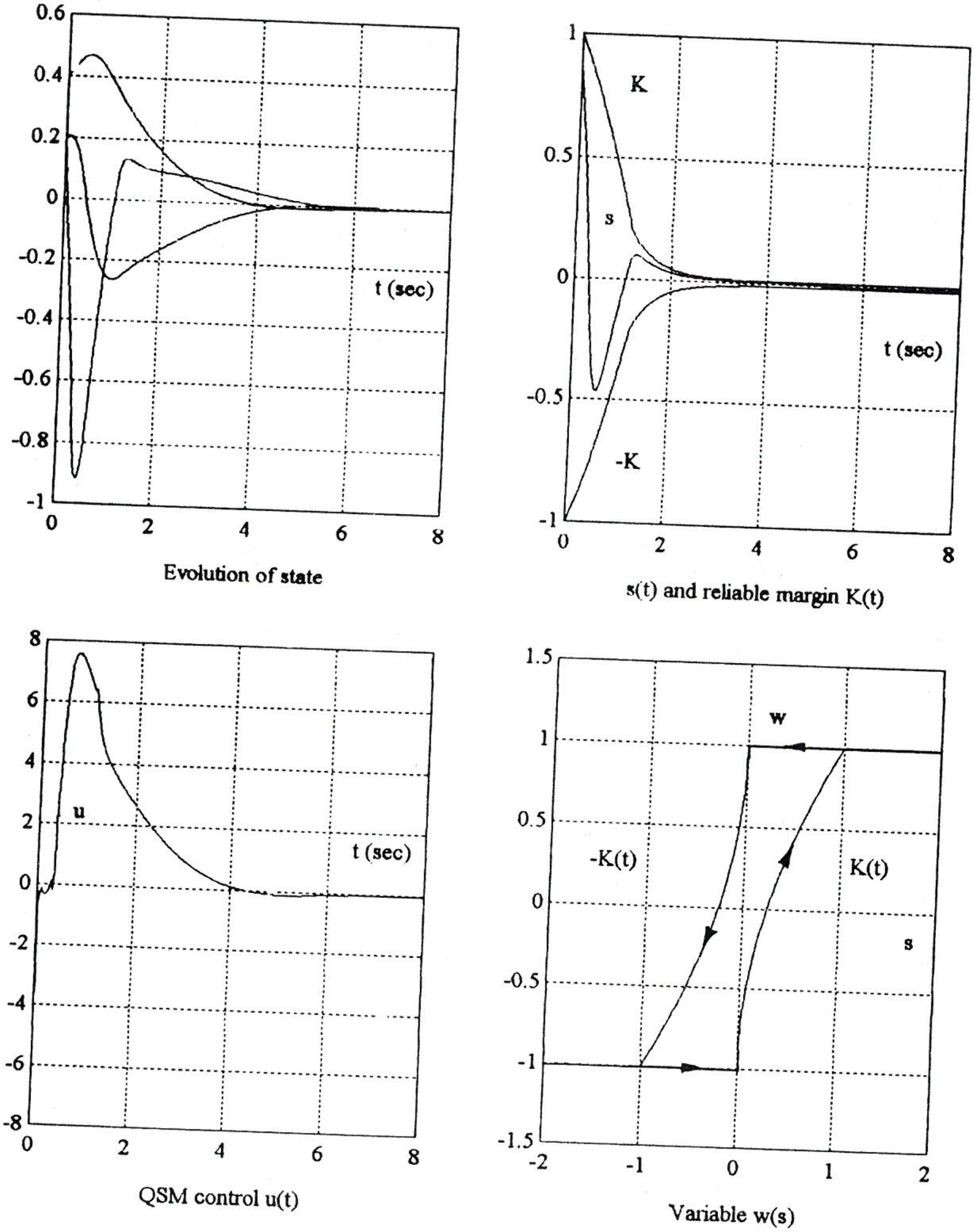
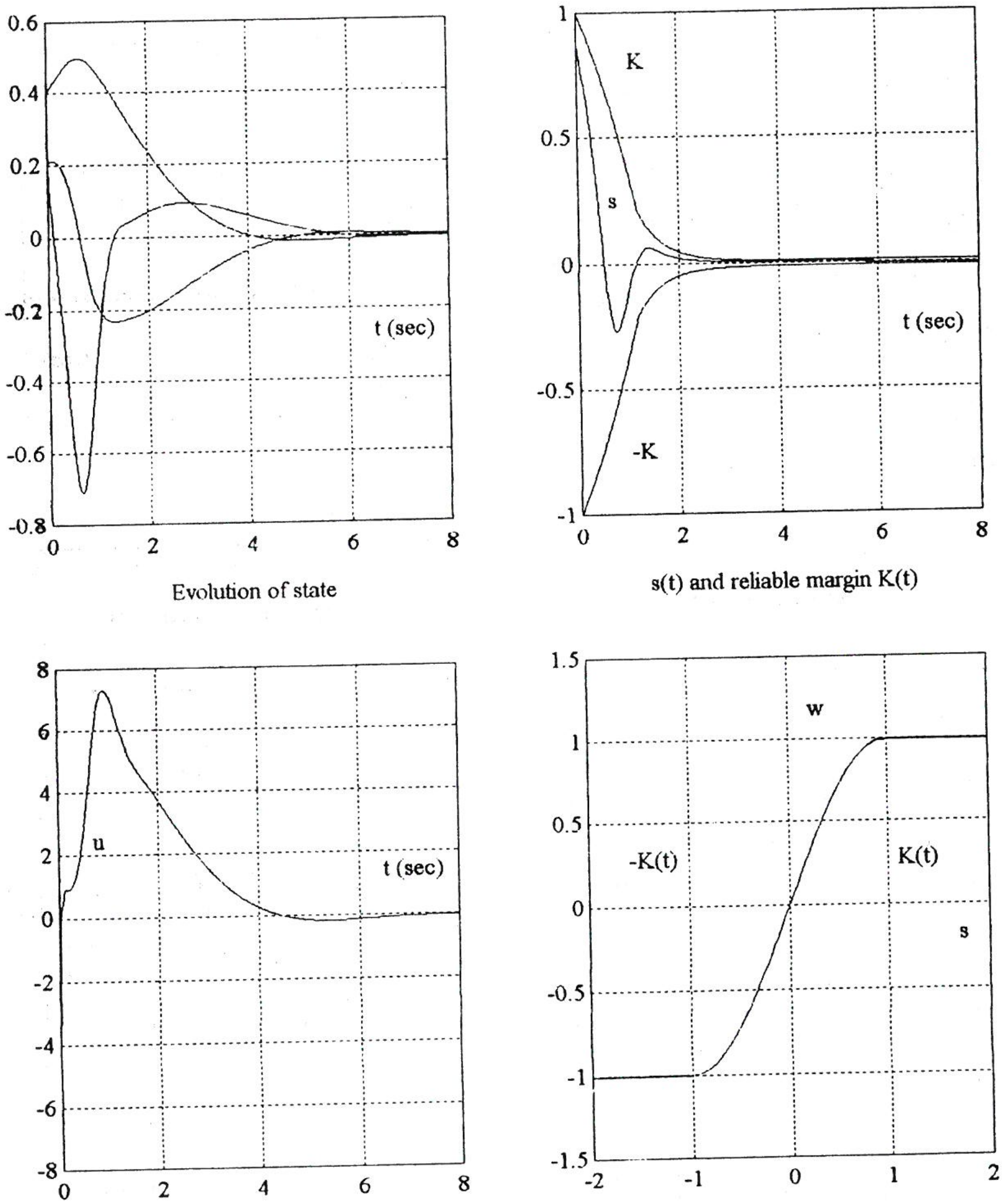


Figure 4.3 QSM for Example 4.3, with Smooth Hysteresis w



Evolution of state

s(t) and reliable margin K(t)

QSM control u(t)

Variable w(s)

Figure 4.4 QSM for Example 4.3, with Univalent w

$s(x) = cx, c = [1\ 1.4\ 1]$,

yielding the equivalent form, corresponding to (4.9):

$$\begin{bmatrix} \dot{x}^1 \\ \ldots \\ \dot{s} \end{bmatrix} = \begin{bmatrix} 9.5 & 20.1 & -9 \\ 1 & 0 & 0 \\ 10.9 & 21.1 & -9 \end{bmatrix} \begin{bmatrix} x^1 \\ \ldots \\ s \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u \qquad (4.16)$$

$$x^1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \Re^2.$$

Utkin's equivalent control is $u_{eq} = [10.9\ \ 21.1]x^1$, and one can readily see that the asymptotic stability of ideal sliding mode is ensured.

By assuming the previous notations, we get:

$ca^1 + 9c_1 = 10.9,\ ca^2 + 9c_2 = 21.1.$

Let us adopt: $\psi_1 = 8,\ \psi_2 = 25,\ \alpha_1 = 4, \alpha_2 = 6.$ Define K(t) as a decreasing function:

$$K(t) = \begin{cases} K_0\left(2-e^{t/T_1}\right), & \text{if } K(t) \geq \Delta + \Delta_0 \\ \Delta e^{-t/T_2} + \Delta_0, & \text{if } K(t) < \Delta + \Delta_0 \end{cases}, \qquad (4.17)$$

where $\Delta = 0.2, \Delta_0 = 0.01, T_1 = 2, T_2 = 0.5$ and $K_0 > |s(0)|$. Since K(t) is known, it is allowed to take, according to (4.14):

$\alpha_0 \equiv \alpha_0(t) = -\dot{K}(t) - 9K(t) + \delta_0,\ \delta_0 = 0.05$

Simulation results corresponding to $x(0) = [0.2\ 0.2\ 0.5]^T$ and $K(0) \equiv K_0 = 1$, $w(0) = -1$, are shown in Figures 4.2., 4.3, 4.4. Different laws for w(s) have been passed. They correspond to: sharp hysteresis, smooth hysteresis, univalent function .

Generally, the state responses indicate that the system is stable (but not asymptotically, because of a reliable margin) and obviously $R(t) = \{|s| \leq K(t)\}$ is a QSD.

Remark that both stability and QSM existence are not determined by choosing w(s). On the other hand, the quality of response and the oscillations of control u are related with the adopted strategy for controlling the plant into QSD, by means of function w.

5 . Final Remarks

In this paper, sliding mode in single input and multi input dynamical systems, is approached via flow- invariance method. The main results point to a coherent description of reaching and an ideal sliding.

The flow- invariance method was also used to describe the quasi- sliding motion in VSS with scalar control. This induced a procedure for control synthesis in linear VSS, which was illustrated by an example. This approach could be successfully developed for a large class of dynamical systems, liable to disturbances and parametric uncertainties.

For the sake of compactness and only concerned with a general frame of the overall links between VSS and flow- invariance, the paper does not include proofs.

We think that a review of the methods the new theories on VSS bring forth ,will make a good start for further research.

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