ASYMPTOTIC EXPANSIONS FOR ALGEBRAIC RICCATI EQUATIONS OF H∞- CONTROL THEORY FOR SYSTEMS WITH TWO TIME SCALES

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ABSTRACT.
If the systems to be controlled are singularly perturbed, then the Riccati equations which appear in H∞-problems are difficult to solve because of the existence of a small parameter. The paper describes and validates asymptotic expansions while showing how they may be used in H∞-control.

1. INTRODUCTION

Let us consider the system

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1^1 u_1 + B_2^1 u_2 \\ \varepsilon\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_1^2 u_1 + B_2^2 u_2 \\ y_1 &= C_1^1 x_1 + C_1^2 x_2 + D_{12}u_2 \\ y_2 &= C_2^1 x_1 + C_2^2 x_2 + D_{21}u_1 \end{aligned} \qquad (1.1)$$

where $x_i \in R^{n_i}, u_i \in R^{m_i}, y_i \in R^{p_i}, A_{ij}, B_i^j, C_i^j, D_{12}, D_{21}$ $(D_{12}^* D_{12} = I_{m_2}, D_{21}D_{21}^* = I_{p_2})$ are constant matrices of corresponding dimensions; $\varepsilon > 0$ is a small parameter, whose presence describes the fast time scale, y_1 a quality output, y_2 a measured output, u_1 a disturbance and u_2 a control input.

Introduce notation

$$A(\varepsilon) = \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\varepsilon}A_{21} & \frac{1}{\varepsilon}A_{22} \end{pmatrix}, \quad \underset{i=1,2}{B_i(\varepsilon)} = \begin{pmatrix} B_i^1 \\ \frac{1}{\varepsilon}B_i^2 \end{pmatrix}, \; C_1 = (C_1^1\, C_1^2),\; C_2 = (C_2^1\, C_2^2) \;\; x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and write (1.1) in the form

$$\begin{aligned} \dot{x} &= A(\varepsilon)x + B_1(\varepsilon)u_1 + B_2(\varepsilon)u_2 \\ y_1 &= C_1 x + D_{12}u_2 \\ y_2 &= C_2 x + D_{21}u_1 \end{aligned} \qquad (1.1')$$

Recent results of [3] and [1] have circumscribed the solution of the H_∞-control problem to the existence of stabilizing, positive definite solutions to special algebraic Riccati equations

$$\begin{aligned} &A^*(\varepsilon)X + XA(\varepsilon) + \frac{1}{\gamma^2}XB_1(\varepsilon)B_1^*(\varepsilon)X - \\ &-\left[XB_2(\varepsilon) + C_1^* D_{12}\right]\left[B_2^*(\varepsilon)X + D_{12}^* C_1\right] + C_1^* C_1 = 0 \end{aligned} \qquad (1.2)$$

$$A(\varepsilon)Y+YA^*(\varepsilon)+\frac{1}{\gamma^2}YC_1^*C_1Y-$$
$$-\left[YC_2^*+B_1(\varepsilon)D_{21}^*\right]\left[C_2Y+D_{21}B_1^*(\varepsilon)\right]+B_1(\varepsilon)B_1^*(\varepsilon)=0 \qquad (1.2')$$

The existence of a stabilizing compensator such that $\| T_{y_1u_1} \| < \gamma$ is equivalent to the existence of stabilizing solutions $X_\varepsilon \geq 0$, $Y_\varepsilon \geq 0$ to (1.2), (1.2') respectively with $\rho(X_\varepsilon Y_\varepsilon) \leq \gamma^2$.

In order to prove if a γ is admissible, one has to solve (1.2) and (1.2') and test the condition for the spectral radius. In our specific situation the presence of the small parameter ε makes such computations difficult and so, as usually, asymptotic expansions can help in such situations.

It is intended that this paper describes the asymptotic structure of the solutions X_ε Y_ε and shows how one can reduce their computations to lower order equations not including the small parameter any more.

2. ASYMPTOTIC EXPANSIONS

We shall make the fundamental assumption that det $A_{22} \neq 0$; such an assumption allows that the reduced model be constructed.

$$\dot{x}_1=\tilde{A}x_1+\tilde{B}_1u_1+\tilde{B}_2u_2$$
$$y_1=\tilde{C}_1x_1+\tilde{D}_{11}u_1+\tilde{D}_{12}u_2 \qquad (2.1)$$
$$y_2=\tilde{C}_2x_1+\tilde{D}_{21}u_1+\tilde{D}_{22}u_2$$

with $\tilde{A}=A_{11}-A_{12}A_{22}^{-1}A_{21}$

$$\left.\begin{array}{l}\tilde{B}_i=B_i^1-A_{12}A_{22}^{-1}B_i^2\\ \tilde{C}_i=C_i^1-C_i^2A_{22}^{-1}A_{21}\end{array}\right\} \quad i=1,2 \qquad (2.1')$$

$$\tilde{D}_{11}=-C_1^2A_{22}^{-1}B_1^2;\ \tilde{D}_{12}=D_{12}-C_1^2A_{22}^{-1}B_2^2;$$
$$\tilde{D}_{21}=D_{21}-C_2^2A_{22}^{-1}B_1^2;\ \tilde{D}_{22}=-C_2^2A_{22}^{-1}B_2^2$$

The so-called boundary layer system has been associated.

$$x_2=A_{22}x_2+B_1^2u_1+B_2^2u_2$$
$$y_1=C_1^2x_2+D_{12}u_2 \qquad (2.2)$$
$$y_2=C_2^2x_2+D_{21}u_1$$

According to [3] the algebraic Riccati equations have been considered.

$$\tilde{A}^*X_1+X_1\tilde{A}+\tilde{C}_i^*\tilde{C}_1=\left[X_1(\tilde{B}_1\quad \tilde{B}_2)+\tilde{C}_1^*(\tilde{D}_{11}\quad \tilde{D}_{12})\right]\tilde{R}^{-1}\left[\begin{pmatrix}\tilde{B}_1^*\\ \tilde{B}_2^*\end{pmatrix}X_1+\begin{pmatrix}\tilde{D}_{11}^*\\ \tilde{D}_{12}^*\end{pmatrix}\tilde{C}_1\right] \qquad (2.3)$$

and

$$\tilde{A}Y_1+Y_1\tilde{A}^*+\tilde{B}_1\tilde{B}_1^*=\left[Y_1(\tilde{C}_1^* \quad \tilde{C}_2^*)+\tilde{B}_1(\tilde{D}_{11}^* \quad \tilde{D}_{21}^*)\right]\hat{R}^{-1}\left[\begin{pmatrix}\tilde{C}_1\\ \tilde{C}_2\end{pmatrix}Y_1+\begin{pmatrix}\tilde{D}_{11}\\ \tilde{D}_{21}\end{pmatrix}\tilde{B}_1\right] \quad (2.3')$$

respectively

$$A_{22}^*X_2+X_2A_{22}+(C_1^2)^*C_1^2=-\frac{1}{\gamma^2}X_2B_1^2B_1^{2*}X_2+(X_2B_2^2+C_1^{2*}D_{12})\,(B_2^{2*}X_2+D_{12}^*C_1^2) \quad (2.4)$$

and

$$A_{22}Y_2+Y_2A_{22}^*+B_1^2B_1^{2*}=-\frac{1}{\gamma^2}Y_2C_1^{2*}C_1^2Y_2+(Y_2C_2^{2*}+B_1^2D_{21}^*)\,(C_2^2Y_2+D_{21}B_1^{2*}) \quad (2.4')$$

where

$$\tilde{R}=\begin{pmatrix}\tilde{D}_{11}^*\tilde{D}_{11}-\gamma^2 I_{m_1} & \tilde{D}_{11}^*\tilde{D}_{12}\\ \tilde{D}_{12}^*\tilde{D}_{11} & \tilde{D}_{12}^*\tilde{D}_{12}\end{pmatrix}$$

$$\hat{R}=\begin{pmatrix}\tilde{D}_{11}\tilde{D}_{11}^*-\gamma^2 I_{p_1} & \tilde{D}_{11}\tilde{D}_{21}^*\\ \tilde{D}_{21}\tilde{D}_{11}^* & \tilde{D}_{21}\tilde{D}_{21}^*\end{pmatrix} \quad (2.3'')$$

It is advisable that the Riccati equations are replaced by the Lurie-Yakubovici-Popov type systems (see [2])

$$\begin{aligned} &X_1(\tilde{B}_1 \quad \tilde{B}_2)+\tilde{C}_1^*(\tilde{D}_{11} \quad \tilde{D}_{12})=W_1\tilde{R}\\ &\tilde{A}^*X_1+X_1\tilde{A}+\tilde{C}_1^*\tilde{C}_1=W_1\tilde{R}W_1^* \end{aligned} \quad (2.5)$$

respectively

$$\begin{aligned} &Y_1(\tilde{C}_1^* \quad \tilde{C}_2^*)+\tilde{B}_1(\tilde{D}_{11}^* \quad \tilde{D}_{21}^*)=V_1\hat{R}\\ &\tilde{A}Y_1+Y_1\tilde{A}^*+\tilde{B}_1\tilde{B}_1^*=V_1\hat{R}\,V_1^* \end{aligned} \quad (2.5')$$

$$\begin{aligned} &X_2(B_1^2 \quad B_2^2)+C_1^{2*}(O \quad D_{12})=W_2\tilde{S}\\ &A_{22}^*X_2+X_2A_{22}+C_1^{2*}C_1^2=W_2\tilde{S}\,W_2^* \end{aligned} \quad (2.6)$$

$$\begin{aligned} &Y_2(C_1^{2*} \quad C_2^{2*})+B_1^2(O \quad D_{21}^*)=V_2\hat{S}\\ &A_{22}Y_2+Y_2A_{22}^*+B_1^2B_1^{2*}=V_2\hat{S}\,V_2^* \end{aligned} \quad (2.6')$$

with

$$\hat{S}=\begin{pmatrix}-\gamma^2 I_{p_1} & 0\\ 0 & I_{p_2}\end{pmatrix} \qquad \tilde{S}=\begin{pmatrix}-\gamma^2 I_{m_1} & 0\\ 0 & I_{m_2}\end{pmatrix}$$

If $(\tilde{X}_1, \tilde{W}_1)$ is a solution to (2.5) then $\tilde{X}_1$ is a solution to (2.3); $\tilde{X}_1$ is stabilizing if $\tilde{A}-(\tilde{B}_1 \quad \tilde{B}_2)\tilde{W}_1$ is stable (with eigenvalues in C^-).

In the same way, if $(\tilde{Y}_1, \tilde{V}_1)$ is a solution to (2.5') then $\tilde{Y}_1$ is a solution to (2.3'); $\tilde{Y}_1$ is stabilizing if $\tilde{A}-\tilde{V}_1 \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix}$ is stable. If $(\tilde{X}_2, \tilde{W}_2)$ is a solution to (2.6) then $\tilde{X}_2$ is a solution to (2.4); $\tilde{X}_2$ is stabilizing if $A_{22}-(B_1^2 \;\; B_2^2)\tilde{W}_2^*$ is stable; if $(\tilde{Y}_2, \tilde{V}_2)$ is a solution to (2.6') then $\tilde{Y}_2$ is a solution to (2.4') and $\tilde{Y}_2$ is stabilizing if $A_{22}-\tilde{V}_2 \begin{pmatrix} C_1^2 \\ C_2^2 \end{pmatrix}$ is stable.

The main result of the paper is:

Theorem 2.1

a) Assume that there exists $\gamma > 0$ such that equations (2.3) and (2.4) have stabilizing solutions $\tilde{X}_1 > 0$ and $\tilde{X}_2 > 0$, respectively. Then there exists $\tilde{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \tilde{\varepsilon})$ equation (1.2) has a stabilizing solution $X_\varepsilon > 0$. If we write down

$$X_\varepsilon = \begin{pmatrix} X_{11}(\varepsilon) & X_{12}(\varepsilon) \\ X_{12}^*(\varepsilon) & X_{22}(\varepsilon) \end{pmatrix}$$

then $X_{11}(\varepsilon) = \tilde{X}_1 + \varepsilon \hat{X}_{11}(\varepsilon)$

$$X_{12}(\varepsilon) = \varepsilon \tilde{X}_{12} + \varepsilon^2 \hat{X}_{12}(\varepsilon)$$

$$X_{22}(\varepsilon) = \varepsilon \tilde{X}_2 + \varepsilon^2 \hat{X}_{22}(\varepsilon)$$

where $\tilde{X}_{12} = -\Big[A_{21}^*\tilde{X}_2 + \tilde{X}_1 A_{12} + C_1^{1*}C_1^2 + \frac{1}{\gamma^2}\tilde{X}_1 B_1^1 B_1^{2*}\tilde{X}_2 - (\tilde{X}_1 B_2^1 +$

$$+ C_1^{1*}D_{12})(B_2^{2*}\tilde{X}_2 + D_{12}^* C_1^2)\Big]\Big[A_{22} + \frac{1}{\gamma^2} B_2^1 B_2^{1*}\tilde{X}_2 - B_2^2(B_2^{2*}\tilde{X}_2 + D_{12}^* C_1^2)\Big]^{-1}$$

and there exists $c < \infty$ such that $(\forall)\ \varepsilon \in (0, \tilde{\varepsilon})$

$$|\hat{X}_{11}(\varepsilon)| + |\hat{X}_{12}(\varepsilon)| + |\hat{X}_{22}(\varepsilon)| \le c$$

b) Assume that for the same $\gamma > 0$ equations (2.3') and (2.4') have stabilizing solutions $\tilde{Y}_1 > 0$ and $\tilde{Y}_2 > 0$, respectively. Then there exists $\hat{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \hat{\varepsilon})$ equation (1.2') has a stabilizing solution $Y_\varepsilon > 0$. If we write down

$$Y_\varepsilon = \begin{pmatrix} Y_{11}(\varepsilon) & Y_{12}(\varepsilon) \\ Y_{12}^*(\varepsilon) & Y_{22}(\varepsilon) \end{pmatrix}$$ then

$$Y_{11}(\varepsilon) = \tilde{Y}_1 + \varepsilon \hat{Y}_{11}(\varepsilon)$$

$$Y_{12}(\varepsilon) = \tilde{Y}_{12} + \varepsilon \hat{Y}_{12}(\varepsilon)$$

$$Y_{22}(\varepsilon) = \frac{1}{\varepsilon}(\tilde{Y}_2 + \varepsilon \hat{Y}_{22}(\varepsilon))$$ where

$$\tilde{Y}_{12} = -\left[A_{12}\tilde{Y}_2+\tilde{Y}_1A^*_{21}+B^1_1B^{2*}_1+\frac{1}{\gamma^2}\tilde{Y}_1C^{1*}_1C^2_1\tilde{Y}_2-(\tilde{Y}_1C^{1*}_2+\right.$$

$$\left.+B^1_1D^*_{21})\,(C^2_2\tilde{Y}_2+D_{21}B^{2*}_1)\right]\left[A^*_{22}+\frac{1}{\gamma^2}C^2_1C^{2*}_1\tilde{Y}_2-C^{2*}_2(C^2_2\tilde{Y}_2+D_{21}B^{2*}_1)\right]^{-1}$$

and there exists $c < \infty$ such that

$$\left|\hat{Y}_{11}(\varepsilon)\right|+\left|\hat{Y}_{12}(\varepsilon)\right|+\left|\hat{Y}_{22}(\varepsilon)\right| \le c$$

Corollary 2.1 Based on the assumptions of Theorem 2.1

$$X_\varepsilon Y_\varepsilon = \begin{pmatrix} \tilde{X}_1\tilde{Y}_1 & \tilde{X}_1\tilde{Y}_{12} + \tilde{X}_{12}\tilde{Y}_2 \\ 0 & \tilde{X}_2\tilde{Y}_2 \end{pmatrix} + O(\varepsilon)$$

Corollary 2.2 Based on the assumptions of Theorem 2.1 let the solutions $\tilde{X}_i$, $\tilde{Y}_i$ satisfy $\rho(\tilde{X}_i\tilde{Y}_i) < \gamma^2$ i=1,2.

Assume also a) $D^*_{12}(C^1_1 \;\; C^2_1)=(0 \;\; 0)$

$$D_{21}((B^1_1)^* \;\; (B^2_1)^*) = (0 \;\; 0)$$

b) $(\tilde{A},\tilde{B}_1,\tilde{C}_1)$ and (A_{22},B^2_1,C^2_1) are stabilizable and detectable.

Then there exists $\varepsilon_0>0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there will be a stabilizing compensator for (1.1) such that $||T_{y_1u_1}|| \le \gamma$.

The corollary derives from Theorem 2.1, Theorem 3 in [1] and from the fact that assumption b) implies that $(A(\varepsilon), B(\varepsilon), C_1)$ is stabilizable and detectable.

Remark. Such assumptions as a), b) above might be replaced by those asking for detectability of $((I-D_{12}D^*_{12})C_1, A(\varepsilon)-B_1(\varepsilon)D^*_{12}C_1)$ and observability of $(A(\varepsilon)-B_1(\varepsilon)D^*_{21}C_2, B_1(\varepsilon)(I-D^*_{21}D_{21}))$ but these assumptions derogate from the general idea of this paper, i.e. working with systems not depending on the small parameter ε.

3. PROOF OF THEOREM 2.1

A. Let $\tilde{S},\hat{S}$ be defined as above. Let X_ε be a solution to (1.2) and define $W_\varepsilon = \left[X_\varepsilon(B_1(\varepsilon) \;\; B_2(\varepsilon))+C^*_1(0 \;\; D_{12})\right]\tilde{S}^{-1}$

Then $(X_\varepsilon, W_\varepsilon)$ is a solution to $A^*(\varepsilon)X+XA(\varepsilon)+C^*_1C_1 = W\tilde{S}W^*$

$$X(B_1(\varepsilon) \;\; B_2(\varepsilon))+C^*_1(0 \;\; D_{12}) = W\tilde{S} \tag{3.1}$$

and if $(X_\varepsilon,W_\varepsilon)$ is a solution to (3.1), X_ε is a solution to (1.2). The same way

$$Y(C^*_1 \;\; C^*_2)+B_1(\varepsilon)\,(0 \;\; D^*_{21}) = V\hat{S} \tag{3.2}$$

$A(\varepsilon)Y+YA^*(\varepsilon)+B_1(\varepsilon)B^*_1(\varepsilon) = V\hat{S}V^*$ can be associated with (1.2')

B. Let now $X_\varepsilon = \begin{pmatrix} X_{11}(\varepsilon) & X_{12}(\varepsilon) \\ X_{12}^*(\varepsilon) & X_{22}(\varepsilon) \end{pmatrix}$

$W_\varepsilon = \begin{pmatrix} W_1(\varepsilon) \\ W_2(\varepsilon) \end{pmatrix}$ Then (3.1) is written as

$$X_{11}(B_1^1 \;\; B_2^1)+\frac{1}{\varepsilon}X_{12}(B_1^2 \;\; B_2^2)+C_1^{1*}(0 \;\; D_{12}) = W_1\tilde{S}$$

$$X_{12}^*(B_1^1 \;\; B_2^1)+\frac{1}{\varepsilon}X_{22}(B_1^2 \;\; B_2^2)+C_1^{2*}(0 \;\; D_{12}) = W_2\tilde{S} \qquad (3.3)$$

$$A_{11}^*X_{11}+\frac{1}{\varepsilon}A_{21}^*X_{12}^*+X_{11}A_{11}+\frac{1}{\varepsilon}X_{12}A_{21}+C_1^{1*}C_1^1 = W_1\tilde{S}\,W_1^*$$

$$A_{11}^*X_{12}+\frac{1}{\varepsilon}A_{21}^*X_{22}+X_{11}A_{12}+\frac{1}{\varepsilon}X_{12}A_{22}+C_1^{1*}C_1^2 = W_1\tilde{S}\,W_2^*$$

$$A_{12}^*X_{12}+\frac{1}{\varepsilon}A_{22}^*X_{22}+X_{12}^*A_{12}+\frac{1}{\varepsilon}X_{22}A_{22}+C_1^{2*}C_1^2 = W_2\tilde{S}\,W_2^*$$

If $\tilde{X}_{12}(\varepsilon) = \frac{1}{\varepsilon}X_{12}(\varepsilon)$, $\tilde{X}_{22}(\varepsilon) = \frac{1}{\varepsilon}X_{22}(\varepsilon)$ is denoted,

for $(X_{11},\tilde{X}_{12},\tilde{X}_{22},W_1,W_2)$ the system

$$X_{11}(B_1^1 \;\; B_2^1)+\tilde{X}_{12}(B_1^2 \;\; B_2^2)+C_1^{1*}(0 \;\; D_{12}) = W_1\tilde{S}$$

$$\varepsilon\tilde{X}_{12}^*(B_1^1 \;\; B_2^1)+\tilde{X}_{22}(B_1^2 \;\; B_2^2)+C_1^{2*}(0 \;\; D_{12}) = W_2\tilde{S}$$

$$A_{11}^*X_{11}+X_{11}A_{11}+A_{21}^*\tilde{X}_{12}^*+\tilde{X}_{12}A_{21}+C_1^{1*}C_1^1 = W_1\tilde{S}\,W_1^* \qquad (3.4)$$

$$\varepsilon A_{11}^*\tilde{X}_{12}+A_{21}^*\tilde{X}_{22}+X_{11}A_{12}+\tilde{X}_{12}A_{22}+C_1^*C_1^2 = W_1\tilde{S}\,W_2^*$$

$$\varepsilon A_{12}^*\tilde{X}_{12}+\varepsilon\tilde{X}_{12}A_{12}+A_{22}^*\tilde{X}_{22}+\tilde{X}_{22}A_{22}+C_1^{2*}C_1^2 = W_2\tilde{S}\,W_2^*$$ is achieved.

For $\varepsilon = 0$ system (3.4) reduces to

$$X_{11}(B_1^1 \;\; B_2^1)+\tilde{X}_{12}(B_1^2 \;\; B_2^2)+C_1^{1*}(0 \;\; D_{12}) = W_1\tilde{S}$$

$$\tilde{X}_{22}(B_1^2 \;\; B_2^2)+C_1^{2*}(0 \;\; D_{12}) = W_2\tilde{S}$$

$$A_{11}^*X_{11}+X_{11}A_{11}+A_{21}^*\tilde{X}_{12}^*+\tilde{X}_{12}A_{21}+C_1^{1*}C_1^1 = W_1\tilde{S}\,W_1^* \qquad (3.5)$$

$$A_{21}^*\tilde{X}_{22}+X_{11}A_{12}+\tilde{X}_{12}A_{22}+C_1^{1*}C_1^2 = W_1\tilde{S}\,W_2^*$$

$$A_{22}^*\tilde{X}_{22}+\tilde{X}_{22}A_{22}+C_1^{2*}C_1^2 = W_2\tilde{S}\,W_2^*$$

C. Within the system (3.5) the second and the fifth equations correspond to the Lurie-Yakubovici-Popov system associated with (2.2).

Let $(\tilde{X}_2,\tilde{W}_2)$ be a solution such that $A_{22}-(B_2^1 \;\; B_2^2)\tilde{W}_2^*$ is invertible (this happens, for instance, if the matrix is stable).

Lemma 3.1 Let $(\tilde{X}_{11},\tilde{X}_{12},\tilde{X}_{22},\tilde{W}_1,\tilde{W}_2)$ be a solution to (3.5) such that $A_{22}-(B_1^2\ \ B_2^2)\tilde{W}_2^*$ is invertible. Then:

a) $\tilde{X}_{11}=\tilde{X}_1$ where $(\tilde{X}_1,\tilde{\tilde{W}}_1)$ is a solution to (2.5)

$$\text{b) } \tilde{W}_1=\tilde{\tilde{W}}_1\left[I-(B_1^2\ \ B_2^2)^*(A_{22}^{-1})^*\tilde{W}_2\right]+A_{21}^*(A_{22}^{-1})^*\tilde{W}_2 \tag{3.6}$$

$$\text{c) } \tilde{X}_{12}=\left[\tilde{W}_1\tilde{S}\tilde{W}_2^*-A_{21}^*\tilde{X}_{22}-\tilde{X}_{11}A_{12}-C_1^{1*}C_1^2\right]A_{22}^{-1}$$

Proof. The fourth equation in (3.5) yields

$$\tilde{X}_{12}=\tilde{W}_1\tilde{S}\tilde{W}_2^*A_{22}^{-1}-A_{21}^*\tilde{X}_{22}A_{22}^{-1}-\tilde{X}_{11}A_{12}A_{22}^{-1}-C_1^{1*}C_1^2A_{22}^{-1} \tag{3.7}$$

So, the first equation is substituted for:

$$\tilde{X}_{11}(\tilde{B}_1\ \ \tilde{B}_2)-A_{21}^*\tilde{X}_{22}A_{22}^{-1}(B_1^2\ \ B_2^2)+C_1^{1*}(\tilde{D}_{11}\ \ \tilde{D}_{12})=\tilde{W}_1\tilde{S}\left[I-\tilde{W}_2^*A_{22}^{-1}(B_1^2\ \ B_2^2)\right] \tag{3.8}$$

On the other hand, the fifth and the second equations yield

$$A_{21}^*\tilde{X}_{22}A_{22}^{-1}(B_1^2\ \ B_2^2)=A_{21}^*(A_{22}^{-1})^*\tilde{W}_2\tilde{S}\tilde{W}_2^*A_{22}^{-1}(B_1^2\ \ B_2^2)-A_{21}^*(A_{22}^{-1})^*\tilde{X}_{22}(B_1^2\ \ B_2^2)-$$
$$-A_{21}^*(A_{22}^{-1})^*(C_1^2)^*C_1^2A_{22}^{-1}(B_1^2\ \ B_2^2)=-A_{21}^*(A_{22}^{-1})^*\tilde{W}_2\tilde{S}\left[I-\tilde{W}_2^*A_{22}^{-1}(B_1^2\ \ B_2^2)\right]+$$
$$+A_{21}^*(A_{22}^{-1})^*(C_1^2)^*(\tilde{D}_{11}\ \ \tilde{D}_{12})$$

We deduce that (3.8) may be written as

$$\tilde{X}_{11}(\tilde{B}_1\ \ \tilde{B}_2)+\tilde{C}_1^*(\tilde{D}_{11}\ \ \tilde{D}_{12})=\left[\tilde{W}_1-A_{21}^*(A_{22}^{-1})^*\tilde{W}_2\right]\tilde{S}\left[I-\tilde{W}_2^*A_{22}^{-1}(B_1^2\ \ B_2^2)\right] \tag{3.8'}$$

A direct calculation shows that based on our assumptions

$I-\tilde{W}_2^*A_{22}^{-1}(B_1^2\ \ B_2^2)$ is non-singular and the inverse is

$$I+\tilde{W}_2^*\left[A_{22}-(B_1^2\ \ B_2^2)\tilde{W}_2^*\right]^{-1}(B_1^2\ \ B_2^2)$$

We may thus define

$$\tilde{\tilde{W}}_1=\left[\tilde{W}_1-A_{21}^*(A_{22}^{-1})^*\tilde{W}_2\right]\left[I-(B_1^2\ \ B_2^2)^*(A_{22}^{-1})^*\tilde{W}_2\right]^{-1}$$

$$\check{R}=\left[I-(B_1^2\ \ B_2^2)^*(A_{22}^{-1})^*\tilde{W}_2\right]\tilde{S}\left[I-\tilde{W}_2^*A_{22}^{-1}(B_1^2\ \ B_2^2)\right] \tag{3.9}$$

and (3.8') becomes

$$\tilde{X}_{11}(\tilde{B}_1\ \ \tilde{B}_2)+\tilde{C}_1^*(\tilde{D}_{11}\ \ \tilde{D}_{12})=\tilde{\tilde{W}}_1\check{R} \tag{3.10}$$

By substituting (3.7) into the third equation (3.5) we get

$$\tilde{A}^*\tilde{X}_{11}+\tilde{X}_{11}\tilde{A}-A_{21}^*(A_{22}^{-1})^*[A_{22}^*\tilde{X}_{22}+\tilde{X}_{22}A_{22}]A_{22}^{-1}A_{21}+(C_1^1)^*C_1^1-(C_1^1)^*C_1^2A_{22}^{-1}A_{21}-$$
$$-A_{21}^*(A_{22}^{-1})^*C_1^{2*}C_1^1=\tilde{W}_1\tilde{S}\tilde{W}_1^*-A_{21}^*(A_{22}^{-1})^*\tilde{W}_2\tilde{S}\tilde{W}_1^*-\tilde{W}_1\tilde{S}\tilde{W}_2^*A_{22}^{-1}A_{21}$$

By using (2.6) we obtain

$$\tilde{A}^*\tilde{X}_{11}+\tilde{X}_{11}\tilde{A}+\tilde{C}_1^*\tilde{C}_1 = \left[\tilde{W}_1-A_{21}^*(A_{22}^{-1})^*\tilde{W}_2\right]\tilde{S}\left[\tilde{W}_1-A_{21}^*(A_{22}^{-1})^*\tilde{W}_2\right]^*$$

and with (3.9) we finally have

$$\tilde{A}^*\tilde{X}_{11}+\tilde{X}_{11}\tilde{A}+\tilde{C}_1^*\tilde{C}_1 = \approx{W}_1\check{R}\,\approx{W}_1^* \tag{3.10'}$$

Further

$$\check{R} = \tilde{S}-(B_1^2\ \ B_2^2)^*(A_{22}^{-1})^*\tilde{W}_2\tilde{S} - \tilde{S}\,\tilde{W}_2^*A_{22}^{-1}(B_1^2\ \ B_2^2)+$$

$$+(B_1^2\ \ B_2^2)^*(A_{22}^{-1})^*\tilde{W}_2\tilde{S}\,\tilde{W}_2^*A_{22}^{-1}(B_1^2\ \ B_2^2)$$

By using (2.6) and after some simple calculations $\check{R} = \tilde{R}$ and (3.10), (3.10') become

$$\tilde{X}_{11}(\tilde{B}_1\ \ \tilde{B}_2)+\tilde{C}_1^*(\tilde{D}_{11}\ \ \tilde{D}_{12}) = \approx{W}_1\tilde{R}$$

$$\tilde{A}^*\tilde{X}_{11}+\tilde{X}_{11}\tilde{A}+\tilde{C}_1^*\tilde{C}_1 = \approx{W}_1\tilde{R}\,\approx{W}_1^* \tag{3.11}$$

and Lemma 3.1 is proved.

Lemma 3.2 Let $(\tilde{X}_1,\approx{W}_1),(\tilde{X}_2,\approx{W}_2)$be stabilizing solutions to (2.5), (2.6), respectively. Then the system

$$X_{11}(B_1^1\ \ B_1^2)+X_{12}(B_1^2\ \ B_2^2) = W_1\tilde{S}$$

$$X_{22}(B_1^2\ \ B_2^2) = W_2\tilde{S}$$

$$A_{11}^*X_{11}+X_{11}A_{11}+A_{21}^*X_{12}^*+X_{12}A_{21} = \tilde{W}_1\tilde{S}\,W_1^*+W_1\tilde{S}\,\tilde{W}_1^*$$

$$A_{21}^*X_{22}+X_{11}A_{12}+X_{12}A_{22} = \tilde{W}_1\tilde{S}\,W_2^*+W_1\tilde{S}\,\tilde{W}_2^*$$

$$A_{22}^*X_{22}+X_{22}A_{22} = \tilde{W}_2\tilde{S}\,W_2^*+W_2\tilde{S}\,\tilde{W}_2^*$$

is only let the solution $X_{11}=0, X_{12}=0, X_{22}=0, W_1=0, W_2=0$,(with $\tilde{W}_1$ from (3.6)).

Proof. The second and the fifth equations yield

$$\left[A_{22}-(B_1^2\ \ B_2^2)\tilde{W}_2^*\right]^*X_{22}+X_{22}\left[A_{22}-(B_1^2\ \ B_2^2)\tilde{W}_2^*\right] = 0 \text{ and since}$$

$A_{22}-(B_1^2\ \ B_2^2)\tilde{W}_2^*$ is stable, we deduce $X_{22}=0$; the second equation yields to $\tilde{W}_2$=0.

Solve the fourth equation to obtain

$$X_{12} = -X_{11}A_{12}A_{22}^{-1}+W_1\,\tilde{S}\,\tilde{W}_2^*\,A_{22}^{-1} \tag{3.12}$$

and then the first equation gives

$$X_{11}(\tilde{B}_1\ \ \tilde{B}_2) = W_1\tilde{S}\left[I-\tilde{W}_2^*A_{22}^{-1}(B_1^2\ \ B_2^2)\right] \tag{3.13}$$

On the other hand, by substituting (3.12) into the third equation we obtain

$$\tilde{A}^*X_{11}+X_{11}\tilde{A} = W_1\tilde{S}\left[\tilde{W}_1-A_{21}^*(A_{22}^{-1})^*\tilde{W}_2\right]^*+\left[\tilde{W}_1-A_{21}^*(A_{22}^{-1})^*\tilde{W}_2\right]\tilde{S}\,W_1^*.$$

By using (3.9) we further have

$$\tilde{A}^*X_{11}+X_{11}\tilde{A} = W_1\tilde{S}\left[I-\tilde{W}_2^*A_{22}^{-1}(B_1^2\ \ B_2^2)\right]\approx{W}_1^*+\approx{W}_1\left[I-(B_1^2\ \ B_2^2)^*(A_{22}^{-1})^*\tilde{W}_2\right]\tilde{S}W_1^*$$

From (3.13) we deduce now

$$\left[\tilde{A}-(\tilde{B}_1 \quad \tilde{B}_2)\tilde{\tilde{W}}_1^*\right]^* X_{11}+X_{11}\left[\tilde{A}-(\tilde{B}_1 \quad \tilde{B}_2)\tilde{\tilde{W}}_1^*\right] = 0$$

and since $\tilde{A}-(\tilde{B}_1 \quad \tilde{B}_2)\tilde{\tilde{W}}_1^*$ is stable, it follows that $X_{11} = 0$; then by (3.13) we have $W_1 = 0$ and by (3.12) $X_{12} = 0$.

D. The proof of Theorem 2.1 ends by an implicit function argument. System (3.4) has the form $F(\varepsilon, X_{11}, X_{12}, X_{22}, W_1, W_2) = 0$, the number of scalar equations being equal to the one of scalar unknowns. System $F(0, X_{11}, X_{12}, X_{22}, W_1, W_2) = 0$ is (3.5).

Assumptions in Theorem 2.1 and Lemma 3.1 give a solution $(\tilde{X}_1, \tilde{X}_{12}, \tilde{X}_2, \tilde{W}_1, \tilde{W}_2)$ such that

$$F(0, \tilde{X}_1, \tilde{X}_{12}, \tilde{X}_2, \tilde{W}_1, \tilde{W}_2) = 0 \tag{3.14}$$

and $\tilde{X}_1>0, \tilde{X}_2>0$..

Lemma 3.2 shows that $(DF)(0, \tilde{X}_1, \tilde{X}_{12}, \tilde{X}_2, \tilde{W}_1, \tilde{W}_2)$ is injective.

The implicit function theorem gives life to $\tilde{\varepsilon}>0$ and to smooth functions $X_{11}(\varepsilon), X_{12}(\varepsilon), X_{22}(\varepsilon), W_1(\varepsilon), W_2(\varepsilon)$ defined for $|\varepsilon| < \tilde{\varepsilon}$ such that $F(\varepsilon, X_{11}(\varepsilon), X_{12}(\varepsilon), X_{22}(\varepsilon), W_1(\varepsilon), W_2(\varepsilon)) = 0$, $X_{11}(0) = \tilde{X}_1$, $X_{12}(0)=\tilde{X}_{12}$, $X_{22}(0)=\tilde{X}_2$, $W_1(0)=\tilde{W}_1$, $W_2(0)=\tilde{W}_2$. These functions lead to the solution $X_\varepsilon > 0$.

E. For equation (1.2') denote $Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{pmatrix}$ $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ and we obtain the system

$$\begin{aligned}
&Y_{11}(C_1^{1*} \quad C_2^{1*})+Y_{12}(C_1^{2*} \quad C_2^{2*})+B_1^1(0 \quad D_{21}^*) = V_1\hat{S} \\
&Y_{12}^*(C_1^{1*} \quad C_2^{1*})+Y_{22}(C_1^{2*} \quad C_2^{2*}) + \frac{1}{\varepsilon}B_1^2(0 \quad D_{21}^*) = V_2\hat{S} \\
&A_{11}X_{11}+A_{12}Y_{12}^*+Y_{11}A_{11}^*+Y_{12}A_{12}^*+B_1^1B_1^{1*} = V_1\hat{S}V_1^* \\
&A_{11}Y_{12} + A_{12}Y_{22} + \frac{1}{\varepsilon}Y_{11}A_{21}^* + \frac{1}{\varepsilon}Y_{12}A_{22}^* + \frac{1}{\varepsilon}B_1^1B_1^{2*} = V_1\hat{S}V_2^* \\
&\frac{1}{\varepsilon}A_{21}Y_{12} + \frac{1}{\varepsilon}A_{22}Y_{22} + \frac{1}{\varepsilon}Y_{12}^*A_{21}^* + \frac{1}{\varepsilon}Y_{22}A_{22}^* + \frac{1}{\varepsilon^2}B_1^2B_1^{2*} = V_2\hat{S}V_2^*
\end{aligned} \tag{3.15}$$

If $(Y_{11}(\varepsilon),Y_{12}(\varepsilon),Y_{22}(\varepsilon),V_1(\varepsilon),V_2(\varepsilon))$ is a solution to (3.15) put $Y_{22}(\varepsilon) = \frac{1}{\varepsilon}\tilde{Y}_{22}(\varepsilon)$, $V_2(\varepsilon) = \frac{1}{\varepsilon}\tilde{V}_2(\varepsilon)$; then $(Y_{11}(\varepsilon),Y_{12}(\varepsilon), \tilde{Y}_{22}(\varepsilon),V_1(\varepsilon)\tilde{V}_2(\varepsilon))$ is a solution to (3.16)

$$Y_{11}(C_1^{1*} \quad C_2^{1*})+Y_{12}(C_1^{2*} \quad C_2^{2*}) + B_1^1(0 \quad D_{21}^*) = V_1\hat{S}$$

$$\varepsilon Y_{12}^*(C_1^{1*} \quad C_2^{1*}) + \tilde{Y}_{22}(C_1^{2*} \quad C_2^{2*}) + B_1^2(0 \quad D_{21}^*) = \tilde{V}_2\hat{S}$$

$$A_{11}Y_{11} + Y_{11}A_{11}^* + A_{12}Y_{12}^* + Y_{12}A_{12}^* + B_1^1B_1^{1*} = V_1\hat{S}V_1^*$$

$$\varepsilon A_{11}Y_{12} + A_{12}\tilde{Y}_{22} + Y_{11}A_{21}^* + Y_{12}A_{22}^* + B_1^1B_1^{2*} = V_1\hat{S}\tilde{V}_2^* \tag{3.16}$$

$$\varepsilon A_{21}Y_{12} + \varepsilon Y_{12}^*A_{21}^* + A_{22}\tilde{Y}_{22} + \tilde{Y}_{22}A_{22}^* + B_1^2B_1^{2*} = \tilde{V}_2\hat{S}\tilde{V}_2^*$$

System (3.16) has the form (3.4) with A_{ij} replaced by A_{ij}^* and B_i^j replaced by $(C_i^j)^*$. With this remark the proof ends.

4. CONCLUSIONS

State - space methods in H_∞- control have been developed among others for computational purposes. The computations mainly rely on solving specific algebraic Riccati equations.

If the systems to be controlled are singularly perturbed (which corresponds to the presence of two time-scales), these Riccati equations are difficult to solve due to the presence of the small parameter. In such situations, the conventional procedures must be complemented by asymptotic expansions showing how they may be used in H_∞ -control.

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