GENERAL THEORY OF RICCATI EQUATION

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ABSTRACT

The existence of the stabilizing solution of the algebraic Riccati equation is investigated in terms of frequency domain conditions involving an associated Popov function. Necessary and sufficient conditions are derived for both continuous and discrete-time cases under the weakest possible assumptions imposed on the coefficients.

INTRODUCTION AND NOTATION

Let's consider in the sequel a triplet of the form

$\sum = (A,B,P) \in R^{nxn} x R^{nxm} x R^{(n+m)x(n+m)}$ with $P = P^T$ partitioned as :

$$P = \begin{bmatrix} Q & L \\ L^T & R \end{bmatrix} \qquad (1)$$

If R is nonsingular we associate to $\sum$ the continuous-time algebraic Riccati equation (CTARE)

$$A^TX + XA - (XB + L)R^{-1}(L^T + B^TX) + Q = 0 \qquad (2)$$

Any symmetric matrix $X \in R^{nxn}$ that satisfies (2) is called a **stabilizing solution** if

$$F = -R^{-1}(L^T + B^TX) \qquad (3)$$

makes $A + BF$ exponentially stable in the sense that all its eigenvalues are in the left open halfplane.

Similarly, we can associate with $\sum$ the discrete-time algebraic Riccati equation (DTARE)

$$A^TXA - X - (A^TXB + L)(R + B^TXB)^{-1}(L^T + B^TXA) + Q = 0 \qquad (2')$$

A symmetric matrix X making $R + B^TXB$ invertible and satisfying (2') is called a **stabilizing solution** if

$$F = -(R + B^TXB)^{-1}(L^T + B^TXA) \qquad (3)$$

makes $A + BF$ stable in the sense that all its eigenvalues are in the open unit disk. Notice that in the discrete case no assumption is made on the invertibility of R.

It can be easily proved for both continuous and discrete cases that if the stabilizing solution exists then it is unique.

The present paper presents the basic results of a complete theory for the existence of the stabilizing solutions giving necessary and sufficient conditions in terms of associated matrix pencils and in terms of associated rational functions of Popov type satisfying frequency domain conditions. The proofs are omitted as they will be reported elsewhere.

Let's introduce now some notations. The transfer function of a linear system will be denoted

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](\lambda) = C(\lambda I - A)^{-1}B + D$$

and the transfer function of a generalized (descriptor) system

$P(\lambda)z = Q(\lambda)u$

$y = R(\lambda)z + W(\lambda)u$

will be denoted

$$\left[\begin{array}{c|c} P(\lambda) & Q(\lambda) \\ \hline R(\lambda) & W(\lambda) \end{array}\right] = W(\lambda) + R(\lambda)P(\lambda)^{-1}Q(\lambda)$$

For a rational function G, we define the adjoint

$$G^*(\lambda) = \begin{cases} G^T(-\lambda) & \text{for the continuous case} \\ G^T(1/\lambda) & \text{for the discrete case.} \end{cases}$$

BASIC NOTIONS AND STATEMENT OF THE MAIN RESULT

We shall associate with the triplet Σ the following objects:

The **continuous Popov function** is the rational function

$$\Pi_{\Sigma}(s) = [B^T(-sI-A^T)^{-1}\ I]\ P \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix} = R + B^T(-sI-A^T)^{-1}L + L^T(sI-A)^{-1}B + B^T(-sI-A^T)^{-1}Q(sI-A)^{-1}B \tag{4}$$

The **extended Hamiltonian pencil** (EHP) is the matrix pencil $\lambda M-N$ with M, $N \in R^{(2n+m)x(2n+m)}$ where

$$M = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} A & 0 & B \\ -Q & -A^T & -L \\ -L^T & -B^T & -R \end{bmatrix} \tag{5}$$

and their discrete counterparts:

The **discrete Popov function** is the rational function

$$\Pi_{\Sigma}(z) = [B^T(1/zI-A^T)^{-1}\ I]\ P \begin{bmatrix} (zI-A)^{-1}B \\ I \end{bmatrix} = R + B^T(1/zI-A^T)^{-1}L + L^T(zI-A)^{-1}B + B^T(1/zI-A^T)^{-1}Q(zI-A)^{-1}B \tag{4'}$$

The **extended symplectic pencil** (ESP) is the matrix pencil $\lambda M-N$ with M, $N \in R^{(2n+m)x(2n+m)}$ where

$$M = \begin{bmatrix} I & 0 & 0 \\ 0 & -A^T & 0 \\ 0 & -B^T & 0 \end{bmatrix}, \quad N = \begin{bmatrix} A & 0 & B \\ Q & -I & L \\ L^T & 0 & R \end{bmatrix} \tag{5'}$$

Before stating our results, let's give some definitions of the matrix pencils introduced above. We assume as being familiar such notions as generalized eigenvalues, deflating subspace, etc. For an extensive treatment of the general theory of matrix pencils, one can consult [1], [2], [3]. We shall assume in the sequel that EHP and ESP are regular i.e. det. $(\lambda M-N)\neq 0$. We shall say that EHP is **dichotomic** if it has no generalized eigenvalues on the imaginary axis and exactly m generalized eigenvalues at infinity. In this case we can prove that there exists an n-dimensional stable deflating subspace $\vartheta \in R^{2n+m}$ i.e. a deflating subspace such that the spectrum of EHP restricted to it lies in the open left halfplane. This is, in turn, equivalent to the fact that for any basis matrix V of ϑ, there exists an nxn stable matrix S such that

$$NV = MVS \tag{6}$$

Let V be partitioned as

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \begin{matrix} \}n \\ \}n \\ \}m \end{matrix} \tag{7}$$

Then, if V_1 is nonsingular, we shall say that EHP is **disconjugate.** We shall say that ESP is **dichotomic** if it has no generalized eigenvalues on the unit circle. In this case, we can prove that there exists an n-dimensional **stable deflating subspace** i.e. a deflating subspace such that the spectrum of ESP restricted to it lies inside the unit disk. Exactly as for the continuous case, there exists a basis V and a stable (in the discrete acceptance) S such that (6) is satisfied and considering the partition (7) a disconjugate ESP is defined in the same manner.

We can now state our results.

Theorem 1. The following statements are equivalent:

1. The following hold:

   a) the pair (A,B) is stabilizable;

   b) $\Pi_{\sum}(s)$ is invertible over the rationals and has no zeros on $j\overline{R}$;

   c) The realization

$$\Pi_{\sum}(s) = \left[\begin{array}{cc|c} A & 0 & B \\ -Q & -A^T & -L \\ \hline L^T & B^T & R \end{array}\right] \tag{8}$$

has no uncontrollable and/or unobservable modes on $j\overline{R}$.

2. R is nonsingular and CTARE (2) associated with $\sum$ has a stabilizing solution X.

3. EHP (5) is disconjugate.

4. The following hold:

   a) the pair (A,B) is stabilizable;

b) R is nonsingular and EHP is dichotomic.

5. There exists a sign matrix

$$J = \begin{bmatrix} -I_{m1} & \\ & I_{m2} \end{bmatrix}, \quad m_1 + m_2 = m \tag{9}$$

and a biproper matrix transfer function

$$G = \left[\begin{array}{c|c} A & B \\ \hline W & V \end{array} \right]$$

and with stable inverse such that

$$\Pi_{\Sigma}(s) = G^{*}(s)JG(s) \tag{10}$$

The corresponding result for the discrete-time case can be formulated as

Theorem 1'. The following statements are equivalent:

1. The following hold:

a) the pair (A,B) is stabilizable;

b) $\Pi_{\Sigma}(z)$ is invertible over rationals and has no zeros on the unit circle;

c) The descriptor realization

$$\Pi_{\Sigma}(z) = \left[\begin{array}{cc|c} zI-A & 0 & B \\ -Q & I-zA^{T} & L \\ \hline L^{T} & zB^{T} & R \end{array} \right] \tag{8'}$$

has no uncontrollable and/or unobservable modes on the unit circle.

2. DTARE (2') associated with Σ has a stabilizing solution X.

3. ESP (5') is disconjugate.

4. The following hold:

a) the pair (A,B) is stabilizable;

b) ESP is dichotomic.

5. There exists a sign matrix (9) and a biproper matrix transfer function

$$G = \left[\begin{array}{c|c} A & B \\ \hline W & V \end{array} \right]$$

with discrete stable inverse such that (10) is satisfied with the adjoint defined according to the discrete case.

Theorems 1 and 1' are, in fact, generalizations of the celebrated Positivity Theorem in the continuous and discrete cases (see[4]), where the positivity condition was replaced by an invertibility condition. Relation (10) is called the J-quasispectral factorization of Π_{Σ}. If stability for G were also imposed, we would have a spectral factorization of the Popov function.

We shall not give the proofs here, but only their main lines. The equivalence 2.⟺3. was proved for the discrete case in [5]. The continuous case is even simpler. With this equivalence, the implication 2.→4. becomes obvious. The implications 5.→1. and 4.→1. are quite simple. The implication 2.→5. can be easily obtained from the so-called spectral identity. J is taken the sign matrix of R in the continuous case and the sign matrix of $R + B^TXB$ in the discrete–time case. The implication 5.→2. can be obtained through the techniques used in [4] for the proof of the Positivity Theorem. The hard part of the results is 1.→2. This is proved first for the particular case when A is stable and then the general case is reduced to it with the aid of a stabilizing feedback which exists due to assertion a).

CONCLUSIONS

Necessary and sufficient conditions for existence of the stabilizing solution to CTARE (2) and DTARE (2') are derived under the most general assumptions imposed on the initial data: A, B, $Q = Q^T$, L and $R = R^T$. These conditions are expressed in three equivalent forms, namely: frequency domain conditions, disconjugacy of an associated matrix pencil and stabilizability of the pair (A,B) in conjunction with dichotomy for the same matrix pencil.

REFERENCES

[1] GANTMAHER, F.R., **Theory of Matrices**, Chelsea Publishing Co., NY, 1959.

[2] STEWART, G., **Introduction to Matrix Computations**, Academic Press, 1973.

[3] GOLUB, G.P., VAN LOAN, G.F., **Matrix Computations**, The Johns Hopkins University Press, 1989.

[4] POPOV, V.M., **L'Hyperstabilité des systèmes automatiques**, Dunod Paris, 1973.

[5] IONESCU, V., WEISS, M., **On Computing the Stabilizing Solution of the Discrete-Time Riccati Equation**, to appear in Linear Algebra and Its Applications, 1992.