Discrete-time Integral Sliding Mode Control with Anti-windup

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Abstract: This paper presents a discrete-time integral sliding mode control with anti-windup for single-input single-output linear systems with external disturbances whose upper bound is not required to be known. The proposed scheme is designed by using a linear quadratic regulator approach in order to ensure the stability of the closed-loop system in the quasi-sliding mode. The robustness with respect to external disturbances, the follow-up of a reference model and the elimination of the reaching phase are guaranteed. The chattering phenomenon is avoided using the saturation function in the controller design. Furthermore, the anti-windup method is used in order to improve system performance and to reduce the control input values in the initial phase. Simulation results show that the developed controller achieves better results than the conventional discrete-time sliding mode controller and the discrete-time integral sliding mode controller.

Keywords: Discrete-time integral sliding mode control, Anti-windup method, Chattering phenomenon, Robust tracking and model following.

1. Introduction

Since the increasing use of computers and digital signal processors (DSPs) in control implementation, discrete-time sliding mode control (DSMC) has known a growing interest from the automation community (Bartoszewicz & Latosiński, 2016; Milosavljevic, 1985; Sarpturk, Istefanopulos & Kaynak, 1987). However, due to the finite sampling rate, extending the concept of continuous-time sliding mode control to discrete-time affects the robustness properties of the conventional sliding mode with respect to parametric uncertainties, modelling errors, and external disturbances (Bandyopadhyay, Deepak & Kim, 2009; Utkin, 1977; Young, Utkin & Ozguner, 1996). This can be explained by the fact that the controller is updated at each sampling time in such a way that it cannot be changed when the system trajectory crosses the sliding surface during the sampling period (Utkin & Drakunov, 1989).

Later, the quasi-sliding mode (QSM) notion was introduced by Gao, Wang and Homaifa (1995). It consists in bringing the state trajectory, from any initial state, towards the sliding surface, crossing it in finite-time, following a non-increasing zigzag motion around it, and then staying within a boundary layer in its vicinity. Hence, the so-called chattering phenomenon will be attenuated but not avoided. Actually, taking into consideration the existence of switching imperfections in practice, the chatter effect is caused by the discontinuity of the sign function used as a switching function in the controller design. This effect might excite disregarded high-frequency dynamics, badly affect system performance, and even damage control devices (Perruquetti & Barbot, 2002).

In order to overcome the above-mentioned drawback, several solutions were proposed in the literature. Furuta (1990) developed a stable DSMC based on a discrete Lyapunov function. In (Bartolini, Ferrara & Utkin, 1995), an adaptive sliding mode control was developed for discrete-time linear systems. It consists in defining the equivalent control as a piecewise-constant control. Further on, in (Alanis et al., 2013; Mihoub, Nouri & Abdennour, 2008; Mihoub, Nouri & Abdennour, 2009), a discrete-time second-order sliding mode control was developed, by referring to a continuous-time second-order sliding mode controller, for both linear and nonlinear systems. Yet, the simple solution to avoid the chattering phenomenon is to replace the sign function by a smooth one such as a hyperbolic tangent function (Khandekar, Malwatkar & Patre, 2013; Khandekar & Patre, 2014) or a saturation function (Kim, Oh & Hedrick, 2000; Kim & Cho, 2000).

Discrete-time integral sliding mode control (DISMC) was proposed by Abidi, Xu and Xinghuo (2007) as an improved technique of DSMC designed on the basis of the integral sliding mode (ISM) concept (Utkin & Shi, 1996). Subsequently, it was developed to control nonlinear systems for better tracking performances and response

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characteristics (Chihi et al., 2017), and uncertain and time-delayed linear systems for robust tracking and model following (Pai, 2008; Pai, 2009; Pai, 2014). Its main advantages are the full order of the motion equation in the quasi-sliding mode, the stability of the closed-loop system, the elimination of the reaching phase and the excellent tracking performance. However, the integral term in the sliding function is at the origin of the error accumulation and hence of the so-called windup phenomenon. The latter can lead to inappropriate characteristics of system response such as high overshoot and settling time (Barambones, Garrido & Maseda, 2007; Oliveira et al., 2016).

In this paper, a discrete-time integral sliding mode controller with anti-windup (DISMC-AW) is developed for single-input single-output (SISO) linear systems subject to external disturbances in order to improve control performance. As in the case of the anti-windup method proposed with the continuous-time integral sliding mode controller (Oliveira et al., 2016), a gain based on Gaussian function is assigned to the integrative portion of the sliding function in order to set its activity area such that the effect of error accumulation on system performance and on control input in the initial phase is limited. Moreover, the sliding surface is designed on the basis of the linear quadratic regulator (LQR) approach in order to guarantee the stability of the closed-loop system in the quasi-sliding mode. Without requiring knowledge of the disturbances' upper bound, the control law is designed so that the robustness in the presence of external disturbances, the follow-up of the reference model, and the elimination of the chattering phenomenon are ensured. In order to demonstrate its efficiency, the developed controller will be compared with the conventional discrete-time sliding mode controller and with the discrete-time integral sliding mode controller designed using the linear matrix inequalities (LMI) and the LQR methods.

This paper is organized as follows. Section 2 presents the problem formulation of model following as well as the development of DSMC and DISMC controllers for SISO linear systems with external disturbances. Section 3 provides the proposed DISMC-AW scheme. Numerical simulation results are illustrated in section 4. Section 5 offers concluding remarks.

2. Problem Formulation

2.1 The problem of model following

Consider the following SISO discrete-time linear system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + d(k) \\ y(k) = Cx(k) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}$ is the control input, $y(k) \in \mathbb{R}$ is the system output and $d(k) \in \mathbb{R}^n$ is the external disturbance. A, B and C are matrices of appropriate dimensions.

The disturbance $d(k)$ is supposed to satisfy the so-called matched condition so that it will be expressed in the form of $d(k) = Bf(k)$.

The discrete-time reference model is given by

$$\begin{cases} x_m(k+1) = A_m x_m(k) \\ y_m(k) = C_m x_m(k) \end{cases} \tag{2}$$

where $x_m(k) \in \mathbb{R}^{n_m}$ and $y_m(k) \in \mathbb{R}$ are the state vector and the output of the reference model respectively. $y(k)$ and $y_m(k)$ must have the same dimension.

The follow-up of the reference model (2) requires the existence of matrices $G \in \mathbb{R}^{n \times n_m}$ and $H \in \mathbb{R}^{1 \times n_m}$ which satisfy the following relation (Hopp & Schmitendorf, 1990; Shyu & Chen, 1995)

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} G A_m \\ C_m \end{bmatrix}. \tag{3}$$

The tracking controller has the following structure

$$u(k) = H x_m(k) + v(k), \tag{4}$$

where $v(k)$ is defined as the auxiliary control.

An auxiliary state vector $z(k)$ is introduced to facilitate further development. It is given by

$$z(k) = x(k) - G x_m(k). \tag{5}$$

Hence, as $CG = C_m$ in (3), the tracking error $e(k) = y(k) - y_m(k)$ can be expressed as follows

$$e(k) = C z(k) \tag{6}$$

Using (1) and (5), the auxiliary system is given by

$$z(k+1) = A z(k) + B v(k) + d(k). \tag{7}$$

Later on, the auxiliary control $v(k)$ will be designed using DISMC and DISMC-AW controllers such that the auxiliary system (7) is asymptotically stable in the quasi-sliding mode.

2.2 Discrete-time Sliding Mode Control

For the conventional discrete-time sliding mode controller, the sliding function is expressed as follows

$$\sigma(k) = S(x(k) - x_m(k)), \tag{8}$$

where $S \in \mathbb{R}^{1\times n}$ is the sliding matrix chosen so that SB is non-singular.

The reaching law is given by

$$\sigma(k+1) = \phi\sigma(k) - \eta \text{ sat}\left(\frac{\sigma(k)}{\varphi}\right), \tag{9}$$

where $0 < \phi < 1$, $\eta > 0$, $\varphi > 0$ is the boundary layer width of sliding manifold and sat is the saturation function defined by

$$\text{sat}\left(\frac{\sigma(k)}{\varphi}\right) = \begin{cases} \dfrac{\sigma(k)}{\varphi} & \text{if} \left|\dfrac{\sigma(k)}{\varphi}\right| \le 1 \\ \text{sign}(\sigma(k)) & \text{else} \end{cases}, \tag{10}$$

where sign is the sign function.

Using the forward expression of the sliding function (8) and the reaching law (9), the control law $u(k)$ is given by

$$u(k) = (SB)^{-1}\left(\phi\sigma(k) - \eta \text{ sat}\left(\frac{\sigma(k)}{\varphi}\right)\right) - (SB)^{-1} S\left(Ax(k) + d(k) - A_m x_m(k)\right). \tag{11}$$

Yet, under practical considerations, the control law cannot be implemented in the form given in (11) due to the lack of priori knowledge of the external disturbance $d(k)$.

The control law and the disturbance estimation law for system (1) are given, respectively, by (Pai, 2012; Pai, 2014)

$$u(k) = (SB)^{-1}\left(\phi\sigma(k) - \eta \text{ sat}\left(\frac{\sigma(k)}{\varphi}\right)\right) - (SB)^{-1} S\left(Ax(k) + \hat{d}(k) - A_m x_m(k)\right), \tag{12}$$

$$\hat{d}(k) = \hat{d}(k-1) + S^{+}\left(\sigma(k) - \phi\sigma(k-1) + \eta \text{ sat}\left(\frac{\sigma(k-1)}{\varphi}\right)\right), \tag{13}$$

where $\hat{d}(k)$ is the estimate of $d(k)$ and S^{+} denotes the pseudo-inverse of S.

Theorem 1: Consider the system (1) with the control law (12). If the sliding function (8), the reaching law (9), and the disturbance estimation law (13) are used, the control law (12) will drive the state trajectory arbitrarily close to the quasi-sliding mode band Δ given by

$$\Delta = \frac{1}{1-\phi}(\eta + \delta), \tag{14}$$

where δ is the maximum changing rate of the disturbance defined by

$$\left|S\left(d(k+1) - d(k)\right)\right| < \delta \text{ for all } k \tag{15}$$

Proof: Using the system (1) and the forward expression of the sliding function (8) yields

$$\sigma(k+1) = SBu(k) + Sd(k) + SAx(k) - SA_m x(k). \tag{16}$$

Substituting (12) into (16) yields

$$\sigma(k+1) = \phi\sigma(k) - \eta \text{ sat}\left(\frac{\sigma(k)}{\varphi}\right) + S\left(d(k) - \hat{d}(k)\right). \tag{17}$$

Therefore, substituting the backward expression of (17) into (13) yields

$$\hat{d}(k) = d(k-1) \tag{18}$$

For $k = 1, 2$, using (17) and (18), the sliding function $\sigma(k)$ is expressed as follows:

$$\sigma(1) = \phi\sigma(0) - \eta \text{ sat}\left(\frac{\sigma(0)}{\varphi}\right) + S\left(d(0) - \hat{d}(0)\right),$$

$$\sigma(2) = \phi\sigma(1) - \eta \text{ sat}\left(\frac{\sigma(1)}{\varphi}\right) + S\left(d(1) - d(0)\right), \tag{19}$$

where $\hat{d}(0)$ is an arbitrary bounded estimated disturbance.

For $k>1$, the sliding function $\sigma(k)$ is written as follows

$$\sigma(k)=\phi^{k-1}\sigma(1)-\eta\sum_{j=0}^{k-2}\phi^{j}\,\mathrm{sat}\left(\frac{\sigma(k-2-j+1)}{\varphi}\right)+\sum_{j=0}^{k-2}\phi^{j}\,S\big(d(k-2-j+1)-d(k-2-j)\big). \quad (20)$$

Knowing that $\left|S\big(d(k+1)-d(k)\big)\right|<\delta$ for all k, the above expression of $\sigma(k)$ is bounded by

$$\left|\sigma(k)\right|<\phi^{k-1}\left|\sigma(1)\right|+\sum_{j=0}^{k-2}\phi^{j}\,(\eta+\delta) \quad (21)$$

As $0<\phi<1$, it is easy to verify that, when k approaches infinity,

$$\left|\sigma(k)\right|<\frac{1}{1-\phi}(\eta+\delta) \quad (22)$$

Therefore, the control law (12) will drive the state trajectory arbitrary close to the quasi-sliding mode band $\Delta=\frac{1}{1-\phi}(\eta+\delta)$. The proof is completed.

2.3 Discrete-time Integral Sliding Mode Control

The sliding function is expressed as

$$\sigma(k)=S\,z(k)-S\exp(-\beta k)\,z(0)-\varepsilon(k),\beta>0, \quad (23a)$$

$$\varepsilon(k)=\varepsilon(k-1)+S(A+BK)\,z(k-1),\varepsilon(0)=0, \quad (23b)$$

where $K\in\mathbb{R}^{1\times n}$ is the feedback matrix ensuring the stability of the auxiliary system (7) in the quasi-sliding mode, $S\exp(-\beta k)z(0)$ is the exponential term ensuring the elimination of the reaching phase, and $\varepsilon(k)$ is the integrative portion ensuring the zero-tracking error.

Using the equivalent control concept (Utkin & Shi, 1996), the equivalent control v_{eq} is obtained by setting $\sigma(k+1)=0$. It is expressed as

$$v_{eq}(k)=K\,z(k)-(SB)^{-1}S\,d(k)+(SB)^{-1}S\exp\big(-\beta(k+1)\big)z(0)+(SB)^{-1}\,\varepsilon(k). \quad (24)$$

In the quasi-sliding mode, solving $\sigma(k)=0$ gives

$$\varepsilon(k)=S\,z(k)-S\exp(-\beta k)\,z(0). \quad (25)$$

Then, substituting (25) into (24) yields

$$v_{eq}(k)=\left(K+(SB)^{-1}S\right)z(k)-(SB)^{-1}S\,d(k)+\Gamma(k), \quad (26)$$

with

$$\Gamma(k)=(SB)^{-1}S\big(\exp\big(-\beta(k+1)\big)-\exp(-\beta k)\big)z(0).$$

Using the equivalent control law (26), the auxiliary system is expressed in the quasi-sliding mode as follows

$$z(k+1)=A_c\,z(k)+B\,\Gamma(k), \quad (27)$$

with $A_c=A+B(SB)^{-1}S+BK$.

Remark 1: The exponential term $\Gamma(k)$ in (27) tends towards zero when k approaches infinity, i.e. $\lim_{k\to\infty}\Gamma(k)=0$, so that the auxiliary system (27) is asymptotically stable in the quasi-sliding mode.

2.3.1 Linear matrix inequalities approach

Using the LMI method, the auxiliary system (27) is asymptotically stable in the quasi-sliding mode if there exist a positive-definite matrix $X\in\mathbb{R}^{n\times n}$ and a matrix $W\in\mathbb{R}^{1\times n}$ such that the following inequality is satisfied

$$\begin{bmatrix} -X & \left(A_{eq}X-B_{eq}W\right)^{T} \\ A_{eq}X-B_{eq}W & -X \end{bmatrix}<0 \quad (28)$$

where $A_{eq}=A+B(SB)^{-1}S$ and $B_{eq}=-B$.

The feedback matrix K is expressed by

$$K=W\,X^{-1} \quad (29)$$

2.3.2 Linear quadratic regulator approach

Using the infinite time horizon discrete-time LQR approach, the auxiliary system (27) is asymptotically stable in the quasi-sliding mode if there exist positive definite matrices $Q\in\mathbb{R}^{n\times n}$ and $R\in\mathbb{R}$ that minimize the following objective function

$$J = \sum_{k=0}^{\infty}\left(z(k)^T Q z(k) + v_{eq}(k)^T R v_{eq}(k)\right) \quad (30)$$

The feedback matrix K is given by

$$K = \left(B_{eq}^T P B_{eq} + R\right)^{-1} B_{eq}^T P A_{eq}, \quad (31)$$

where P is a positive-definite matrix solution of the discrete-time algebraic Riccati equation

$$A_{eq}^T P A_{eq} - P - A_{eq}^T P B_{eq}\left(B_{eq}^T P B_{eq} + R\right)^{-1} B_{eq}^T P A_{eq} + Q = 0. \quad (32)$$

Using the forward expression of the sliding function (23) and the reaching law (9), the auxiliary control law $v(k)$ is given by

$$\begin{aligned} v(k) = {} & (SB)^{-1}\left(\phi\sigma(k) - \eta\,\text{sat}\left(\frac{\sigma(k)}{\varphi}\right)\right) \\ & + K z(k) - (SB)^{-1} S d(k) \\ & + (SB)^{-1} S \exp\left(-\beta(k+1)\right) z(0) \\ & + (SB)^{-1} \varepsilon(k). \end{aligned} \quad (33)$$

However, as in the case of the control law (11), the implementation of the auxiliary control (33) requires a priori knowledge of the disturbance $d(k)$.

The auxiliary control law (33) for the auxiliary system (7) is given by

$$\begin{aligned} v(k) = {} & (SB)^{-1}\left(\phi\sigma(k) - \eta\,\text{sat}\left(\frac{\sigma(k)}{\varphi}\right)\right) \\ & + K z(k) - (SB)^{-1} S \hat{d}(k) \\ & + (SB)^{-1} S \exp\left(-\beta(k+1)\right) z(0) \\ & + (SB)^{-1} \varepsilon(k). \end{aligned} \quad (34)$$

where $\hat{d}(k)$ is the estimate of $d(k)$ given by the disturbance estimation law (13).

Theorem 2: Consider the auxiliary system (7) with the auxiliary control law (34). If the sliding function (23), the reaching law (9), and the disturbance estimation law (13) are used, the control law (34) will derive the state trajectory arbitrarily close to the quasi-sliding mode band (14).

Proof: Using the auxiliary system (7) and the forward expression of the sliding function (23) yields

$$\begin{aligned} \sigma(k+1) = {} & SBv(k) + Sd(k) - SBK z(k) \\ & - \varepsilon(k) - S\exp\left(-\beta(k+1)\right) z(0). \end{aligned} \quad (35)$$

Substituting (34) into (35) yields the same forward expression of the sliding function given in (17). Then, substituting the backward expression of the latter into the disturbance estimation law (13) leads to the same estimate of $d(k)$ given in (18). One can follow the same development from (19) to (22).

Thus, the auxiliary control law (34) will drive the state trajectory arbitrary close to the quasi-sliding mode band (14). The proof is completed.

3. Discrete-time Integral Sliding Mode Control with Anti-windup

For the discrete-time integral sliding mode controller with anti-windup, the sliding function is defined as follows

$$\begin{aligned} \sigma(k) = {} & S z(k) - S\exp(-\beta k) z(0) \\ & - \alpha(k)\varepsilon(k), \beta > 0, \end{aligned} \quad (36a)$$

$$\begin{aligned} \varepsilon(k) = {} & \varepsilon(k-1) \\ & + S(A + BK) z(k-1), \varepsilon(0) = 0, \end{aligned} \quad (36b)$$

$$\alpha(k) = \exp\left(\frac{-\left(\varepsilon(k)\right)^2}{\mu}\right), \mu > 0, \quad (36c)$$

where $\alpha(k)$ is a positive function assigned to the integrative portion $\varepsilon(k)$ in order to set its activity area.

Setting $\sigma(k+1) = 0$, the equivalent control $v_{eq}(k)$ is given by

$$\begin{aligned} v_{eq}(k) = {} & -\left(1 - \alpha(k+1)\right)(SB)^{-1} SAz(k) \\ & + \alpha(k+1) Kz(k) - (SB)^{-1} S d(k) \\ & + (SB)^{-1} S\exp\left(-\beta(k+1)\right) z(0) \\ & + (SB)^{-1} \alpha(k+1)\varepsilon(k). \end{aligned} \quad (37)$$

Remark 2: The auxiliary system (7) is asymptotically stable in the quasi-sliding mode if the integrative portion $\varepsilon(k)$ is bounded when k approaches infinity, i.e. $\lim_{k\to\infty}\left\|\varepsilon(k)\right\| < \rho$ with $\rho > 0$. Hence, the gain $\alpha(k)$ tends towards a non-zero value, i.e. $\lim_{k\to\infty}\alpha(k) \neq 0$.

Solving $\sigma(k)=0$ leads to

$$\varepsilon(k)=\frac{1}{\alpha(k)}\left(Sz(k)-S\exp(-\beta k)z(0)\right) \quad (38)$$

Substituting (38) into (37) yields

$$\begin{aligned} v_{eq}(k) &= -\left(1-\alpha(k+1)\right)(SB)^{-1}SAz(k) \\ &+\frac{\alpha(k+1)}{\alpha(k)}(SB)^{-1}Sz(k) \\ &+\alpha(k+1)Kz(k)-(SB)^{-1}Sd(k) \\ &+\Gamma_{\alpha}(k). \end{aligned} \quad (39)$$

with

$$\Gamma_{\alpha}(k)=(SB)^{-1}S \left(\exp(-\beta(k+1))-\frac{\alpha(k+1)}{\alpha(k)}\exp(-\beta k)\right)z(0).$$

Hence, substituting (39) into (7), the auxiliary system (7) in the quasi-sliding mode is expressed as follows

$$z(k+1)=A_{c\alpha}(k)z(k)+B\Gamma_{\alpha}(k), \quad (40)$$

with

$$\begin{aligned} A_{c\alpha}(k) &= A-\left(1-\alpha(k+1)\right)B(SB)^{-1}SA \\ &+\frac{\alpha(k+1)}{\alpha(k)}B(SB)^{-1}S+\alpha(k+1)BK. \end{aligned}$$

Let's suppose that in the quasi-sliding mode $\lim_{k\to\infty}\alpha(k)=\gamma$, $\gamma>0$. Thus, the matrix $A_{c\alpha}(k)$ and the exponential term $\Gamma_{\alpha}(k)$ in (40) tend towards the constant term $A-(1-\gamma)B(SB)^{-1}SA+B(SB)^{-1}S+\gamma BK$ and zero, respectively, when k approaches infinity, i.e.

$$\begin{aligned} \lim_{k\to\infty}A_{c\alpha}(k) &= A-(1-\gamma)B(SB)^{-1}SA \\ &+B(SB)^{-1}S+\gamma BK \end{aligned}$$

and $\lim_{k\to\infty}\Gamma_{\alpha}(k)=0$.

Lemma: If there exists a gain matrix K that ensures the stability of the auxiliary system (40) in the quasi-sliding mode, the auxiliary state vector $z(k)$ tends towards zero when k approaches infinity, i.e.

$$\lim_{k\to\infty}\|z(k)\|=0 \quad (41)$$

Proof: The gain matrix K ensures the stability of the auxiliary system (40) in the quasi-sliding mode. Then, the norm of all eigenvalues of matrix $A_{c\alpha}$ in (40) is lower than one, i.e. $\|\lambda_j\|<1$, $j=1,\ldots,n$. In other words, the matrix $A_{c\alpha}$ can be expressed using the similarity transformation matrix T by

$$A_{c\alpha}=TDT^{-1}, \quad (42)$$

with $D=\text{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$. Subsequently, the auxiliary system (40) can be rewritten as

$$z(k)=TD^kT^{-1}z(0), \quad (43)$$

and $\lim_{k\to\infty}\|z(k)\|=0$. The proof is completed.

Remark 3: Since $\lim_{k\to\infty}\|z(k)\|=0$, the integral term $\varepsilon(k)$ given in (38) tends towards zero as $k\to\infty$, i.e. $\lim_{k\to\infty}\varepsilon(k)=0$, so that the gain $\alpha(k)$ tends towards one when k approaches infinity, i.e. $\lim_{k\to\infty}\alpha(k)=1$. Therefore, the matrix $A_{c\alpha}(k)$ of the auxiliary system (40) tends towards the constant term $A+B(SB)^{-1}S+BK$ in the quasi-sliding mode, i.e. $\lim_{k\to\infty}A_{c\alpha}(k)=A_c(k)=A+B(SB)^{-1}S+BK$.

Using the LQR method, the feedback matrix K is the same as that given in (31).

The parameter μ in (36) is adjusted such that the gain $\alpha(k)$ tends towards zero for the significant error accumulation and then tends to one when the state trajectory of the system reaches the sliding surface as shown in Figure 1. Hence, the effect of the integrative portion $\varepsilon(k)$ on system performance and on control input in the initial phase is limited.

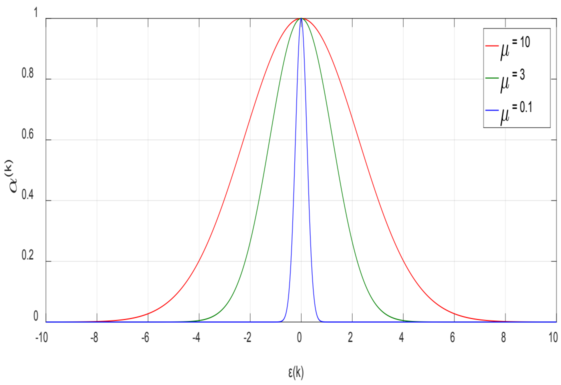


Figure 1. Variation of the gain $\alpha(k)$ with different values of μ

Remark 4: Since $\|e(k)\| < \|C\|\|z(k)\|$, $\|C\| < \infty$ and $\lim_{k\to\infty}\|z(k)\| = 0$, the tracking error $e(k)$ converges to zero when k approaches infinity, i.e. $\lim_{k\to\infty} e(k) = 0$, so that the zero-tracking error is guaranteed in the quasi-sliding mode.

Using the forward expression of the sliding function (36) and the reaching law (9), the auxiliary control law is given by

$$\begin{aligned} v(k) = (SB)^{-1}\left(\phi\sigma(k) - \eta\,\mathrm{sat}\left(\frac{\sigma(k)}{\varphi}\right)\right) \\ -(SB)^{-1}SA\,z(k) - (SB)^{-1}S\,d(k) \\ +(SB)^{-1}S\exp(-\beta(k+1))z(0) \\ +(SB)^{-1}\alpha(k+1)\varepsilon(k+1). \end{aligned} \quad (44)$$

As in the case of DSMC and DISMC controllers, the disturbance $d(k)$ in (44) is replaced by its estimate $\hat{d}(k)$ given by the disturbance estimation law (13). Hence, the auxiliary control law (44) is rewritten as follows

$$\begin{aligned} v(k) = (SB)^{-1}\left(\phi\sigma(k) - \eta\,\mathrm{sat}\left(\frac{\sigma(k)}{\varphi}\right)\right) \\ -(SB)^{-1}SA\,z(k) - (SB)^{-1}S\,\hat{d}(k) \\ +(SB)^{-1}S\exp(-\beta(k+1))z(0) \\ +(SB)^{-1}\alpha(k+1)\varepsilon(k+1). \end{aligned} \quad (45)$$

Theorem 3: Consider the auxiliary system (7) with the auxiliary control law (45). If the sliding function (36), the reaching law (9), and the disturbance estimation law (13) are used, the auxiliary control law (45) will derive the state trajectory arbitrarily close to the quasi-sliding mode band (14).

Proof: Using the auxiliary system (7) and the forward expression of the sliding surface (36) yields

$$\begin{aligned} \sigma(k+1) = SBv(k) + Sd(k) + SA\,z(k) \\ -\alpha(k+1)\varepsilon(k+1) \\ -S\exp(-\beta(k+1))z(0). \end{aligned} \quad (46)$$

Substituting (45) into (46) yields the same forward expression of the sliding function (17), and then substituting its backward expression into (13) yields the same disturbance estimate $\hat{d}(k)$ (18). One can follow the same development from (19) to (22).

Hence, the auxiliary control law (45) will drive the state trajectory arbitrary close to the quasi-sliding mode band (14). The proof is completed.

4. Numerical Simulation Results

Consider the nominal system sampled with a sampling period $T_s = 0.02\,\mathrm{s}$ (Oucheriah, 1999; Pai, 2014)

$$\begin{cases} x(k+1) = \begin{bmatrix} 0.9802 & 0.04 \\ 0 & 1.0202 \end{bmatrix} x(k) \\ \qquad + \begin{bmatrix} 0.0004 \\ 0.0202 \end{bmatrix} u(k) + d(k) \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{cases} \quad (47)$$

with $d(k) = \begin{bmatrix} 0.0004 \\ 0.0202 \end{bmatrix} \times (0.1\sin(k))$.

The reference model is given by

$$\begin{cases} x_m(k+1) = \begin{bmatrix} 0.9998 & 0.02 \\ -0.02 & 0.9958 \end{bmatrix} x_m(k) \\ y_m(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_m(k) \end{cases} \quad (48)$$

The initial conditions for (47) and (48) are chosen as follows

$$x(0) = \begin{bmatrix} -1 & 0 \end{bmatrix}^T, x_m(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \quad (49)$$

The obtained matrices G and H, solutions of (3), are

$$G = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5011 \end{bmatrix}, H = \begin{bmatrix} -1.0011 & 0.1102 \end{bmatrix}. \quad (50)$$

For all controllers, the sliding matrix is selected as follows

$$S = \begin{bmatrix} 3 & 1 \end{bmatrix} \quad (51)$$

The reaching law parameters in (9) are given by

$$\phi = 0.9, \eta = 0.5, \varphi = 0.5 \quad (52)$$

The design parameters for the sliding functions (23) and (36) are

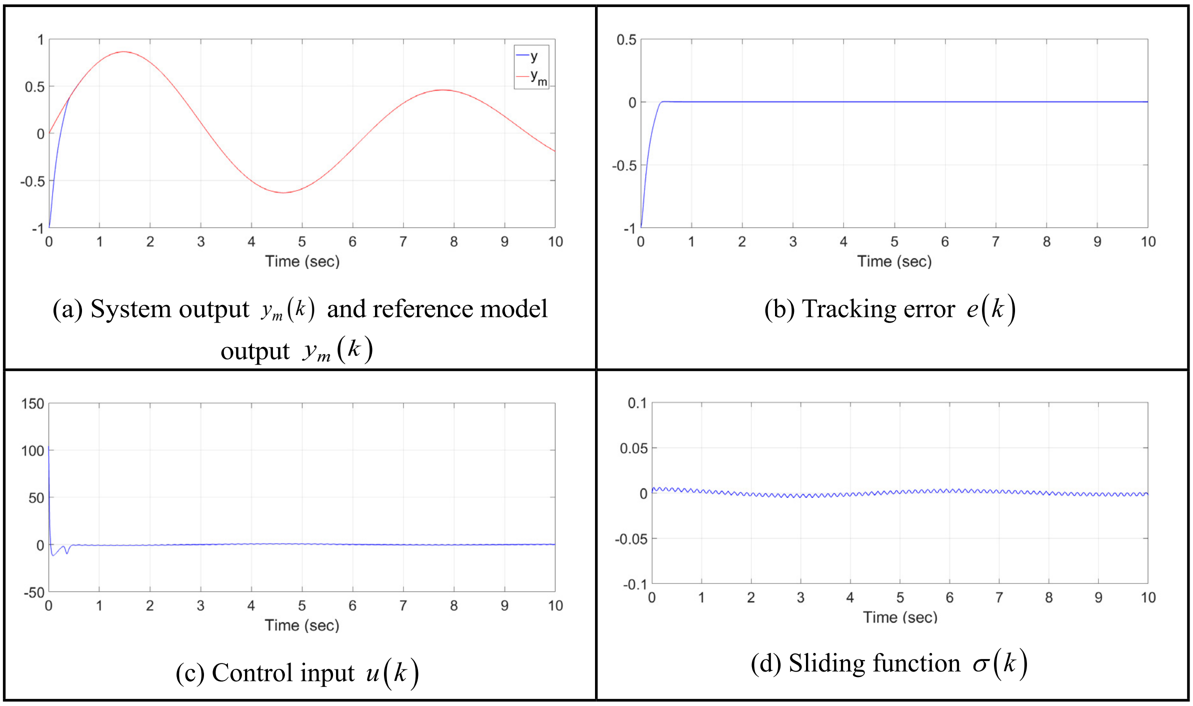
$$\beta = 0.9, \mu = 1.13 \quad (53)$$

Solving the LMI (28), the obtained solutions are

$$\begin{aligned} X = \begin{bmatrix} 0.9271 & -0.2371 \\ -0.2371 & 1.0729 \end{bmatrix}, \\ W = \begin{bmatrix} -119.3733 & -67.8097 \end{bmatrix}. \end{aligned} \quad (54)$$

Hence, the feedback matrix K is given by

$$K = \begin{bmatrix} -153.6018 & -97.1424 \end{bmatrix} \quad (55)$$



(a) System output $y_m(k)$ and reference model output $y_m(k)$

(b) Tracking error $e(k)$

(c) Control input $u(k)$

(d) Sliding function $\sigma(k)$

Figure 2. Simulation results for DISMC-AW using the LQR method

Concerning the LQR approach, the weighting matrices R and Q are selected as follows

$$R = 1, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{56}$$

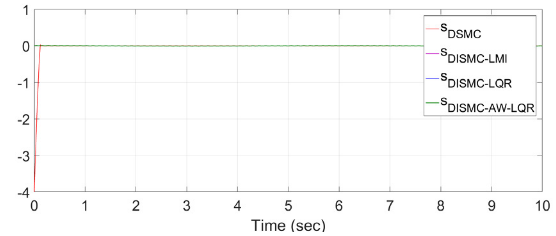
Then, the obtained feedback matrix K is given by

$$K = \begin{bmatrix} -202.7857 & -77.4525 \end{bmatrix} \tag{57}$$

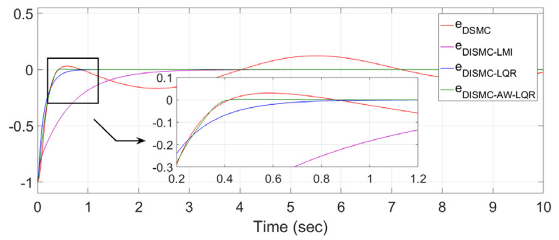
The implementation of all algorithms was done with Matlab 2016a on Windows 7 using an Intel® Core™ i5-2430M CPU running at 2.4GHz with 6GB in RAM.

Figure 2 shows the numerical simulation results for the discrete-time integral sliding mode controller with anti-windup using the LQR approach. Figure 2(a) illustrates the system output $y(k)$ and the reference model output $y_m(k)$. It shows that the proposed algorithm ensures the stability of the closed-loop system in the quasi-sliding mode as well as the robust tracking and the follow-up of the reference model. Figure 2(b) depicts the tracking error $e(k)$. It shows that the developed controller guarantees the zero-tracking error in the presence of external disturbance. Figure 2(c) presents the control input $u(k)$. It shows that the chattering phenomenon is avoided using the saturation function as a smooth switching function in the control law design. Figure 2(d) depicts the sliding function $\sigma(k)$. It shows that the reaching phase is eliminated due to the exponential term in the sliding function, and that the state trajectory converges to the sliding surface and therefore the existence of the quasi-sliding mode.

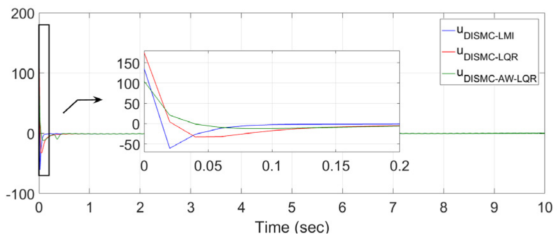
Figure 3 provides a comparison between four algorithms: conventional DSMC, DISMC using the LMI and LQR approaches and the developed DISMC-AW using the LQR approach. Figure 3(a) depicts the time variation of the sliding functions of all controllers. It shows that all state trajectories converge to the sliding surface and that the reaching phase is eliminated for DISMC and DISMC-AW controllers due to the exponential term in the sliding functions. Figure 3(b) illustrates a comparison of tracking errors. It shows that, contrary to the conventional DSMC, DISMC and DISMC-AW controllers guarantee the zero-tracking error in the quasi-sliding mode, which is due to the integral term $\varepsilon(k)$ of the sliding surface. Moreover, it shows that the proposed DISMC-AW controller has the fastest response compared to DISMC controller. Figure 3(c) presents a comparison of control inputs. It shows that the chattering phenomenon is eliminated for all controllers using the saturation function in their designs. Furthermore, the control input corresponding to DISMC-AW has lower values than DISMC in the initial phase. These improvements can be attributed to the gain $\alpha(k)$, assigned to the integrative portion $\varepsilon(k)$, which initially tends towards zero because of the significant error accumulation and therefore the impact of the latter on system performance and on control input is restricted.



(a) Comparison of the variation of sliding functions



(b) Comparison of tracking errors



(c) Comparison of control inputs

Figure 3. Comparison between conventional DSMC, DISMC using the LMI and LQR approaches and DISMC-AW using the LQR method

Table 1. Summary table of numerical resutls

	Rise time (s)	**Settling time (s)**	**u_{min}**	**u_{max}**
DISMC-LMI	1.82	2.88	-60.6	135.8
DISMC-LQR	0.44	0.72	-32.7	175.1
DISMC-AW-LQR	0.32	0.38	-11.1	104.5

For summarizing the results, Table 1 allows the comparison of the proposed scheme with the existing ones. It shows that best results are obtained for DISMC-AW controller with the least rise and settling times of $0.32s$ and $0.38s$ respectively, and with the lowest minimum and maximum values of control input of -11.1 and 104.5 respectively. Figure 3 and Table 1 prove that DISMC-AW controller outperforms DSMC and DISMC controllers.

5. Conclusion

This work presents a discrete-time integral sliding mode control with anti-windup for SISO linear systems with matched disturbance. The proposed algorithm ensures the stability of the closed-loop system in the quasi-sliding mode and the robustness in the presence of external disturbance. Moreover, it guarantees the elimination of the reaching phase as well as the robust tracking and the follow-up of the reference model. The chattering phenomenon is avoided using the saturation function in the controller design. According to numerical simulation results, the developed algorithm offers not only a faster response than the discrete-time integral sliding mode controller, but also lower values of the control input in the initial phase. This is due to the anti-windup method which is used to limit the effect of error accumulation on system performance and control input. Future work will focus on developing the DISMC-AW controller for uncertain, time-delayed and multi-input multi-output linear systems. Furthermore, the effectiveness of the proposed algorithm will be verified by experiments.

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