Introduction to Signal Processing with Wavelets

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Abstract: The Signal Processing (SP) domain is a well-delimited branch with strong connexions in both Applied Mathematics and Electronics. The present viewpoint is rather mathematics-oriented than electronics-oriented. After the main SP problem statement - in a mathematical manner - a new approach of this problem is made (from a software or engineering viewpoint).Problem- solving by classical harmonic analysis - based on **Fourier's** ideas- resulted in a certain type of algorithms called **"Fast Fourier Transform"** (FFT). This class of algorithms is used for a large set of usual signals and it is very productive. However, this tool seems to be inadequate for several signals, including non-stationary signals. Signals of a certain type, as those associated with seismology, cardiography, speech processing or image processing, are improper for frequency modelling based on the **Fourier's** series because of their not few high instantaneous frequencies. The number of computations which these signals are subject to in the **Fourier** analysis is large enough and the slow convergence of this series makes the results be not so precise as they usually are. The proposed solution for error recovery is that of the "wavelets" ("ondelettes" (Fr), "undine" (Rom)) - a new family of functions by means of which the signals can be represented more exactly. It is not the differential equations or the differences that do this representation more exactly, but the "Biscalar Dilation Equations" (BDE) or "Two Scale Difference Equations", as in the following example:

<BDE> $\Phi(x) = \sum_{n\in Z} c_n \Phi(c\alpha + \beta_n), \ \forall x \in R,$

There, $\{c_n\}_{n\in Z}$ is a family of complex numbers with finite support, $\alpha > 1$, $\{\beta_n\}_{n\in Z}$ is a real increasing numbers family and Φ is the solution of equation (the so-called "scaling function"). Usually, $\alpha = 2$, $\beta_n = -n$, for $n = \overline{o,N}$,and $\beta_n = 0$ for $n \notin \overline{0,N}$. The coefficients $\{c_n\}_{n\in Z}$ are used for generating the "wavelet-mother" Φ by the following expression:

$$\Psi(x) = \sum_{n\in Z} (-1)^n c_{1-n} \Phi(\alpha x + \beta_n) \ \forall x \in R.$$

The family of wavelets:

$$\Psi_{nm}(x) \stackrel{def}{=} \alpha^{-m/2} \Psi\left(\frac{x - \beta_n \alpha^m}{\alpha^m}\right), \forall x \in R, \ n,m \in Z$$

can be constructed as an orthogonal basis with compact support functions of **Hilbert** space ($L^2(\tau)$, $<.,.>$) (finite energy signals). Consequently, any signal $f \in L^2(\tau)$ can be codified by a denumerable sequence of coefficients:

$C_{nm} = <f, \Psi_{nm}> \ \forall n,m \in Z.$

If the signal f is a finite frequency band, then only a finite number of such coefficients will be far from zero. One effect of using wavelets will be in high frequencies or in any local irregularity of time signal's evolution producing higher coefficients values than low frequencies do. So, f can be codified by means of a computer, no high frequency information being lost.

Keywords: harmonic analysis, Fast Fourier Transform, non-stationary signals, instantaneous frequency, wavelets, Biscalar Dilation Equations, scaling function, wavelet-mother, finite energy signals.

Abbreviations:

BDE - Biscalar Dilation Equation
BLDE - Biscalar Laticeal Dilation Equation
DFT - Discrete Time Fourier Transform
EP - Engineering Problem
FFT - Fast Fourier Transform
MP - Mathematical Problem
SP - Signal Processing
$\eta = (2\pi)^{-1/2}$

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1. Signal Processing Problem

Nowadays, the Signal Processing (SP) field has been dedicated a very well-defined theory, called **"Signals Theory"**. Famous mathematicians, say, **Fourier, Dirichiet, Weierstrass** and **Hilbert**, studied the properties of certain complicated functions by using other known functions,and eventually, set the basis of the SP domain. Claimed as much by mathematics as by engineering, this space has lately developed explosively. The last half of our century findings in Applied Mathematics research led reputed scientists like **Walsh, Erdös, de Rham, Gabor, Wigner, Vil'e,** Moyal and others to revealing interesting possibilities of the initial functions for being decomposed into other simpler ones - called "components" - and for reconstructing the initial function by its components. The next step was towards generating efficient decomposition and rebuilding algorithms. This step transposes the pure SP mathematical bases to signal engineering.

Signal Theory is a vast domain not to be reduced to SP. The theory works with the "signal" concept as proposed by various scientific branches.

The definition of this concept is always unicompletely given, but, nowadays, both mathematicians and engineers are going to accept a general convention [2]:

● Definition 1 ●

A "**signal**" is an application as in the following shape:

$$f;\tau \rightarrow M,$$

where:

o τ is a (real) set having a total ordering relation ("$\leq$");

τ is also called "**the moments set**" (although, physically,it is not necessary to include time moments, but, for example, spatial distances);

o M is any (real or complex) set: "**the signal's values set**".

The natural development framework closely related to the signal concept is called " Theory of Distributions". It represents an important branch of the Functional Analysis (and- collaterally- Theory of differential equations/equations with differences). However, from an engineering point of view, the "signal" is a specific and natural notion of **Lebesgue's** spaces: $L^p(\tau)$ where $p \geq 1$. Thus, $t \leq R_+$(not necessarily bounded), whose generic element is marked by "t" or "x" and $M \leq C$ (signals with complex values).Just before speaking of the SP problem, this work is to be circumscribed.(For more details, see [1] and [2]).

□ Notes:

1. $L^p(\tau)$ is the space of the function f: $\tau \rightarrow C$ **Lebesgue** p- integrable on τ; l^p is the space of real/complex p-summable sequences $\{x_n\}_{n \in Z}$. These spaces are normate and complete, having infinite dimensions and countable vectorial bases. If p=2, then they belong to **Hilbert** spaces too, having scalar produces:

$$<f,g> \underline{def} \sqrt{\int_\tau f(x)\overline{g(x)}dx} \; ; \quad \forall \; f,g \in L^2(\tau);$$

$$<f,g> \underline{def} \sqrt{\sum_{n \in Z} f_n \overline{g_n}} \quad \forall \; f,g \in l^2.$$

Given $p \geq q \geq 1$ we have: $L^p(\tau) \leq L^q(\tau)$ and $l^p \leq l^q$, the usual framework is represented by the signal class with "**frequency spectrum**" ($L^1(\tau)$ or l^1) or by the signal class with "**energy spectrum**" (finite energy signals sets: ($L^2(\tau)$ or l^2). For convenience, we will mark by "S^p" any of the two signal spaces types (with $p \in \{1,2\}$).

2. It is well- known that $L^p(\tau)$ is the space of the "**continual**" signals (i.e. having τ homeomorphic with a certain interval from R_+; these signals can be or not uncontinuous, but bounded functions). l^p is the "**discrete-time**" signals space (i.e. having τ homeomorphic with a countable set of R_+; $f \in l^p$ is known due to its samples $f_n = f(t_n)$, $t_n \in \tau$, $\forall n \in N$). Usually, either continuous or discrete time bounded signals are considered.

□

In this context, the main mathematical problem (MP) of the SP, in a largely accepted form, is the following:

<MP> **a. The direct problem ("decomposition problem")**

Knowing the signal $f \in S^p$ and the fact that in S^p there has been a countable basis $\varepsilon = [e_n]_n \in N$, established, the problem will be how to decompose f into its components:

$$f = \sum_{n \in Z} f_{<n>},$$

where $f_{<n>} = c_n e_n$, with $c_n \in C$, $\forall \; n \in N$.

(That is to determine the coefficients $\{c_n\}_{n \in N} \leq C$).

b. The inverse problem ("reconstruction problem")

We are supposed to know a basis in S^p (as above) and the components of a signal $f \in S^p$, $\{f_{<n>} \text{ or } c_n\}_{n \in N}$, relatively to this basis, the problem being how to reconstruct the signal f from its components.

If **b.** is an easy to solve problem (knowing the basis), **a.** represents the essential problem, its solving being practically far enough from simple, due to the variety of countable bases in S^p . But, from an engineering point of view, this two-fold problem cannot be accepted as such. It is important for a mathematician to almost prove the existence of the signal decomposition and, eventually, of the uniqueness of this decomposition, whereas for an engineer the interesting part will be the modality of components effective construction or the signal reconstruction by its components. Given all the above, the problem can be reformulated as an engineering problem (EP):

<EP> a. The decomposition problem

Knowing a signal $f \in S^p$ (often by its samples over a finite number of time moments) and supposing a basis of S^p: $\{e_n\}_{n \in Z}$ has been chosen , we are asked to find a new signal the properties of which are the following:

1. f can be expressed like this: $f = \sum_{n=0}^{N} c_n e_n$, where $N \in N$ is fixed and finite and $c_0, c_1, \ldots, c_N \in C$ can be derived from the values of f;

2. f better approximates f, in the sense of the S^p's norm, i.e.

*$\forall$ $\varepsilon > 0$ $\exists f$ (depending on ε) so that $\| f - f \| < \varepsilon$; ε represents an approximation error and "$\| \cdot \|$" - the norm of S^p.

b. The reconstruction problem

Knowing the "mathematical pattern" f (from the preceding problem), for the unknown f signal, we are asked to find a way to mark the value of f at a certain moment $t_0 \in \tau$, with a precision degree given by f.

Both **a.** and **b.** become more complicated in case the coefficients "c_n" cannot be constructed directly (but recursively, for example) or in case the elements of the ε basis are not explicitly defined, but implicitly, by means of some functional equations. On constructing these coefficients from f (or f from them), it seems obvious that an algorithmisation will be carried out.

The EP solution is conditioned by finding a basis $\varepsilon = \{e_n\} n \in N$ of S^p with more interesting properties. Having a precise formula for each element of the basis (as the **Fourier** analysis asks for) is not so important for our work. Two other properties would be very significant for this approach:

A ***The efficient approximation**

The same precision of approximation ε will be required and, in comparison with other bases, this ε basis gets a minimum number N into <EP> a.1. and this determines - for certain algorithms- a minimum number of operations for the signal decomposition/reconstruction.

O **Orthogonality**

As a rule, there is the S^p context within which, by using the **Gramm-Schmidt** orthogonalization procedure, - an orthonormate basis, $\varepsilon^{\perp}$: $< e_i, e_j > = \delta_{ij}$, $\forall i,j \in N$ where "δ_{ij}" is the **Kronecker** symbol, can be eventually built. So, obtaining coefficients: $c_n = < f, e_n >$, $n \in N$ will be simplified.

Generally speaking, solving <MP> takes place in $L^p(\tau)$, while <EP> is solved in the space l^p or, more exactly, in "l_0^p ", which is the subset of l^p with elements having such finite support as: $\{x_0, x_1, \ldots, x_N, 0, 0, \ldots, 0, \ldots\}$.

This paper will deal with *O* and *A* will be the subject of another paper.

Trigonometrical Bases Problem-solving

The main underlying idea was that of any continuous or discrete time bounded stationary signal being viewed as an additive superposing of monofrequential different magnitude signals. These frequencies do not vary with $t \in \tau$, but their totality depends on τ. So, the notions called "**Fourier** series attached to a signal" and the "frequency analysis of a signal" have been introduced. Not only should the frequency content of a signal be remarked, but also a certain frequency "relevance" in the multifrequential signal.

The signal decomposition/reconstruction into/from the associated Fourier series will be further explained. Here is the middle of S^2 space. By applying **Weierstrass-Stone** theorems [1], the set of polynomials with complex coefficients is dense in C and the latter is denser in S^2. "C" denotes any of the sets:

C not the continuous functions (continuous signals) set;

c not the set of **Cauchy** strings from C (discrete bounded signals).

With the trigonometrical functions belonging to C, the set of trigonometrical polynomials is also dense in S^2. So, a remarkable orthonormate basis reveals:

$L^2: \tau = [-\pi,\pi]$;

$$\varepsilon=\left\{e_0(t)=\eta,\ e_{n1}(t)=\sqrt{2}\eta\cos(nt),\ e_{n2}(t)=\sqrt{2}\eta\sin(nt)\right\}_{n\in N}$$

$l^2: \tau = \{2\pi n/N\}_{k\in N}$;

$$\varepsilon=\left\{\left\{\cos\left(2\pi nk/N\right)\right\}_{n\in N},\ \left\{\sin\left(2\pi nk/N\right)\right\}_{n\in N}\right\}_{k\in N}.$$

The orthogonality is expressed as follows [3]:

$$\int_{-\pi}^{\pi}\cos(nx)\cos(mx)dx=\pi\delta_{nm};$$

$$\int_{-\pi}^{\pi}\sin(nx)\sin(mx)dx=\pi\delta_{nm};$$

$$\int_{-\pi}^{\pi}\sin(nx)\cos(mx)dx=0$$

$$\sum_{k=1}^{N}\cos\omega_{kn}\cos\omega_{km}=\frac{N}{2}\delta_{nm};$$

$$\sum_{k=1}^{N}\sin\omega_{kn}\sin\omega_{km}=\frac{N}{2}\delta_{nm};$$

$$\sum_{k=1}^{N}\sin\omega_{kn}\cos\omega_{km}=0$$

$\forall$ n,m$\in$N,

where $\omega_{kn}=\frac{2kn}{N}\pi$, $\forall\ k=\overline{1,N}$, $\forall n\in N$.

Mathematically speaking [1], and having $f\in L^2([-\pi,\pi])$, its decomposition and reconstruction formulae (which prove both existence and uniqueness) are, respectively:

$$\alpha_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx;$$

$$\alpha_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)dx;$$

$$\beta_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)dx\ \forall n\geq 1$$

$$f(x)=\alpha_0+\sum_{n\geq 1}\left(\alpha_n\cos nx+\beta_n\sin nx\right),\quad \forall x\in R.$$

The most current frequencies are pointed out by the maxima of string $\left\{\sqrt{\alpha_n^2+\beta_n^2}\right\}_{n\in N}$, of "harmonics power", in which $b_0=0$.

The infinite sum above is a "**Fourier** series" associated with the signal f and is converging f (in the norm sense). Normally, *f* derives from <EP> by fixing up N$\in$*N* \{0} and defining [4]:

$$\langle f\rangle \quad f(x)=\alpha_0+\sum_{n=1}^{N}\left(\alpha_n\cos(nx)+\beta_n\sin(nx)\right).$$

When referring this, we must have in mind one of the **Parseval**'s theorems:

$$\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\beta_n=0$$

The sum above includes trigonometrical functions with fixed growing frequencies. In studying the frequency structure, the "**Fourier** Transform" of a signal can be used, as defined below [2]:

• Definition 2 •

> **"The Fourier Transform"** of a continual signal $f\in L^p(\tau)$ is:
>
> $$\begin{cases} F_f: R\to C \\ \omega\to F_f(\omega)\overset{def}{=}\int_{-\infty}^{+\infty}f(x)e^{-j\omega x}dx, \text{ with } j^2=-1. \end{cases}$$
>
> For a discrete-time signal $f\in l^p$, where $f=\left\{f_n\right\}_{n\in N}$, the integral becomes series:
>
> $$\begin{cases} F_f: R\to C \\ \omega\to F_f(\omega)\overset{def}{=}\sum_{n\in Z}f_n e^{-j\omega n}, \text{ with } j^2=-1. \end{cases}$$

ω varies continuously with R and F_f is, in either case, an analytical function.

From an engineering point of view, and in order to solve these problems, a "Discrete **Fourier** Transform" (DFT) will be used, with its algorithms called "Fast **Fourier** Transform" (FFT) [4].Let $f\in l^2$, be

$f=\left\{\ldots,0,f_0,f_1,\ldots,f_{N-1},0,\ldots,0,\ldots\right\}$.

We mark by w not exp $(-2\pi j/N)$ and give the following definition:

● Definition 3 ●

> **"The DFT"** of $f \in l^2$ is the sequence (string):
>
> $$DFT\,[f] \stackrel{def}{=} \{\ldots,0,0,\} \cup \left\{\sum_{n=0}^{N-1} f_n w^{kn}\right\}_{k\in\overline{0,N-1}} \cup \{0,0,\ldots\}$$

This time, DFT variation for "w^k" will not be continuous, but discrete variation, in equally spaced points on the unit circle of the complex plane. Consequently, "analyticity" of DFT cannot be invoked.

FFT is a set of algorithms providing an efficient computation of DFT in the special case: N=2. Generally, FFT yields "Wf=F", in which: $f=\left\{f_n\right\}_{n=\overline{0,N-1}}$ (column vector), $F=\left\{F_n\right\}_{n=\overline{0,N-1}}$ is the column vector composed of the non-zero elements of DFT and W is a symmetric matrix: $W=\left[w^{kn}\right]_{k,\,n=\overline{0,N-1}}$.

Given $w=\cos\left(2\frac{\pi}{N}\right)+j\sin\left(2\frac{\pi}{N}\right)$, it follows that $DFT[f]\in l^2$ and its expression is represented by the ε basis elements. The "Wf=F" formula makes f be decomposed, while the property of W is of being symmetrical and of satisfying the following relation:

$$W\overline{W}=\overline{W}w=(1/N)\,I_N,$$

where I_N is the N-unit matrix, and makes signal f be reconstructed from F,according to the formula:

$$f=W^{-1}F=(1/N)\,\overline{W}F$$

The number of operations necessary for computing signal f decomposition and reconstruction is $\sigma(N)=N^2$ for DFT, but $\sigma(N)=N\log_2 N$ and even $\sigma(N)=N\log_2\sqrt{N}$ for FFT. Avoiding the inversion of a big order array like W is noticed. If $N=2^m$, then $\sigma(m)=2^m m$.

General Aspects of Dilation Equations

2.1. The "Dilation Equation" Concept

Although FFT needs simple and few operations, using it has many drawbacks which generate new researches for the SP problem- solving. These drawbacks can be directly charged to the type of vectorial basis used or ,more significantly, to Fourier series idea. Can any signal be looked upon as a multifrequential superposing of cosines by time constant frequencies? Some seismological researches as well as some chapters of the Fractals Theory or of the Image Processing Theory show that such a hypothesis is too restrictive. The **Fourier** analysis does not apply to a set of signals-the so-called "non-stationaries".

A new idea will define these signals: the frequencies which compose the signal are "instantaneous" frequencies, i.e. they can also vary during time moments (even if not always explicitly). The signals not supporting the classical **Fourier** analysis are of a special type and are represented by a famous example: **de Rham's** functions. We mark by $\{f_n\}_{n\in N}$ these functions and give their definitions below:

$$\left[\begin{array}{l} f_0:R\to R \\ x\to f_0(x) \stackrel{def}{=} \begin{cases} 1+x, & x\in[-1,0) \\ 1-x, & x\in[0,+1] \\ 0, & |x|>1 \end{cases} \end{array}\right.$$

f_1: R→R is obtained from f_0 such as:

$$\left[\begin{array}{l} \forall m\in Z \Rightarrow f_1(m)=f_0(m)\,\&\,f_1\left((m+1)/3\right)= \\ \quad =f_0\left((m+2)/3\right)\&\,f_1\left((m+2)/3\right)=f_0\left((m+1)/3\right) \\ \forall x\in[m/3,(m+1)/3)\Rightarrow \\ \Rightarrow f_1(x)=3\left[f_1\left((m+1)/3\right)-f_1(m/3)\right]x+ \\ +f_1(m/3)-m\left[f_1\left((m+1)/3\right)-f_1(m/3)\right]. \end{array}\right.$$

f_n: R→R is obtained from f_{n-1}, $\forall n\geq 1$ as above.

Figure 1.a presents the evaluation process generated by the above definitions. For n large enough, f_n will be described as in Figure 1.b. So, just little magnitude high frequency oscillations superpose over a low frequency curve. Such signals are typical in seismology and cardiology and processing them by using a **(Fourier)** multifrequential model assumes that:

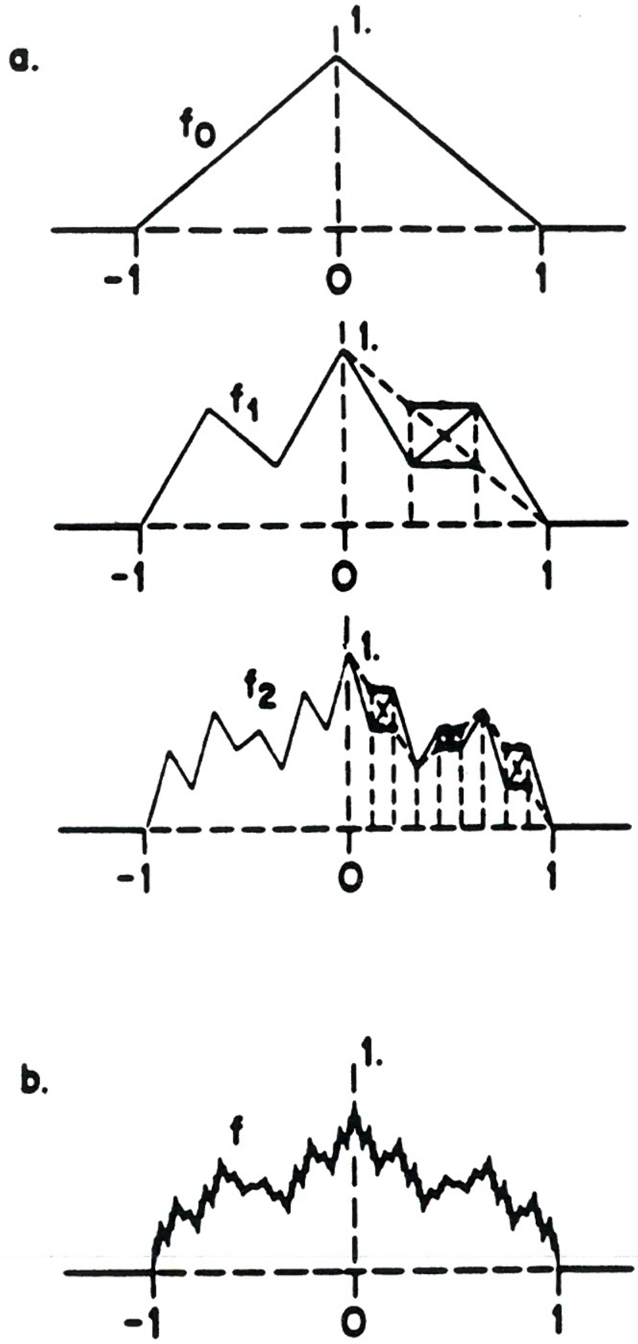

Figure 1.
a. The First Three Approximations of de Rham's Function
b. The 8th Approximation of de Rham's Function; at the Figure Scale, f_8 is very close to f- the Solution of de Rham's Dilation Equation

a) in a continual case - N of <f> is very large for an acceptable error e between f and f (since, in the sum of <f>, the α_n and β_n coefficients hardly approach 0, high frequency harmonics having high magnitudes; the strings of these coefficients slowly approach 0)

b) in a discrete-time case - the sampling frequency is high for better observing the local "vibrations" of the signal.

The effort to compute is great, complex and, implicitly, time-consuming in either case. Furthermore, technical problems arise if high sampling frequencies (with the same precision of samples)are to be obtained.

Based on these observations and on others, no less convincing, as **Daubechies', Meyer's, Mallat's** [5], [6], [7], a new framework has been created for only studying the SP problem solutions (including this set of signals). The §1 context being maintained, we are searching for another construction of the S^P's basis, namely, using one property of the above functions : if f is the pointwise limit of the string $\{f_n\}_{n\in N}$, then any approximation of f from $\{f_n\}_{n\in N}$ can be made recursively: $f_{n+1}=vf_n$ "v" is a functional operator; for example, in the case of **de Rham** functions, the v operator satisfies:

$$<R_r> v\Phi(x)=\Phi(3x)+\frac{1}{3}\Phi(3x+1)+\frac{1}{3}\Phi(3x-1)+$$

$$+\frac{2}{3}\Phi(3x+2)+\frac{2}{3}\Phi(3x-2),\ x\in R.$$

Giving a new basis to S^P requires a preparatory phase. Now, the"Two Scale Difference Equations", also called "Biscalar Dilation Equations" (BDE) [5] are the new mathematical framework.

• Definition 4 •

A "biscalar dilation equation" (BDE) is a two-scale difference equation of this shape:

$$<BDE>\quad \Phi(x)=\sum_{n=0}^{N} c_n\Phi\left(\alpha x+\beta_n\right),$$

where: $\alpha>1$; $\beta_0<\beta_1<...<\beta_N$; $c_0,c_1,...,c_N\in C$; $\Phi\in S^P$, $x\in R$.

The above equation is a "dilation" equation due to the omothetia by coefficient $\alpha>1$ applied to the argument of "**scaling functions**" Φ. This argument appears to be amplified ("dilated") after all. By convention, we will consider $\tau\equiv R$; this is a not too restrictive condition: $f(x)=0$ is possible for any $x\in\tau$.

The "two-scale" term comes from obtaining Φ from a sum of its values by arguments affecting two operations : a translation (on the "scale" of coefficients $\beta_0,...,\beta_N$) and an 0-centred, α-multiplied omothetia, of which variation yields a second "scale". Usually, we say that the translation affects the time domain and the omothetia - the frequency (or scale) domain behaviour of the signal. In fact, the "scale" is the inverse of the frequency.

• Definition 5 •

If into $<BDE>: \alpha=k\in N\setminus\{0,1\}$ and $\beta_n=-n, \forall n\in\overline{1,N}$ then the equation is called "biscalar laticeal dilation equation":

$$<BLDE>\Phi(x)=\sum_{n=0}^{N} c_n\Phi\,(kx-n),\quad \forall\in R.$$

Such an equation can be written as below:

$$<BLDE'>\ \Phi(x)=\sum_{n\in Z} c_n\Phi(kx-n),\ \forall x\in R,$$

where $\{c_n\}n\in Z\leq C$ has a "finite support" $\left(\exists N_1\leq N_2\in Z \text{ such as } c_n=0, \forall n\notin\{N_1,N_1+1,...,N_2\}\right)$ and, using a translation of axes and a recount of indices of coefficients "c_n", it looks like a BLDE.

The BDE/BLDE's solutions (existence and uniqueness) can be studied using either the **Fourier** transform or Theory of operators [5]. The quantity:

$$\Delta \overset{not}{=} \frac{c_0+c_1+...+c_N}{\alpha}=\frac{c_{N1}+..+c_{N2}}{\alpha}$$

is essential to the solutions existence. Thus, [5] is the proof of the following result:

Theorem

a. If $|\Delta|<1$ *or* $|\Delta|=1$, *with* $\Delta\neq 1$, *then BDE has only a trivial solution;*

b. If $|\Delta| > 1$ then BDE cannot get a trivial solution, it can only have a unique solution or an infinity of solutions, without compact support;

c. If $\Delta = 1$ (the most interesting case), then BDE has a unique solution included in $[0,N|(\alpha-1)]$, with compact support.

Note that a function f "support" is marked by "Supp(f)" and is defined as a closure of the set $\{x \in \tau \mid f(x) \neq 0\}$.

For the EP, the most interesting framework is provided by <BLDE'> with $\Delta = 1$.

The existence and uniqueness of the solution is thus guaranteed, and, furthermore, the solution

$F: R \rightarrow C$ is "normalized": $\int_R \Phi(x)dx = 1$. Using these equations, certain signals can recursively be modelled. Some examples are of a special interest:

1st Example

Dirac distribution:: $\Phi(x) = \delta(x), x \in R$ ($\delta(x) = \infty$ if $x = 0$ and $\delta(x) = 0$ elsewhere is the solution of the equation: $\Phi(x) = 2\Phi(2x)$, which corresponds to the coefficients: $c_0 = 2 \& (c_k = 0, \forall k \neq 0) \& \alpha = 2$.

2nd Example

Box-function::

$$\Phi(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & \text{elsewhere} \end{cases}$$ is the solution of the equation:

$\Phi(x) = \Phi(2x) + \Phi(2x-1)$ with the coefficients $c_0 = c_1 = 1$ and $\alpha = 2$.

3rd Example

Hat - function::

$$\Phi(x) = \begin{cases} x & , x \in [0,1) \\ 2-x, & x \in [1,2] \\ 0 & , \text{elsewhere} \end{cases}$$ verifies the equation:

$$\Phi(x) = \frac{1}{2}\Phi(2x) + \Phi(2x-1) + \frac{1}{2}\Phi(2x-2),$$

with the coefficients: $c_0 = c_2 = \frac{1}{2}$ and $\alpha = 2$.

A BLDE aware of a certain scaling function Φ has

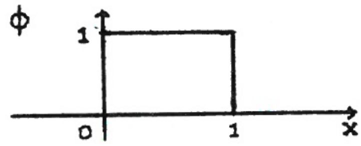

Figure 2. Box Function

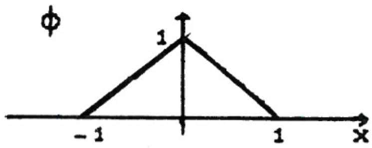

Figure 3. Hat Function

been mentioned in these examples . The inverse problem, finding solutions to a BLDE would be interesting. There are several methods for solving a BLDE. For example, a solution can be reached recursively, using the following operator (the shape being as suggested by the BLDE):

$$v\Phi(x) = \sum_{n \in Z} c_n \Phi(kx-n), \quad \forall x \in R.$$

The scaling function Φ verifies the equation $v\Phi = \Phi$. That means Φ is a fixed point for the operator v. According to the famous fixed point **Banach-Picard** theorem in Functional Analysis [1], the construction will develop this way:

- Initially, Φ is considered as being satisfactorily approximated by a knowing function Φ_0;
- An iterative process follows where $\{\Phi_m\}_{m \in N}$ is an approximations string of Φ, so that:
  $\Phi_{m+1}(x) = v\Phi_m(x) = \sum_{n \in Z} c_n \Phi_m(kn-n) \forall x \in R.$
- The process stops after a number of steps, M, so that Φ_M is a "fine" approximation of Φ;

the accuracy degree is user's choice.

It is not difficult to see that if Φ is the pointwise limit of the approximations string $\{\Phi_m\}_{m\in N}$, then it will verify <ELDB'> (by passing to limit in the recursive equation $\Phi_{m+1} = v\Phi_m$). [5] shows that the string above is pointwise convergent to a function $\Phi \in S^1$, if $\Phi_0 \in S^1$. The approximation process goes on as in Figure 1.a. This scaling function (**de Rham**'s function) has a fractal shape (between any two support points, there is at least a third point in which a unique value of the first derivative, even if the function is continuous, cannot be specified).

2.2. Grouping Frequencies into Octaves

An interesting method for BLDE solving is proposed by the connection between the omothetia effect on the scaling function's argument and a **Weber-Fechner** physiology law on the model of human auditive perception. The law states that the noise intensity is logarithmically perceived by the human being: "*The sensation is proportional to the logarithm of excitation*". This fact suggests that the logarithm of excitation should represent sensation as a linear function. The graph can be traced using a semilogarithmical technique: the horizontal axis is logarithmically (base 2) scaled and is called "the sonorous frequencies octaves axis"; the vertical axis is linearly scaled. The interval between two consecutive integer values of the "$\log_2$" is denoted by "$[d_m, d_{m+1}]$" and is called "octave". Frequentially speaking, all the eight musical sounds are to be found in one of these octaves.

On getting back to the dilation equation, we will assume that any point $x\in Z$ values of the scaling function are known, and define the points "$x/\aleph_m$" with $x\in Z$, $m\in N$, $\aleph_m = k^m$ and $k\in N$, $k\geq 2$ (the scale factor) fixed, by "k-dic points". If $k=2$, these points will be called "dyadic points". In this (most frequent) case, the term "octave" corresponds to the usual intuitive term (presented above). In general, an "octave" has the shape $[x/\aleph_m, x/\aleph_{m-1}]$, and will be called "generalised octave" or "k-octave". Using the recursive relation as above, the Φ values in any "k-dic point" will be determined:

$$\Phi\left(x/\aleph_{m+1}\right) = \sum_{n\in Z} c_n \Phi\left(x/\aleph_m - n\right), \ \forall\, m\in N.$$

By considering Φ as continuous in zero point, Φ can also be determined by passing to the limit for $m\to\infty$ in this relationship.

But, the main problem left is to determine the values of Φ on integer points, for initiating this process. At this point, the compact support of Φ can successfully solve the problem. The dilation equation can be several times written: one relation for each $x\in Z$. Given *Theorem*, for $k=2$, there is $N\in N$ so that:

$$\Phi(x) = \sum_{n=0}^{N} c_n \Phi(kx-n), \ \forall x\in R$$

and $\operatorname{Supp}(\Phi) \leq \left[0, \frac{N}{k-1}\right]$.

A finite number of such relations will not express $0\equiv 0$ identities. They are:

$$\begin{cases} \Phi(1) = c_0\Phi(k) + c_1\Phi(k-1) + \ldots + c_N\Phi(k-N) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \Phi(M) = c_0\Phi(Mk) + c_1\Phi(Mk-1) + \ldots + \\ \qquad + \ldots + c_N\Phi(Mk-N) \end{cases},$$

where M is the cardinal of set: $N\cap(0,N/(k-1))$ and the $(M+1)^{th}$ equation is an identity. This system can be represented as a matrix as below:

$$\underbrace{[\Phi(j)]_{j=\overline{1,M}}}_{not_\Phi} = \underbrace{[\alpha_{ij}]_{i,j=\overline{1,M}}}_{not_L} = \underbrace{[\Phi(j)]_{j=\overline{1,M}}}_{not_\Phi}$$

or, with such new notations as: $\Phi = L\Phi$. The coefficients of L matrix are to be obtained from $c_0,\ldots,c_N$ by an obvious new arrangement. So, Φ is the eigenvector of L for the eigenvalue $\lambda = 1$. The *Theorem* shows that the spectrum of L matrix covers the eigenvalue $\lambda = 1$. Now, we can evaluate all the values of Φ on integer points of the interval $[0,N/(k-1)]$ (all the rest will be null).

We shall take into consideration the **Daubechies** dilation equation [5], [6]:

$$\Phi(x) = c_0\Phi(2x) + c_1\Phi(2x-1) + \\ + c_2\Phi(2x-2) + c_3\Phi(2x-3), \ \forall\, x\in R,$$

where:

$N=3$, $k=2$, $\operatorname{Supp}\Phi \leq [0,3]$, $c_0 = \left(1+\sqrt{3}\right)/4$,

$c_1=\left(3+\sqrt{3}\right)/4,\ c_2=\left(1-\sqrt{3}\right)/4,\ c_3=\left(3-\sqrt{3}\right)/4.$

Daubechies showed in [5] that the solution of this equation is continuous, so $\Phi(0)=\Phi(3)=0$. Then $M=2$ and:

$$\begin{bmatrix}\Phi(1)\\ \Phi(2)\end{bmatrix}=\begin{bmatrix}c_1 & c_0\\ c_3 & c_2\end{bmatrix}\begin{bmatrix}\Phi(1)\\ \Phi(2)\end{bmatrix}$$

The spectrum of this matrix is $\{0.5,1\}$, and, in this case, the minimal eigenvector corresponding to the unit eigenvalue offers this solution: $\Phi(1)=2c_0$, $\Phi(2)=2c_3$. Now, the scaling function can be evaluated in any dyadic point of $\{0.3\}$ interval, using the dilation equation. The shape of this scaling function can be seen in Figure 4.

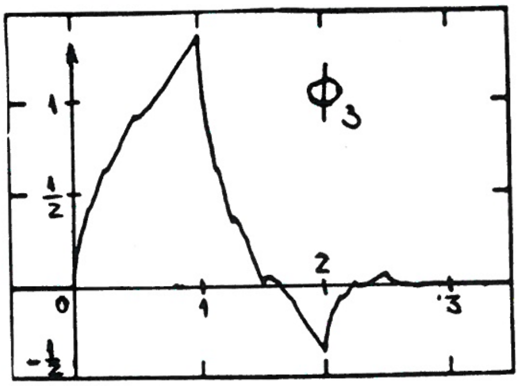

Figure 4. The Daubechies Scaling Function

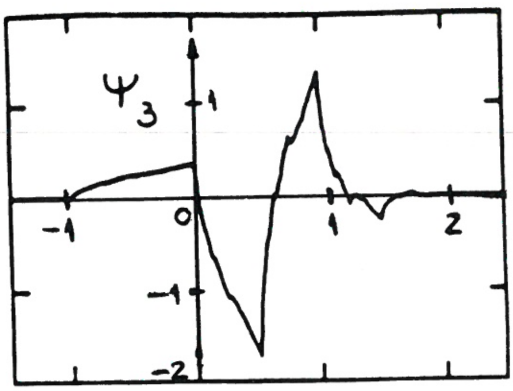

Figure 5. The Daubechies Wavelet

3.Wavelets

3.1 Wavelet Definition and Examples

The BDEs proposed a new pattern for solving the SP problem. However, this problem has not been solved yet. Let us now return, for an instant, to the graph of Figure 1.b. One can clearly observe the vibrations of the last graph round about an average instantaneous value. The vibrations are called "small waves" and they are the measure of a signal's local unsteadiness. The BDEs offered a new pattern to signals' processing , but with some restrictions. One restriction asks for the values of any scaling function to be positive, i.e. only the positive signals can be processed. Of course, because of any natural signal being bounded, using a simple translation on axes can make any signal be positive. One most important requirement is the orthogonality of the scaling functions family:

$\nu=\{\Phi_{mn}\}_{m,n\in Z}$, where:

$$\begin{cases}\Phi_{mn}:R\to C & \forall m,n\in Z.\\ \quad x\to\Phi_{mn}(x)\overset{def}{=}\Phi(k^{-m}x-n)\end{cases}$$

Unfortunately, it is impossible that this requirement is met. So, a new concept has been defined [6].

● Definition 6 ●

A **"wavelet (-mother)"** is any application of the shape:

$$\begin{cases}\Psi:R\to C\\ \quad x\to\Psi(x)\overset{def}{=}\sum_{n\in Z}(-1)^n c_{1-n}\Phi(kx-n),\end{cases}$$

where Φ is the unique solution of the BLDE (having $\Delta=1$):

$$\Phi(x)=\sum_{n=0}^{N}c_n\Phi(kx-n)\qquad \forall x\in R.$$

The number of non-zero terms in the sum defining the wavelet- mother is obviously finite. If $0\notin\{c_0,\ldots,c_N\}$, then $c_{1-n}=0$, for all $n>1$ or $n<1-N$. The following wavelets examples are deduced from the few BLDEs already presented.

4th Example

Box - function ::

$$\Phi(x)=\begin{cases}1, & x\in[0,1)\\ 0, & \text{elsewhere}\end{cases} \qquad (\Phi(x)=\Phi(2x)+\Phi(2x-1))$$

leads to Haar wavelet:

$$\Psi(x)=c_1\Phi(2x)-c_0\Phi(2x-1)=\Phi(2x)-\Phi(2x-1), \quad \forall x\in R.$$

(The associated graph is shown in Figure 6).

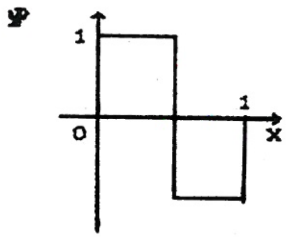

Figure 6. Haar's Wavelet

5th Example

Hat-function::

$$\Phi(x)=\begin{cases}x & , x\in[0,1)\\ 2-x, & x\in[1,2]\\ 0 & , \text{elsewhere}\end{cases} \quad \left(\Phi(x)=\frac{1}{2}\Phi(2x)+\Phi(2x-1)+ +\frac{1}{2}\Phi(2x-2)\right)$$

leads to the wavelet in Figure 7, described by the relation:

$$\Psi(x) = -c_2\Phi(2x+1) + c_1\Phi(2x)-c_0\Phi(2x-1) = = -\Phi(2x+1)/2 + \Phi(2x)-\Phi(2x-1)/2$$

6th Example

The **Daubechies** scaling function leads to the wavelet in Figure 5.

The wavelet support is also compact (like the support of the scaling function); for $k=2$, if Supp $(F)\geq[N_-,N_+]$, then

$$\text{Supp}(\Psi)\leq\frac{1}{2}\left[1-N_+ + N_-, 1+N_+ - N_-\right].$$

Given the correspondence between the scaling function Φ and the wavelet Ψ, Φ is also called "wavelet-father".

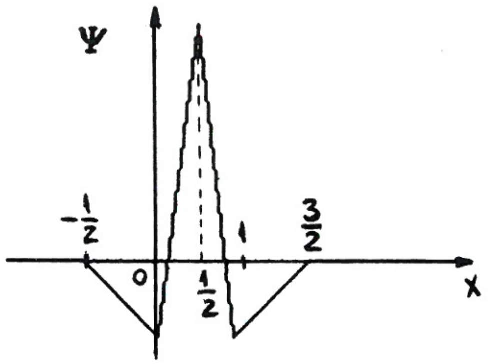

Figure 7. "Hat" Wavelet

3.2. Wavelet and Dilation Subspaces in S^2. Orthogonality

We will use the scaling function and the associated wavelet- mother for generating two bases in S^2. The bases can produce a solution to the SP problem in the field of non-stationary signals.

Let us consider the BLDE as in Definition 5 or Definition 6, with $\Delta = 1$. The corresponding scaling function Φ is generated for both the family ν and the wavelet-mother Ψ. For each fixed $m\in Z$, "v_m"denotes the subspace spun in S^2 by the "partial" family: $\nu_m = \{\Phi_{mn}\}n\in Z$. V_m is called "**dilation subspace**" and its structure is the following:

$$v_m = \left\langle \{\Phi_{mn}\}_{n\in Z} \right\rangle =$$

$$= \left\{ \sum_{n=-N_1}^{N_2} \alpha_n \Phi_{mn} \mid \alpha_n\in C, \forall n=\overline{-N_1,N_2}, N_1,N_2\in N \right\}$$

(finite linear combinations (expansions) with elements from v_m). Given the relation:

$$\Phi(k^{-(m+1)}x-n)=\sum_{p\in Z} c_p\Phi(k^{-m}x-kn-p), \ \forall x\in R,$$

which holls for each fixed m, $n\in Z$, we can write that $\Phi_{m+1,n}\in v_m$, so $v_{m+1}\leq v_m$. Consequently, it is obvious that:

$$\ldots\leq v_m\leq\ldots\leq v_1\leq v_0\leq v_{-1}\leq\ldots\leq v_{-m}\leq\ldots.$$

Daubechies and **Mallat** [6] showed that:

$\bigcup_{m\in Z} v_m$ is dense in S^2 and $\bigcap_{m\in Z} v_m = \{0\}$.

That means ν is a basis in S^2, so for any $f\in S^2$ there is a finite support family of coefficients $\{f_{mn}\}_{m,n\in Z}\leq C$ expressing f as below:

$$f(x)=\sum_{m\in Z}\sum_{n\in Z} f_{mn}\Phi(k^{-m}x-n),\ \forall x\in R.$$

In this relation, the two-scale idea and the frequency grouping by k-octaves ($[k^{-m},k^{-m+1}]$) are both suggested once again.

Now, we can define $W=\{\Psi_{mn}\}_{m,n\in Z}$, where:

$$\begin{cases}\Psi_{mn}:R\to C \\ x\to\Psi_{mn}(x)\overset{def}{=}k^{-m/2}\Psi(k^{-m}x-n)\end{cases},$$

that is the family of wavelets generated by the wavelet-mother Ψ with the same technique as in the case of ν family, but with the additional factor "$k^{-m/2}$" for normalisation reasons. For any fixed $m\in Z$, the "partial" family $W=\{\Psi_{mn}\}_{n\in Z}$ spun in a subspace denoted by "W_m" in S^2. Its structure is:

$$W_m=\langle W_m\rangle=$$

$$=\left\{\sum_{n=-N_1}^{N_2}\beta_n\Psi_{mn}\,\middle|\,\beta_n\in C,\ \forall n\in\overline{-N_1,N_2}, N_1,N_2\in N\right\}$$

(finite expansions with elements from W_m). As above, W_m is called "**wavelet subspace**" and the relation:

$$\Psi(k^{-(m+1)}x-n)=\sum_{p\in Z}(-1)^p c_{1-p}\Phi(k^{-m}x-kn-p),$$

$$\forall x\in R,$$

leads towards $W_{m+1}\leq v_m$. The most important property of wavelet subspaces derives from its orthogonal structure [6].

For clarity sake, the orthogonality will be studied in the **Hilbert** space $L^2(\tau)$. Be $\Phi_0\in L^2(\tau)$ with the property:

$$\int_{-\infty}^{+\infty}\Phi_0(kx-n)\overline{\Phi_0(kx-m)}dx=\delta_{m,n}\quad\forall m,n\in Z,$$

where $\delta_{m,n}$ is the **Kronecker** symbol. For example, the box-function satisfies this requirement. Φ_0 is used to initialize the approximation process in solving BLDE. Then, the first approximation of the scaling function Φ is

$$\Phi_1(x)=\sum_{n\in Z}c_n\Phi_0(kx-n),\ \forall x\in R,$$

and we can write the following result:

$$\int_{-\infty}^{+\infty}\Phi_1(x)\overline{\Phi_1(x-m)}dx=k^{-1}\|\Phi_0\|^2\sum_{p\in Z}\overline{c_{p-km}}c_p,$$

where

$$\|\Phi_0\|^2\overset{def}{=}\int_{-\infty}^{+\infty}|\Phi_0(x)|^2dx.$$

"$\Phi_1(x-m)$" will be orthogonal with "Φ_1" if and only if:

$$\langle 11\rangle\quad \|\Phi_0\|^2\sum_{p\in Z}c_p\overline{c_{p-km}}=k\,\delta_{0,m}\quad\forall m\in Z.$$

Also, we can choose a normalised version of $\Phi_0(\|\Phi_0\|=1)$, so that <11> is equivalent to:

$$\langle 11'\rangle\quad\sum_{p\in Z}c_p\overline{c_{p-km}}=k\,\delta_{0,m}\quad\forall m\in Z.$$

Using an induction process, we can show that <11'> is closer to

$$\Phi_n(x-m)\perp\Phi_n(x)\Rightarrow\Phi_{n+1}(x-m)\perp\Phi_{n+1}(x),$$
$$\forall m\in Z\setminus(0).$$

To the limit $n\to\infty$, $\Phi_n\to\Phi$, so that <11 bis> $\Leftrightarrow\Phi(x-m)\perp\Phi(x), \forall m\in Z\setminus(0)$ (Φ is orthogonal upon all its translates). Furthermore, if $\Phi(x-m)\perp\Phi(x), \forall m\in Z\setminus(0)$, then any two different translates of Φ will be orthogonal, because of the following obvious relation:

$$\int_{-\infty}^{+\infty}\Phi(x-m)\overline{\Phi(x-n)}dx=\int_{-\infty}^{+\infty}\Phi(x)\overline{\Phi(x+m-n)}dx,$$

$$\forall m\neq n\in Z.$$

So, only a certain kind of BLDE produces orthogonal solutions. In this case, the subspace v_m is spun by an orthogonal family. Nevertheless, this is not a sufficient condition for the orthogonality between the family of subspaces $\{v_m\}_{m\in Z}$. Let us consider that Φ is an "orthogonal scaling function" (in the above sense). Then the corresponding wavelet will satisfy the properties:

$$\int_{-\infty}^{+\infty}\Psi(x)\overline{\Phi(x-m)}dx=$$

$$= \sum_{p\in Z} (-1)^n c_{1-p} \int_{-\infty}^{+\infty} \Phi(kx-p)\overline{\Phi(x-m)}dx =$$

$$= k^{-1} \|\Phi\|^2 \sum_{p\in Z} (-1)^n c_{1-p}\overline{c_{p-km}} =$$

$$= k^{-1} \sum_{p\in Z} (-1)^n c_{1-p}\overline{c_{p-km}} \quad \forall m\in Z \quad \left(\|\Phi\|^2=1\right);$$

<12> $\Psi(x)\ \Phi(x-m)\ \forall m\in Z\backslash\{0\} \Leftrightarrow$

$$\Leftrightarrow \sum_{p\in Z} (-1)^n c_{1-p}\overline{c_{p-km}} = 0\ \forall m\in Z\backslash\{0\}$$

In the same manner, it follows that:

$$\int_{-\infty}^{+\infty} \Psi(x-m)\overline{\Psi(x)}dx = k^{-1} \sum_{p\in Z} (-1)^n c_p\overline{c_{p-km}}, \forall m\in Z$$

and: $\Psi(x)\perp\Psi(x\text{-}m)\ \forall m\in Z\backslash\{0\} \Leftrightarrow$ <12>. Like in the scaling functions case, any of two different translates could be orthogonal: $\Psi(x\text{-}n)\perp\Psi(x\text{-}m)$, $\forall n\neq m\in Z$ provided (12). In this case, W_m is spun by an orthogonal wavelets family. Furthermore, even the entire family of wavelet subspaces can be orthogonal, by mutual orthogonality. To prove that, a new concept is used: "the multiresolution of $L^2(\tau)$" [6, 8].

If Φ is recursively obtained from box-function and the coefficients of BLDE satisfy the conditions (11) and (12), then it is obvious that $W_m\perp v_m$, $\forall m\in Z$, and with a certain set of requirements [9], we can write: $W_m\oplus v_m = v_{m-1}$ ("$\oplus$" denotes the direct sum of two subspaces). This fact leads to: $\oplus W_{m+1}\oplus W_m = v_{m-1}\leq v_{m-2}$ for any $m\in Z$ and it follows that:

$$\bigoplus_{m\in Z} W_m = L^2(\tau) \Leftrightarrow V_0 \oplus \left(\bigoplus_{m\in N} W_{-m}\right) = L^2(\tau).$$

The structure of the space $L^2(\tau)$ is depicted in Figure 8 and the orthogonal dashes suggest the mutual orthogonality between the wavelet subspaces.

• Definition 7 •

> A topological linear space has the **"multiresolution"** property that if a dense subspace expressed by a direct sum of other subspaces exists, each subspace indicates a certain precision degree of representation of any space element. This precision is called **"resolution"** and corresponds to a constant "scaling factor" k^m ($m\in Z$ is constant).

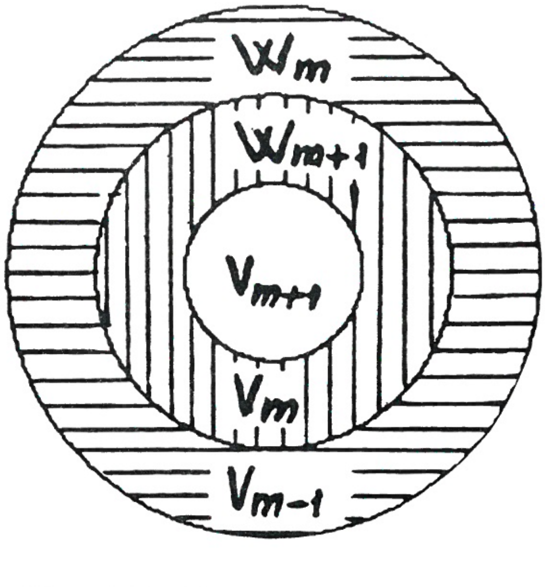

Figure 8. The Multiresolution Structure of $L^2(\tau)$. The Mutual Orthogonality Is Suggested by the Position of the Dashed Lines

The relations above show that $L^2(\tau)$ is a multiresolution space. This fact implies that any $f\in L^2(\tau)$ multiresolution space can be uniquely expressed as:

$$f = f_\Delta + f_{<0>} + f_{<1>} + \ldots + f_{<m>} + \ldots,$$

where $f_\Delta\in v_0$, $f_{<m>}\in W_{-m}$, $\forall m\in N$. So, there exists $\alpha_n\in C$, $\beta_{mn}\in C$ ($\forall m,n\in Z$), for expressing f as a finite expansion with scaling function and wavelets translates:

$$f(x) = \sum_{n\in Z} \alpha_n \Phi(x-n) + \sum_{m\in Z}\sum_{n\in Z} \beta_{mn}\Psi\left(k^{-m}x-n\right) \quad \forall x\in R.$$

The coefficients' finite support is due to the compact support of Φ and Ψ. This formula is similar to **Fourier**'s series, the orthogonality of the basic functions being also satisfied. However, Φ and Ψ could not have been explicitly mathematical formulae, their construction being recursive. Theoretically, our problem is solved: we have constructed a new basis in $L^2(\tau)$- wavelets basis - so that the SP problem has a unique solution.

Referring to the formula above, the approximations $f_\Delta + f_0 + \ldots + f_{<M>}$ with $M\in N$ are called "details" [8], [10]. This approximation process is very similar to the film developing process. If f is an image, then f_Δ is "the coarsest detail" [8], i.e. the most veiled (unclear) version of this, meanwhile $f_{<m>}$ ($m \in N$) are additive corrections to f_Δ making the details express the real image more and more crearly. All these results are easily generalised in **Hilbert** space l^2 and determine one of the most remarkable algorithms with wavelets: **Mallat's** algorithm (implemented and used in image processing). This algorithm solves the EP and its description can be found in [8], [9], [10].

The meaning of the "multiresolution" concept is subtle and to point it out needs time.

However, we can emphasize that every correction $f_{<m>}$ is a member of W_{-m}, which is the wavelet subspace with a resolution associated with the scale factor "k^m". The action of W_{-m} is limited in the frequency domain to a single k- octave: $[k^{-(m+1)}, k^{-m}]$. Practically, this resolution is the correction power of the subspace W_{-m}. The m growing strengthens the resolution, that is the image becomes more and more precise, and thanks to that the correction will be more refined, at a sharper local level. The image quality improves with each correction. The correction process should be ended if the image proves accurate enough.There is no need to proceed on very high resolutions for a good image quality. This result is due to the capacity of wavelets of obtaining their support around a point, with a simultaneous increase in the frequency, imposed by the sharpness of the k-octave specific to the wavelet space including the wavelets. Actually, this is a translation and omothetia consequence.

The use of wavelets is not to be universal, but, for a certain kind of non-stationary signals and with a cleverly chosen wavelets set, the feature above seems to be very powerful.

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