An Approach to Compute Low-Order H_∞ Controllers

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Abstract: If H_∞ control laws are built on high-order models, then reduced computational speed and memory problems are likely to arise. This article proposes an algorithm for computing H_∞ controllers which is similar to the Matlab procedure *hinfsyn.m*, but of low order. The method consists in introducing dependencies between the differential equations of the controller by means of a rank minimisation problem. As a consequence, all differential equations of the controller are expressed as a linear combination of a smaller number of differential equations. Redundant states are then removed.

Keywords: H_∞ control, Linear matrix inequalities, Rank minimization, Model order reduction.

1. Introduction

Several articles, books and PhD theses in the literature address model order reduction. References (Mohammadpour & Grigoriadis, 2010), (Schilders, Van der Vorst & Rommes, 2008), (Nouri, 2014) present a state-of-the-art of the existing techniques.

Modal approximation is one of the first model order reduction methods (Davison, 1966), (Vandendorpe, 2004). If the transfer function of the system is given, the reduction is made by removing the poles that are close to the imaginary axis. The advantage of this method lies in its simplicity. The disadvantage is that the approximation error does not have a guaranteed bound.

Balancing (Antoulas, 2005), (Wittmuess et al., 2016) consists in the simultaneous diagonalization of two positive (semi-) definite matrices. A survey on model reduction by balanced truncation can be found in (Gugercin & Antoulas, 2004). Examples of balancing methods are Lyapunov balancing, Positive real balancing, Bounded real balancing, Stochastic balancing and Frequency weighted balancing. The advantage of these methods, except for Frequency weighted balancing, consists in a computable error bound between the high-order and approximated low-order transfer functions. Their disadvantages are computational complexity $O(n^3)$ and memory requirements $O(n^2)$ that are due to solving Lyapunov and Riccati equations.

The Hankel approximation method, (Glover, 1984), utilizes the Hankel norm, an indicator for the quantity of energy that can be transferred through a system from previous inputs to future outputs. The problem is to compute, for a given high-order system, a system of lower order which satisfies the following condition: the difference between the high-order and reduced systems is minimum in Hankel norm. A lower bound for the error in Hankel norm is contained in the Schmidt-Eckart-Young-Mirsky theorem (Schmidt, 1907), (Eckart & Young, 1936), (Mirsky, 1963). According to the Adamjan-Aron-Krein theorem (Adamjan, Arov & Krein, 1971), (Adamjan, Arov & Krein, 1978), this lower bound can be attained. The method involves computing the reachability and observability gramians. Each gramian is computed as a solution of a Lyapunov equation. The advantage of the method is the availability of an error bound. The disadvantages are computational complexity $O(n^3)$ and memory requirements $O(n^2)$.

Moment matching is described in (Ionescu & Astolfi, 2011), (Ionescu & Astolfi, 2016). One of the first methods which achieves moment matching is Asymptotic Waveform Evaluation (AWE), first proposed in (Pillage & Rohrer, 1990). Numerical problems of this method are due to explicit computation of moments and an increased number of moments. Krylov methods (such as: Padé-via-Lanczos, Arnoldi, PRIMA, IRKA, dual rational Arnoldi and rational Lanczos) can achieve moment matching without the explicit computation of moments at any point. The advantage of Krylov methods is computational efficiency. The disadvantages consist in the absence of global error bounds and non-preservation of stability and passivity. Krylov methods are described in (Grimme, 1997), (Freund, 2003), (Benner, Mehrmann & Sorensen, 2005).

This article proposes an algorithm to compute H_∞ controllers similar to the Matlab procedure hinfsyn.m (Gahinet & Apkarian, 1994), but of low-order. In order to formulate the optimisation

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problem for the control law, linear matrix inequalities (Jedda & Douik, 2018) are used. The proposed algorithm has 7 steps. Steps 1-4 consist in computing a controller, whose order is equal to the order of the process, that has dependencies between its differential equations. At step 4, the dependencies are introduced by considering a rank minimization problem whose solution is represented by the controller matrices. The matrix whose rank is to be minimized is a block matrix whose components are the controller's state and input matrices. The problem is NP-hard. A solution can be obtained heuristically by solving a convex optimization problem instead. Then, the controller is reduced to a lower order by removing the dependencies. At steps 5-6, the block matrix computed to be of minimum rank has linearly dependent rows. The idea is to express all rows of the block matrix in function of the linearly independent ones. The coefficients used to express the linear combinations are computed as the solution of a system of linear equations. If all rows can be expressed as a linear combination of the linearly independent ones, then some differential equations can be expressed as a combination of other differential equations. Dependencies between the differential equations will also imply dependencies between the states. Step 7 consists in reducing the number of rows and columns of the controller matrices. Row reduction is made by removing rows from the block matrix such that only linearly independent rows remain. The row reduction is made for the state and input matrices. The column reduction is made by removing columns from the state and output matrices that correspond to redundant states. The feedthrough matrix remains unmodified.

Section 2 presents the system for which the controller is computed. Section 3 sets forth the proposed control algorithm. Section 4 indicates the simulation results for the proposed method in section 3. In section 5 are the conclusions.

2. System Description

Consider the system:

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ q &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned} \tag{2.1}$$

where:

- x: state of the system, w: exogenous inputs, u: control inputs, q: controlled outputs, y: measured outputs;
- The dimensions of all the vectors and matrices are characterized by: $A \in \mathbb{R}^{n\times n}, D_{11} \in \mathbb{R}^{p_1\times m_1}, D_{22} \in \mathbb{R}^{p_2\times m_2}$.

It is assumed that:

A1: (A, B_2, C_2) is stabilizable and detectable

A2: $D_{22} = 0$.

3. Proposed Control Algorithm

3.1 γ-suboptimal H$_\infty$ Control Problem

The suboptimal H_∞ control problem of parameter γ consists in finding a controller $K(s) = D_K + C_K (sI - A_K)^{-1} B_K$ with $A_K \in \mathbb{R}^{k\times k}$, $B_K \in \mathbb{R}^{k\times p_2}$, $C_K \in \mathbb{R}^{m_2\times k}$ and $D_K \in \mathbb{R}^{m_2\times p_2}$ such that:

- The closed loop system is internally stable;
- The H_∞ norm of the closed-loop transfer function $\mathcal{F}(P,K)$ from w to q is strictly less than γ.

$$\mathcal{F}(P,K) = D_{cl} + C_{cl}(sI - A_{cl})^{-1} B_{cl} \tag{3.1}$$

$$\begin{aligned} A_{cl} &= \begin{pmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{pmatrix} = A_0 + \mathcal{B}\Theta\mathcal{C} \\ B_{cl} &= \begin{pmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{pmatrix} = B_0 + \mathcal{B}\Theta\mathcal{D}_{21} \\ C_{cl} &= \begin{pmatrix} C_1 + D_{12} D_K C_2 & D_{12} C_K \end{pmatrix} = C_0 + \mathcal{D}_{12}\Theta\mathcal{C} \\ D_{cl} &= D_{11} + D_{12} D_K D_{21} = D_{11} + \mathcal{D}_{12}\Theta\mathcal{D}_{21} \end{aligned} \tag{3.2}$$

where:

$$\begin{aligned} &A_0 = \begin{pmatrix} A & 0_{n\times k} \\ 0_{k\times n} & 0_{k\times k} \end{pmatrix},\ B_0 = \begin{pmatrix} B_1 \\ 0_{k\times m_1} \end{pmatrix}, \\ &C_0 = \begin{pmatrix} C_1 & 0_{p_1\times k} \end{pmatrix},\ \Theta = \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}, \\ &\mathcal{B} = \begin{pmatrix} 0_{n\times k} & B_2 \\ I_{k\times k} & 0_{k\times m_2} \end{pmatrix},\ \mathcal{C} = \begin{pmatrix} 0_{k\times n} & I_{k\times k} \\ C_2 & 0_{p_2\times k} \end{pmatrix}, \\ &\mathcal{D}_{12} = \begin{pmatrix} 0_{p_1\times k} & D_{12} \end{pmatrix},\ \mathcal{D}_{21} = \begin{pmatrix} 0_{k\times m_1} \\ D_{21} \end{pmatrix} \end{aligned} \tag{3.3}$$

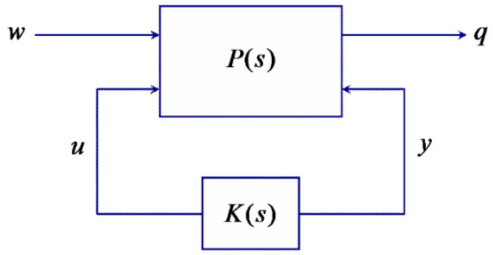

Figure 1. The H_∞ problem

Theorem (Gahinet & Apkarian, 1994):

The continuous-time γ-suboptimal H_∞ problem is solvable if and only if there exist symmetric matrices R, S satisfying the following LMI system:

$$\begin{pmatrix} \mathcal{N}_R & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} AR+RA^T & RC_1^T & B_1 \\ C_1R & -\gamma I & D_{11} \\ B_1^T & D_{11}^T & -\gamma I \end{pmatrix} \begin{pmatrix} \mathcal{N}_R & 0 \\ 0 & I \end{pmatrix} < 0 \quad (3.4)$$

$$\begin{pmatrix} \mathcal{N}_s & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} A^TS+SA & SB_1 & C_1^T \\ B_1^TS & -\gamma I & D_{11}^T \\ C_1 & D_{11} & -\gamma I \end{pmatrix} \begin{pmatrix} \mathcal{N}_s & 0 \\ 0 & I \end{pmatrix} < 0 \quad (3.5)$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0 \quad (3.6)$$

where $\mathcal{N}_R$ and $\mathcal{N}_S$ denote bases of the null spaces of $\begin{pmatrix} B_2^T & D_{12}^T \end{pmatrix}$ and $\begin{pmatrix} C_2 & D_{21} \end{pmatrix}$, respectively.

In addition, there exist γ-suboptimal controllers of order $k < n$ (reduced order) if and only if (3.4)-(3.6) hold for some R, S which further satisfy $\operatorname{rank}(I - RS) \leq k$.

3.2 Proposed γ-suboptimal H_∞ Control Problem

1. Compute solution (R, S, γ) of the optimization problem:

$$\left| \begin{array}{l} \min\limits_{R,S,\gamma} \gamma + Trace(R) + Trace(S) \\ \text{subject to LMIs (3.4)-(3.6)} \end{array} \right. \quad (3.7)$$

where $R \in \mathbb{R}^{n\times n}, S \in \mathbb{R}^{n\times n}, \gamma \in \mathbb{R}$.

2. Compute two full-column-rank matrices $M, N \in \mathbb{R}^{n\times k}$ such that $MN^T = I_{n\times n} - RS$.

By computing $[U, \Sigma, V] = SVD(I_{n\times n} - RS)$, $M = U$ and $N = V\Sigma^T$ can be identified.

3. Compute the positive definite matrix $X_{cl} \in \mathbb{R}^{(n+k)\times(n+k)}$ as the unique solution of the linear equation:

$$\begin{pmatrix} S & I_{n\times n} \\ N^T & O_{k\times n} \end{pmatrix} = X_{cl} \begin{pmatrix} I_{n\times n} & R \\ 0_{k\times n} & M^T \end{pmatrix} \quad (3.8)$$

Note that (3.8) is always solvable when $S > 0$ and M has full column rank.

4. Define the optimization problem:

$$\left| \begin{array}{l} \min\limits_{\Theta} \operatorname{rank}\left(\mathcal{J}_{k,k+m_2}\Theta\right) \\ \text{subject to} \\ \Psi_{X_{cl}} + \mathcal{Q}^T\Theta^T\mathcal{P}_{X_{cl}} + \mathcal{P}_{X_{cl}}^T\Theta\mathcal{Q} < 0 \end{array} \right. \quad (3.9)$$

where $\Theta = \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}$, $\Theta \in \mathbb{R}^{(k+m_2)\times(k+p_2)}$.

The matrices $\mathcal{J}_{k,k+m_2} = \begin{pmatrix} I_{k\times k} & 0_{k\times m_2} \end{pmatrix}$,

$$\Psi_{X_{cl}} = \begin{pmatrix} A_0^TX_{cl} + X_{cl}A_0 & X_{cl}B_0 & C_0^T \\ B_0^TX_{cl} & -\gamma I_{m_1\times m_1} & D_{11}^T \\ C_0 & D_{11} & -\gamma I_{p_1\times p_1} \end{pmatrix},$$

$$\mathcal{Q} = \begin{pmatrix} \mathcal{C} & \mathcal{D}_{21} & 0_{(k+p_2)\times p_1} \end{pmatrix},$$

$$\mathcal{P}_{X_{cl}} = \begin{pmatrix} \mathcal{B}^TX_{cl} & 0_{(k+m_2)\times m_1} & \mathcal{D}_{12}^T \end{pmatrix}$$

are known.

Note the fact that the objective function is $\operatorname{rank}\left(\begin{pmatrix} A_K & B_K \end{pmatrix}\right)$.

$$\mathcal{J}_{k,k+m_2}\Theta = \begin{pmatrix} I_{k\times k} & 0_{k\times m_2} \end{pmatrix} \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix} = \begin{pmatrix} A_K & B_K \end{pmatrix}$$

This is a NP-hard problem, known to be computationally intractable.

Solve the problem defined at 4. using a heuristic method (Fazel, Hindi & Boyd, 2001):

$$\left| \begin{array}{l} \min\limits_{Y,Z,\Theta} \frac{1}{2}\left(Trace(Y) + Trace(Z)\right) \\ \text{subject to} \\ \Psi_{X_{cl}} + \mathcal{Q}^T\Theta^T\mathcal{P}_{X_{cl}} + \mathcal{P}_{X_{cl}}^T\Theta\mathcal{Q} < 0, \\ \begin{pmatrix} Y & \mathcal{J}_{k,k+m_2}\Theta \\ \left(\mathcal{J}_{k,k+m_2}\Theta\right)^T & Z \end{pmatrix} \geq 0 \end{array} \right. \quad (3.10)$$

where $Y = Y^T \in \mathbb{R}^{k\times k}, Z = Z^T \in \mathbb{R}^{(k+p_2)\times(k+p_2)}$, $\Theta \in \mathbb{R}^{(k+m_2)\times(k+p_2)}$.

The variable of interest is Θ which contains the controller's (3.11) matrices:

$$A_K \in \mathbb{R}^{k\times k}, B_K \in \mathbb{R}^{k\times p_2}, C_K \in \mathbb{R}^{m_2\times k}, D_K \in \mathbb{R}^{m_2\times p_2}$$

$$\begin{aligned} \dot{x}_K &= A_K x_K + B_K u_K \\ y_K &= C_K x_K + D_K u_K \end{aligned} \tag{3.11}$$

The matrix $\begin{pmatrix} A_K & B_K \end{pmatrix}$ will have low rank, so a low-order version of the controller can be obtained, namely $\left(A_{K\_red}, B_{K\_red}, C_{K\_red}, D_{K\_red}\right)$ with $x_{K\_red} \in \mathbb{R}^{k\_red\times 1}$ and $k\_red < k$.

The low-order version is computed at steps 5.-7.

5. Compute $\left[U_K, \Sigma_K, V_K\right] = SVD\left(\begin{pmatrix} A_K & B_K \end{pmatrix}\right)$. If there is large distance between the singular values Σ_K, consider the smaller ones as equal to zero and identify the rank of $\begin{pmatrix} A_K & B_K \end{pmatrix}$ which is the order of the proposed controller.

6. Express all rows of $\begin{pmatrix} A_K & B_K \end{pmatrix}$ as a linear combination of the linearly independent ones. When expressing linear combinations, the columns of $\begin{pmatrix} A_K & B_K \end{pmatrix}^T$ are considered instead of the rows of $\begin{pmatrix} A_K & B_K \end{pmatrix}$.

6.1 Compute the reduced row echelon form of $\begin{pmatrix} A_K & B_K \end{pmatrix}^T$ and identify linearly independent columns of $\begin{pmatrix} A_K & B_K \end{pmatrix}^T$.

Each column of $\begin{pmatrix} A_K & B_K \end{pmatrix}^T$ can then be represented as a linear combination of the linearly independent ones:

$$A_{L.I.} * C_{coeff} = \begin{pmatrix} A_K & B_K \end{pmatrix}^T.$$

6.2 Solve the overdetermined system:

$$A_{L.I.} * C_{coeff} = \begin{pmatrix} A_K & B_K \end{pmatrix}^T$$

where:

- $A_{L.I.}$ is composed of the linearly independent columns of $\begin{pmatrix} A_K & B_K \end{pmatrix}^T$;
- Each column $'i'$ of C_{coeff} contains the coefficients that can be used to express column $'i'$ of $\begin{pmatrix} A_K & B_K \end{pmatrix}^T$ as a linear combination of $A_{L.I.}$ columns.

Note that instead of solving the system, C_{coeff} could be obtained directly from the reduced row echelon form.

7. Compute the matrices of the reduced controller of order k_red:

$$\begin{aligned} \dot{x}_{K\_red} &= A_{K\_red} x_{K\_red} + B_{K\_red} u_K \\ y_K &= C_{K\_red} x_{K\_red} + D_{K\_red} u_K \end{aligned}$$

7.1 Remove from $\begin{pmatrix} A_K & B_K \end{pmatrix}$ the rows that can be expressed as a linear combination of the linearly independent rows (the same linearly independent rows as in $A_{L.I.}^T$) $\Rightarrow \left(A_{K\_row\_red}, B_{K\_row\_red}\right)$.

7.2 Compute

$$\begin{aligned} A_{K\_red} &= A_{K\_row\_red} * C_{coeff}^T \\ B_{K\_red} &= B_{K\_row\_red} \\ C_{K\_red} &= C_K * C_{coeff}^T \\ D_{K\_red} &= D_K \end{aligned}$$

In the following, steps 5.-7. of the algorithm are illustrated in a numerical example:

$$\begin{aligned} \dot{x}_K &= A_K x_K + B_K u_K \\ y_K &= C_K x_K + D_K u_K \end{aligned}$$

with:

$$A_K = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 & 1 \\ 4 & 3 & 6 & 4 & 3 \\ 5 & 3 & 6 & 5 & 3 \\ 5 & 4 & 8 & 5 & 4 \end{pmatrix}$$

$$B_K = \begin{pmatrix} 2 & 3 & 7 & 8 & 9 \end{pmatrix}^T$$

$$C_K = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$D_K = 0.1$ and

$$x_K = \begin{pmatrix} x_{K1} & x_{K2} & x_{K3} & x_{K4} & x_{K5} \end{pmatrix}^T$$

- Step 5

$$\left[U_K, \Sigma_K, V_K\right] = SVD\left(\begin{pmatrix} A_K & B_K \end{pmatrix}\right)$$

The obtained singular values are:

$$\Sigma_K = \begin{pmatrix} 23.727 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.410 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.609e-15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.497e-16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.603e-17 & 0 \end{pmatrix}$$

The order of the reduced controller that will be computed is $k\_red = 2$.

- Step 6

The reduced row echelon form of $\begin{pmatrix} A_K & B_K \end{pmatrix}^T$ is

$$\mathrm{R}_{\mathrm{ref}} = \begin{pmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So, in $\begin{pmatrix} A_K & B_K \end{pmatrix}^T$ columns $\mathrm{C}3, C4$ and $C5$ can be expressed as a linear combination of $C1$ and $C2$.

In order to express $\mathrm{C}3, C4$ and $C5$ as a linear combination of $C1$ and $C2$, solve the system $A_{L.I.} * C_{coeff} = \begin{pmatrix} A_K & B_K \end{pmatrix}^T$.

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 2 \\ 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{pmatrix} C_{coeff} = \begin{pmatrix} 1 & 2 & 4 & 5 & 5 \\ 1 & 1 & 3 & 3 & 4 \\ 2 & 2 & 6 & 6 & 8 \\ 1 & 2 & 4 & 5 & 5 \\ 1 & 1 & 3 & 3 & 4 \\ 2 & 3 & 7 & 8 & 9 \end{pmatrix}$$

The solution of the system, C_{coeff} is:

$$C_{coeff} = \begin{pmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 & 1 \end{pmatrix}$$

Thus, the relationship between rows $R1, R2, R3, R4$ and $R5$ of $\begin{pmatrix} A_K & B_K \end{pmatrix}$ (corresponding to columns $C1, C2, \mathrm{C}3, C4$ and $C5$ of $\begin{pmatrix} A_K & B_K \end{pmatrix}^T$) is:

$$R3 = 2R1 + R2$$

$$R4 = R1 + 2R2$$

$$R5 = 3R1 + R2$$

- Step 7

$$\begin{pmatrix} \dot{x}_{K1} \\ \dot{x}_{K2} \\ \dot{x}_{K3} \\ \dot{x}_{K4} \\ \dot{x}_{K5} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 2 & 1 & 3 \\ 4 & 3 & 6 & 4 & 3 & 7 \\ 5 & 3 & 6 & 5 & 3 & 8 \\ 5 & 4 & 8 & 5 & 4 & 9 \end{pmatrix} \begin{pmatrix} x_{K1} \\ x_{K2} \\ x_{K3} \\ x_{K4} \\ x_{K5} \\ u_K \end{pmatrix}$$

$$R3 = 2R1 + R2 \Rightarrow$$

$$\dot{x}_{K3} = 2\dot{x}_{K1} + \dot{x}_{K2} \Rightarrow x_{K3} = 2x_{K1} + x_{K2}$$

$$R4 = R1 + 2R2 \Rightarrow$$

$$\dot{x}_{K4} = \dot{x}_{K1} + 2\dot{x}_{K2} \Rightarrow x_{K4} = x_{K1} + 2x_{K2}$$

$$R5 = 3R1 + R2 \Rightarrow$$

$$\dot{x}_{K5} = 3\dot{x}_{K1} + \dot{x}_{K2} \Rightarrow x_{K5} = 3x_{K1} + x_{K2}$$

The redundancy expressed by $\dot{x}_{K3}$, $\dot{x}_{K4}$ and $\dot{x}_{K5}$ can be removed and x_{K3}, x_{K4} and x_{K5} (that are expressed in function of x_{K1}, x_{K2}) can be replaced.

$$\begin{pmatrix} \dot{x}_{K1} \\ \dot{x}_{K2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_{K1} + 0 * x_{K2} \\ 0 * x_{K1} + x_{K2} \\ 2x_{K1} + x_{K2} \\ x_{K1} + 2x_{K2} \\ 3x_{K1} + x_{K2} \\ u_K \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} \dot{x}_{K1} \\ \dot{x}_{K2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix} x_{K1} + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} x_{K2} \right] +$$

$$+ \begin{pmatrix} 2 \\ 3 \end{pmatrix} u_K \Leftrightarrow$$

$$\begin{pmatrix} \dot{x}_{K1} \\ \dot{x}_{K2} \end{pmatrix} = \left[\begin{pmatrix} 9 \\ 11 \end{pmatrix} x_{K1} + \begin{pmatrix} 6 \\ 8 \end{pmatrix} x_{K2} \right] + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u_K \Leftrightarrow$$

$$\begin{pmatrix} \dot{x}_{K1} \\ \dot{x}_{K2} \end{pmatrix} = \begin{pmatrix} 9 & 6 \\ 11 & 8 \end{pmatrix} \begin{pmatrix} x_{K1} \\ x_{K2} \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u_K$$

The output equation becomes:

$$y_K = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{K1} + 0 * x_{K2} \\ 0 * x_{K1} + x_{K2} \\ 2x_{K1} + x_{K2} \\ x_{K1} + 2x_{K2} \\ 3x_{K1} + x_{K2} \end{pmatrix} + D_K u_K \Leftrightarrow$$

$$y_K = \left[\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix} x_{K1} + \right.$$

$$\left. + \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} x_{K2} \right] + D_K u_K \Leftrightarrow$$

$$y_K = \begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} x_{K1} \\ x_{K2} \end{pmatrix} + D_K u_K$$

The reduced controller of order 2 is:

$$\dot{x}_{K\_red} = A_{K\_red} x_{K\_red} + B_{K\_red} u_K$$
$$y_K = C_{K\_red} x_{K\_red} + D_{K\_red} u_K$$

with:

$$A_{K\_red} = \begin{pmatrix} 9 & 6 \\ 11 & 8 \end{pmatrix}, B_{K\_red} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, C_{K\_red} = \begin{pmatrix} 3 & 1 \end{pmatrix},$$

$$D_{K\_red} = 0.1, \ x_{K\_red} = \begin{pmatrix} x_{K1} \\ x_{K2} \end{pmatrix}.$$

The above computations for step 7 can be summed up as:

1. Remove from $\begin{pmatrix} A_K & B_K \end{pmatrix}$ the rows that can be expressed as a linear combination of the linearly independent rows $\Rightarrow \left(A_{K\_row\_red}, B_{K\_row\_red} \right)$

$$A_{K\_row\_red} = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 & 1 \end{pmatrix}$$

$$B_{K\_row\_red} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

2. Compute:

$$A_{K\_red} = A_{K\_row\_red} * C_{coeff}^T =$$

$$= \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \\ 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 6 \\ 11 & 8 \end{pmatrix}$$

$$B_{K\_red} = B_{K\_row\_red}$$

$$C_{K\_red} = C_K * C_{coeff}^T =$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \\ 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \end{pmatrix}$$

$$D_{K\_red} = D_K .$$

4. Simulation Results

A random system of order 10 is generated using rss function in MATLAB:

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$q = C_1 x + D_{11} w + D_{12} u$$
$$y = C_2 x + D_{21} w + D_{22} u$$

The first controller is computed using Matlab function hinfsyn.m:

$$A_{k\_hinf\,syn} \in \mathbb{R}^{10\times10},\ B_{k\_hinf\,syn} \in \mathbb{R}^{10\times1},$$
$$C_{k\_hinf\,syn} \in \mathbb{R}^{1\times10},\ D_{k\_hinf\,syn} \in \mathbb{R}.$$

The second controller is computed using the proposed algorithm:

$$A_{k\_red} \in \mathbb{R}^{2\times2}, B_{k\_red} \in \mathbb{R}^{2\times1},$$
$$C_{k\_red} \in \mathbb{R}^{1\times2}, D_{k\_red} \in \mathbb{R}.$$

Figure 2 illustrates the outputs of the closed loop system for the hinfsyn.m controller and the proposed controller. For the same input w, the stabilized outputs for the two controllers are similar. Figure 3 indicates a small error between the two outputs and a small standard deviation of the error.

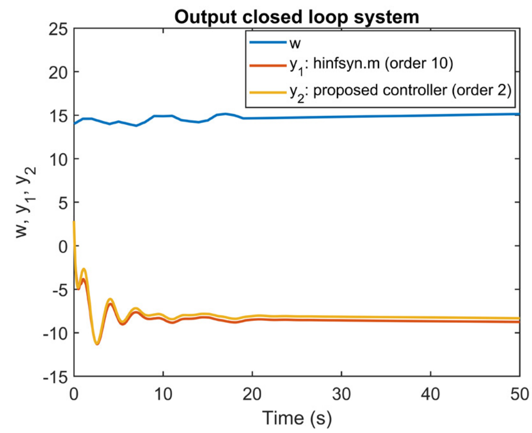

Figure 2. The outputs of the closed loop system

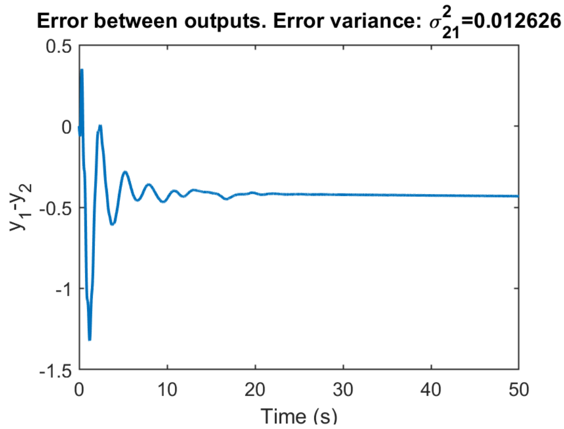

Figure 3. The error between outputs and the standard deviation of error

The results show that the proposed low-order H_∞ controller can be a substitute for the H_∞ controller of higher order that was generated using hinfsyn.m.

5. Conclusion

This article has proposed an algorithm for computing H_∞ controllers in a similar manner as the Matlab procedure hinfsyn.m (Gahinet & Apkarian, 1994), but of low-order. In the proposed algorithm, the controller order reduction was made possible by considering a rank minimization problem which introduces dependencies between the differential equations of the controller. Two controllers were computed and compared in a simulation: the controller computed using Matlab function hinfsyn.m (Riccati) and the low-order controller computed using the proposed algorithm. The controllers were compared by inspecting, for the same input, the outputs of the closed loop system. The results were similar in both cases.

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REFERENCES

1. Adamjan, V. M., Arov, D. Z. & Krein, M. G. (1971). Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur–Takagi problem, *Mathematics of the USSR-Sbornik*, *15*(1), 34-75.

2. Adamjan V. M., Arov, D. Z. & Krein, M. G. (1978). Infinite block Hankel matrices and related extension problems, *Transactions of the American Mathematical Society*, *111*(2), 133-156.

3. Antoulas, A. C. (2005). *Approximation of Large-Scale Dynamical Systems*, 207-246. Series: Advances in Design and Control, SIAM.

4. Benner, P., Mehrmann, V. & Sorensen, D. C. (2005). *Dimension Reduction of Large-Scale Systems*. Series: Lecture Notes in Computational Science and Engineering. Springer.

5. Davison, E. (1966). A method for simplifying linear dynamic systems, *IEEE Transactions on Automatic Control*, *11*(1) , 93-101.

6. Eckart, C. & Young, G. (1936). The approximation of one matrix by another of lower rank, *Psychometrika*, *1*(3), 211-218.

7. Fazel, M., Hindi, H. & Boyd, S. P. (2001). A Rank Minimization Heuristic with Application to Minimum Order System Approximation. In *Proceedings of the 2001 American Control Conference* (pp. 4734-4739).

8. Freund, R.W. (2003). Model reduction methods based on Krylov subspaces, *Acta Numerica*, *12*, 267-319.

9. Gahinet, P. & Apkarian, P. (1994). A Linear Matrix Inequality Approach to H_∞ Control, *International Journal of Robust and Nonlinear Control*, *4*(4) 421-448.

10. Glover, K. (1984). All optimal Hankel-norm approximations of linear multivariable systems and their L_∞ -error bounds, *International Journal of Control*, *39*(6), 1115-1193.

11. Grimme, E. J. (1997). *Krylov projection methods for model reduction*. PhD thesis. University Illinois, Urbana-Champaign, Department of Electrical Engineering.

12. Gugercin, S. & Antoulas, A. C. (2004). A Survey of Model Reduction by Balanced Truncation and Some New Results, *International Journal of Control*, *77*(8), 748-766.

13. Ionescu, T. C. & Astolfi, A. (2011). Moment matching for linear systems – overview and new results, *IFAC Proceedings Volumes*, *44*(1), 12739-12744.

14. Ionescu, T. C. & Astolfi, A. (2016) Nonlinear Moment Matching-Based Model Order Reduction, *IEEE Transactions on Automatic Control*, *61*(10), 2837-2847.

15. Jedda, O. & Douik, A. (2018). Discrete-time Integral Sliding Mode Control with Anti-windup, *Studies in Informatics and Control*, *27*(4), 413-422. DOI: 10.24846/v27i4y201805

16. Mirsky, L. (1963). Results and problems in the theory of doubly-stochastic matrices, *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, *1*(4), 319-334.

17. Mohammadpour, J. & Grigoriadis, K. M. (2010). *Efficient Modeling and Control of Large-Scale Systems*. Control Engineering, Springer.

18. Nouri, S. B. (2014). *Advanced Model-Order Reduction Techniques for Large-Scale Dynamical Systems*. PhD thesis. Ottawa-Carleton Institute for Electrical and Computer Engineering, Department of Electronics, Carleton University, Ottawa, Canada.

19. Pillage, L. T. & Rohrer, R. A. (1990). Asymptotic Waveform Evaluation for Timing Analysis, *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 9*(4) , 352-366.

20. Schilders, W. H., Van der Vorst, H. A. & Rommes, J. (2008). *Model Order Reduction: Theory, Research Aspects and Applications*. The European Consortium for Mathematics in Industry, Springer.

21. Schmidt, E. (1907). Zur Theorie der linearen und nichtlinearen Integralgleichungen. I Tiel. Entwicklung Willkurlichen Funktionen nach System Vorgeschriebener, *Mathematische Annalen, 63*(4), 433-476.

22. Vandendorpe, A. (2004). *Model Reduction of Linear Systems, an Interpolation point of View*. PhD thesis. Université Catholique De Louvain, Faculté des Sciences Appliquées, Departement d'Ingénierie Mathématique.

23. Wittmuess, P., Tarin, C., Keck, A., Arnold, E. & Sawodny, O. (2016). Parametric Model Order Reduction via Balanced Truncation with Taylor Series Representation, *IEEE Transactions on Automatic Control*, *61*(11), 3438-3451.