Observers Design for Discrete-Event Systems Modelled by S-Nets

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Abstract: This paper addresses the design of observers for Discrete-Event Systems modelled by Output Petri nets. The observer is conceived as a copy of the system and a corrective term based on the execution trajectories. The observer performs a tracking of the transition sequence executed by the net. Based on this information, the observer is able to produce approximations of the initial and current state of the system. The focus is a subclass of Petri nets called S-Nets. A Lyapunov criterion is used for testing the stability of the herein proposed scheme. This criterion allows for proving that the observers are asymptotically stable and it supports characterizing the region of stability of the System/Observer pair, as well. An application example is developed through the paper to illustrate the results. Some graphs are provided to show the approximation error of the observer under different initial conditions.

Keywords: Observer Design, Petri Nets, S-Nets, Discrete-Event Systems, Sequence Observer, Lyapunov Stability.

1. Introduction

The observability is an important property of a Discrete-Event System (DES). Different frameworks have been proposed to face the problems related to the study of this property. In the finite automata (FA) framework, automatons are used for modelling a DES, and the analysis of properties is typically done by linguistic approaches. Mainly, the observability notions in the FA framework consist on dividing the automata language into equivalence classes. The control techniques consider that a specification is feasible to be implemented if the classes induced by the observability are finer than those induced by the controllability. This is a class of “static observability” that does not consider the concept of an observer for progressively discover the system’s state, as part of the control scheme [6]-[12].

In Vector Additive Systems (VAS) framework, similar approaches, as those used in FA, are applied. The plant is modelled as a system of vectors, while a set of linear inequalities are the specifications. In a similar way such as in FA, the observability notions require that if two different states of a VAS produce the same output signal, then they must require the same control action. Accordingly, the observability notion in a VAS produces results that are consistent to those of FA. Consequently, a VAS does not consider the notion of an observer for reconstructing the system state [13], [14].

The design of observers in Petri Nets (PN) framework has been addressed in a fewer number of works than those presenting designs of controllers based on this modeling tool. In [16], the problem of discovering the marking of the net is addressed. The proposed scheme produces marking estimations, which are always a lower bound of the current marking of the net. In [17] and [18], the problem of discovering the marking of an Interpreted PN is considered. The sequence invariance and a geometrical approach are used for the analysis and observer design.

Some related works of the authors are reported in the literature. In [1] and [4], the authors state results about the observability analysis focused on the subclass of PN’s known as Free-choice nets. In [5], the authors show that a combination of a PN observer and a supervisor allows for addressing problems that has no solution with the solely use of a supervisor, as in [14] and [15]. In [3] and [2], a framework of practical interest based on Matlab/Simulink for the study of DES, including controllers and observers, is reported.

This work addresses the design of observers for a class of PN known as S-Nets. A Lyapunov criterion is proposed for the stability analysis. The major contributions of this paper are: a) polynomial algorithms for the observer construction; b) the use of a Lyapunov stability criterion for the observer characterization, and c) analysis of the stability region of the pair, system and observer.

The rest of this paper is organized as follows. Section 2 provides some background notions on PN and on the sequence detection in S-Nets. The major contribution of this work is in sections 3 and 4. Section 3 details the proposed observer scheme and the metric space for the measurement of its error. Section 4 proposes a functional term based on the number of sequences that the observer is tracking at every step. It also shows that the observer satisfies the Lyapunov stability criteria. An example at the end of this section illustrates the developed technique. Section 5 provides the conclusions and finally, are the bibliographical references.

2. Background

This section shows basic PN notions, and briefly reviews the main results on sequence-detectability that are relevant for this work.

Output Petri Nets

An OPN is a tuple (B, M_0, φ) where B is a PN structure (P, T, I, O), M_0 is the initial marking and φ is the output function. For simplicity, B also stands for the *incidence matrix* of the PN structure. The *pre-set* $\diamond\, t_j$ and *post-set* $t_j \diamond$ of a transition $t_j \in T$ are as usual. Likewise, are the pre-set $\diamond\, p_i$ and post-set $p_i \diamond$ of a place $p_i \in P$. The operator $\diamond$ is extended in a natural way for sets. This work deals with *well-formed* nets, which in summary are strongly connected, conservative and repetitive nets [19]. The compact representation of a marking $M_k\{xp_i\}$ for $M_k(p_i) = x$, with $p_i \in P$ and $x \in \mathbb{N}^+$, is as in [1]. The state equation of an OPN is:

$$M_{k+1} = M_k + B\vec{u}_k \quad , \quad y_k = \varphi(M_k) \qquad (1)$$

The notation $M_k \overset{t_j}{\rightarrow} M_{k+1}$ means that from M_k the transition t_j is fired reaching M_{k+1}. A net is *safe* if $\forall M_k \in R(B, M_0)$, it holds that $M_k(p_i) \leq 1, \forall p_i \in P$, and *non-safe* otherwise. The length of a sequence of transitions σ is denoted by $|\sigma|$. This sequence or trajectory is denoted by $M_0 \overset{\sigma}{\rightarrow} M_s$.

As in the automata theory, σ^* denotes the Kleen closure of a sequence σ, which extends in a natural way to sets [20]. The *output word* associated to a sequence σ, defined as $\varphi(\sigma) = \varphi(M_k)\varphi(M_{k+1}) \ldots \varphi(M_{k+r})\varphi(M_{k+s})$, is the information that an external observer is able to detect from an OPN.

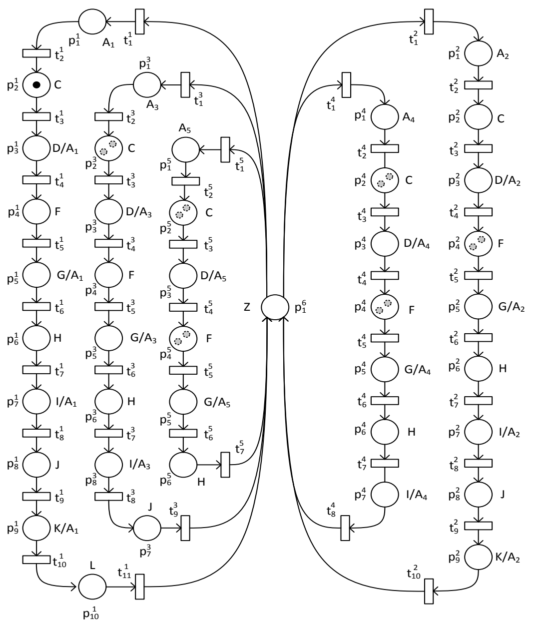

Figure 1. A well-formed OSS.

Thus, by (1) $\varphi M_{k+1} = \varphi M_k + \varphi B\vec{u}_k$. The vector $\varphi B\vec{u}_k$, denoted by $\varphi_B(\vec{u}_k)$, is the change, or increment, in the system sensors due to the firing of u_k, which the observer tries approximate. Notice that it is possible $\varphi_B(\vec{t}_i) = 0$, while $B\vec{t}_i \neq 0$. The transition t_i is known as *silent* [11]. This class of transitions are out of the scope of this work. For additional notions about PN see [19].

Sequence Detectability in S-Nets

This work is devoted to the design of an observer for tracking the transition sequence executed by an OPN. The *Sequence-Detectability (SD)* is a useful concept. An efficient solution for the SD is derived in [1] on *Output S-System* (OSS), as the one shown in Figure 1. A first step in the testing of the SD is the construction of the *Event-Detectability (ED)* table E_B. Given a safe OSS (B, M_0, φ), with $T = \{t_1, \ldots, t_n\}$, E_B is a square arrangement $[n-1 \times n-1]$, where columns represent transitions from t_1 to t_{n-1} and rows represent transitions from t_2 to t_n. The entries of E_B, for $2 \leq i \leq n; 1 \leq j \leq n-1; j < i$, are defined as follows:

$$\begin{gathered} if\ \varphi_B(\vec{t}_i) \neq \varphi_B(\vec{t}_j), then\ E_B(t_i, t_j) = \emptyset \\ otherwise, E_B(t_i, t_j) = \cup\{t_u, t_v\} \qquad (2) \\ \forall t_u \in (t_i \diamond)\diamond, \forall t_v \in (t_j \diamond)\diamond \end{gathered}$$

The Table 1 shows E_B for the net in Figure 1. A further refinement of E_B leads to the *Sequence-Detectability (SD)* table E_B^s. The entries of E_B^s

Table 1. Event-Detectability table of a well-formed OSS.

t_2^1	{}
t_3^1	{}	{}
t_4^1	{}	{}	{}
t_5^1	{}	{}	{}	{}
t_6^1	{}	{}	{}	{}	{}
t_7^1	{}	{}	{}	{}	{}	{}
t_8^1	{}	{}	{}	{}	{}	{}	{}
t_9^1	{}	{}	{}	{}	{}	{}	{}	{}
t_{10}^1	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_{11}^1	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_{12}^1	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_1^2	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_2^2	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_3^2	{}	{}	$\{t_4^2,t_4^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_4^2	{}	{}	{}	$\{t_5^2,t_5^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_5^2	{}	{}	{}	{}	$\{t_6^2,t_6^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_6^2	{}	{}	{}	{}	{}	$\{t_7^2,t_7^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_7^2	{}	{}	{}	{}	{}	{}	$\{t_8^2,t_8^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_8^2	{}	{}	{}	{}	{}	{}	{}	$\{t_9^2,t_9^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_9^2	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_{10}^2,t_{10}^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_{10}^2	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_{11}^2,t_{11}^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_{11}^2	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_1^3	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_2^3	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_3^3	{}	{}	$\{t_4^3,t_4^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_4^3,t_4^2\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_4^3	{}	{}	{}	$\{t_5^3,t_5^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_5^3,t_5^2\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_5^3	{}	{}	{}	{}	$\{t_6^3,t_6^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_6^3,t_6^2\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_6^3	{}	{}	{}	{}	{}	$\{t_7^3,t_7^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_7^3,t_7^2\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_7^3	{}	{}	{}	{}	{}	{}	$\{t_8^3,t_8^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_8^3,t_8^2\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_8^3	{}	{}	{}	{}	{}	{}	{}	$\{t_9^3,t_9^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_9^3,t_9^2\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_9^3	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_{10}^3,t_{10}^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_{10}^3,t_{10}^2\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_{10}^3	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_1^4	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_2^4	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_3^4	{}	{}	$\{t_4^4,t_4^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_4^4,t_4^2\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_4^4,t_4^3\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_4^4	{}	{}	{}	$\{t_5^4,t_5^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_5^4,t_5^2\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_5^4,t_5^3\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_5^4	{}	{}	{}	{}	$\{t_6^4,t_6^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_6^4,t_6^2\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_6^4,t_6^3\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_6^4	{}	{}	{}	{}	{}	$\{t_7^4,t_7^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_7^4,t_7^2\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_7^4,t_7^3\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_7^4	{}	{}	{}	{}	{}	{}	$\{t_8^4,t_8^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_8^4,t_8^2\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_8^4,t_8^3\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}
t_8^4	{}	{}	{}	{}	{}	{}	{}	$\{t_9^4,t_9^1\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_9^4,t_9^2\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}	{}	$\{t_9^4,t_9^3\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}
	t_1^1	t_2^1	t_3^1	t_4^1	t_5^1	t_6^1	t_7^1	t_8^1	t_9^1	t_{10}^1	t_{11}^1	t_{12}^1	t_1^2	t_2^2	t_3^2	t_4^2	t_5^2	t_6^2	t_7^2	t_8^2	t_9^2	t_{10}^2	t_{11}^2	t_1^3	t_2^3	t_3^3	t_4^3	t_5^3	t_6^3	t_7^3	t_8^3	t_9^3	t_{10}^3	t_1^4	t_2^4	t_3^4	t_4^4	t_5^4	t_6^4	t_7^4

are obtained from the entries in E_B by a repetitive procedure as follows:

$$E_B^s(t_i,t_j) = E_B(t_i,t_j) \setminus (t_u,t_v) \quad iff\ E_B(t_u,t_v) = \emptyset \text{ and both, } \diamond t_i \cap \diamond t_j = \emptyset \text{ and } t_i \diamond \cap t_j \diamond = \emptyset \tag{3}$$

The definition (3) is recursively applied until no new empty entries appear. In [1], the authors shown that the non-empty entries in E_B^s lead to circuits of transitions, denoted by ΔE_B^s, that are closely related to the SD of the net.

Theorem 1. *A safe OSS (B, M_0, φ) is SD if and only if $\Delta E_B^s = \emptyset$.* ■

Proposition 1. *Let E_B^s be the SD table of the OSS (B, M_0, φ). Then, for any nonempty entry $E_B^s(t_g, t_h)$ there exist a pair of circuits, say σ_g and σ_h, such that $\varphi(\sigma_g) = \varphi(\sigma_h)$.* ■

When an OSS is non-safe, the results of the safe nets have to be extended.

Theorem 2. *Let (B, M_0, φ) be a non-safe OSS. Then, the net is SD if and only if E_B is empty.* ■

Observability in S-Nets

Besides the tracking of the transition sequence of an OSS, in some cases, the observer is able to determine the marking of the net. Informally, an OPN (B, M_0, φ) is *observable* if its initial and current markings, M_0 and M_s respectively, could be determined in finite time, by using the information of the output word $\varphi(\sigma)$ and the structure of (B, M_0, φ).

The *Firing-Vector-Detectability (FVD)* and the *Marking-Detectability (MD)* are sufficient conditions for the Observability in a PN.

Proposition 2*. An OPN (B, M_0, φ), which is FVD and MD, is Observable.* ■

For the case of OSS nets, the ED and the SD lead to efficient solutions of the FVD and the MD.

Proposition 3. *If a well-formed and safe OSS (B, M_0, φ) is SD then, it is also MD.* ■

Thus, the next holds for the observability.

Corollary 1. *If a well-formed and safe OSS is SD, then it is observable.*

When the OSS is non-safe, the next theorem shows how E_B allows for testing the SD.

Theorem 3*. Let (B, M_0, φ) be a non-safe OSS. Then, the net is SD if and only if E_B is empty.* ■

The SD, as in the case of safe nets, does not directly imply the MD in a non-safe OSS.

Theorem 4. *A non-safe OSS (B, M_0, φ) is MD if and only if $\ker \varphi = \emptyset$.* ■

Corollary 2. *Let (B, M_0, φ) be a non-safe OSS. Then (B, M_0, φ) is observable if and only if* $\ker \varphi = \emptyset$. ■

3. Observer Design

Consider Figure 2, which depicts the block diagram of the proposed observer scheme. The observer is designed as a copy of the system plus a corrective term, as shown.

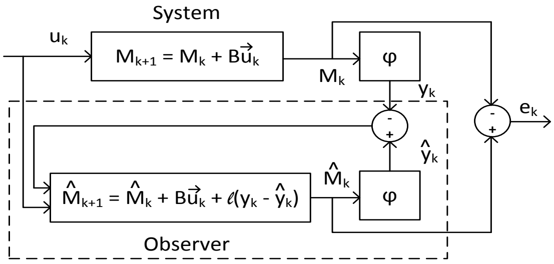

Figure 2. Observer Scheme

Definition 1. *Let (B, M_0, φ) be the OPN as in (1). The equation of the Observer $(B, \widehat{M}_0, \varphi)$ is:*

$$\begin{gathered}\widehat{M}_{k+1} = \widehat{M}_k + B\vec{u}_k + \ell(\hat{y}_k - y_k) \\ \hat{y}_k = \varphi\widehat{M}_k\end{gathered} \tag{4}$$

The term $\ell(\hat{y}_k - y_k)$ is denoted as ℓ_k where no confusion arises. The error e_k between the system and the observer is $e_k = \widehat{M}_k - M_k$. Hence, $e_{k+1} = \widehat{M}_{k+1} - M_{k+1}$. From (1) and (4), $e_{k+1} = \widehat{M}_k + B\vec{u}_k + \ell_k(\varphi\widehat{M}_k - \varphi M_k) - (M_k + B\vec{u}_k)$. By expanding and rearranging the terms, it leads to $e_{k+1} = (\widehat{M}_k - M_k) - \ell_k\varphi(\widehat{M}_k - M_k) = e_k - \ell_k\varphi e_k$. Thus, the equation defining the dynamics of the error system for Figure 2 is as follows:

$$e_{k+1} = (1 - \ell_k\varphi)e_k \tag{5}$$

The initial error in (5) is $e_0 = \widehat{M}_0 - M_0$, where $\widehat{M}_0$ is an initial estimation, or "guess", of the observer. In order to emphasize the initial error e_0, let (E, e_0) denotes the system (5). Notice that a "trajectory", say $e_k \overset{\rho_k}{\to} e_{k+1}$, occurs in (5) if and only if $e_k = \widehat{M}_k - M_k$ and $e_{k+1} = \widehat{M}_{k+1} - M_{k+1}$, such that $\hat{\sigma}_k \in \overline{\mathcal{L}(B, \widehat{M}_0)}: \widehat{M}_k \overset{\hat{\sigma}_k}{\to} \widehat{M}_{k+1}$ and $\sigma_k \in \overline{\mathcal{L}(B, M_0)}: M_k \overset{\sigma_k}{\to} M_{k+1}$. Let $\mathcal{L}(E, e_0)$ be the set of all the trajectories of (5). In this context, we say that $\rho_k \in \mathcal{L}(E, e_0)$ corresponds to σ_k and $\hat{\sigma}_k$ in the system and observer, respectively. Thus, a sequence in (5) may be empty, single or infinite, where $\overline{\mathcal{L}(E, e_0)}$ is the middle set of $\mathcal{L}(E, e_0)$, as with an OPN.

Observer Metrics

In order to measure the error of the system (5), the following notions are considered. Let m be the number of places of the OPN (B, M_0, φ). The *Manhattan distance* ρ of $u, v \in \mathbb{N}^m$, here denoted by $\|u - v\|$, is defined as usual:

$$\rho(u, v) = \sum_{i=1}^{m} |u(i) - v(i)| \tag{6}$$

The distance from an element $u \in \mathbb{N}^m$ to a set $V \subseteq \mathbb{N}^m$ is $\rho(u, V) = \min\{\|u - v\| : v \in V\}$.

Definition 2. *The r-neighbourhood of a subset $\mathcal{M} \subset \mathbb{N}^m$ is $S(\mathcal{M}, r) := \{e \in \mathbb{N}^m : 0 < \|u - v\| < r\}$ for some $r > 0$.*

Since ρ is defined in $\mathbb{N}^m$, then it is supposed that $r \in \mathbb{Z}^+$. The subset $\mathcal{M} \subset \mathbb{N}^m$ is said to be *invariant* w.r.t. (E, e_0), if firstly, $e_0 \in \mathcal{M}$, where $e_0 = (\widehat{M}_0 - M_0)$. Secondly, $\forall e_k \in \mathbb{N}^m$ such that $e_0 \overset{\tau_k}{\to} e_k$. Then, necessarily $e_k \in \mathcal{M}$, for every $k > 0$, whenever exists an infinite sequence γ in (5), that follows from τ_k, i.e., $e_0 \overset{\tau_k}{\to} e_k \overset{\gamma}{\to} \cdots$. Notice that γ implies the existence of infinite sequences, say α and β, such that $\sigma_k \alpha \in \mathcal{L}(B, M_0)$ and $\hat{\sigma}_k \beta \in \mathcal{L}(B, \widehat{M}_0)$, respectively, where σ_k and $\hat{\sigma}_k$ are related to τ_k. In other words, a subset $\mathcal{M} \subset \mathbb{N}^m$ is invariant w.r.t. (E, e_0) if a) e_0 belongs to $\mathcal{M}$ and, b) for any e_k *"reachable"* from e_0 by a finite trajectory $\tau_k, k \geq 0$, then e_k also belongs to $\mathcal{M}$, whenever exist an infinite trajectory γ *"following to"* τ_k in (E, e_0). Notice that if an OSS (B, M_0, φ) is well-formed, then for every $\sigma \in \mathcal{L}(B, M_0): |\sigma| < \infty$, it always exists another infinite sequence, say $\alpha \in \overline{\mathcal{L}(B, M_0)}$, such that $\sigma\alpha \in \mathcal{L}(B, M_0)$ (by Prop. 8.2 Home markings of live S-Nets [19]). Thus, the later requirement is trivially satisfied.

Suppose that the structure (B, M_0, φ) of the OSS is known and the net is SD. Thus, if $e_0 = 0$, i.e. $M_0 = \widehat{M}_0$, then $e_{k+1} = 0$ for any $k \geq 0$. On the contrary, suppose that for some $k > 0$, where $e_k \overset{\tau_k}{\to} e_{k+1}$ for $\tau_k \in \overline{\mathcal{L}(E, e_0)}$, with $e_k = 0$ and $|\tau_k| = 1$, it holds that $e_{k+1} > 0$. Then, there must exist $\hat{\sigma}_k \in \overline{\mathcal{L}(B, \widehat{M}_0)}: \widehat{M}_k \overset{\hat{\sigma}_k}{\to} \widehat{M}_{k+1}$ and

$\sigma_k \in \overline{\mathcal{L}(B, M_0)}: M_k \overset{\sigma_k}{\to} M_{k+1}$, where $|\hat{\sigma}_k| = 1$ and $|\sigma_k| = 1$, with $e_k = M_k - \widehat{M}_k$ and $e_{k+1} = \widehat{M}_{k+1} - M_{k+1}$, corresponding to τ_k. Let say, without loss of generality, that $\sigma_k = t_k$ and $\hat{\sigma}_k = \hat{t}_k$. Thus, since $e_{k+1} > 0$, it implies that $\widehat{M}_{k+1} \neq M_{k+1}$. But, the structure of the OSS (B, M_0, φ) is known, and since the observer has wrongly computed $\widehat{M}_{k+1}$ when $e_k = 0$, this directly implies that there must exists in the net at least one transition besides of t_k, say $t'_k \in T$, such that $\varphi_B(t'_k) = \varphi_B(t_k)$ where $[M_k\rangle t'_k$, and $[M_k\rangle t_k$. However, if the net is safe, then $\diamond\, t'_k = \diamond\, t_k$, which directly contradicts the SD of the net. On the other hand, if the OSS is non-safe, then $\varphi_B(t'_k) = \varphi_B(t_k)$ directly contradicts the ED of the net. Thus, $\{\vec{0}\}$ is invariant w.r.t. (E, e_0), as stated.

The Lyapunov stability criterion for (E, e_0), valid in the metric space $(\mathbb{N}^m, \rho)$, is as follows.

Definition 3. *A closed invariant set $\mathcal{M} \subset \mathbb{N}^m$ of the error system (5) is stable in the sense of Lyapunov w.r.t. $\mathcal{L}(E, e_0)$, if for every $\varepsilon > 0$, it is possible to find $\delta > 0$, such that whenever $\rho(e_0, \mathcal{M}) < \delta$, it holds that $\rho(e_k, \mathcal{M}) < \varepsilon$, where $e_0 \overset{\rho_k}{\rightarrow} e_k$, for all $k > 0$. If furthermore, $\rho(e_k, \mathcal{M}) \to 0$ as $k \to \infty$, then $\mathcal{M}$ is asymptotically stable. The region of asymptotic stability of $\mathcal{M}$ is the subset $\mathcal{M}_a \subset \mathbb{N}^m$ of all $\bar{e}_0 \in \mathbb{N}^m$ for which $\mathcal{M}$ is asymptotically stable.*

The next theorem provides a necessary and sufficient condition to verify asymptotic stability in a neighbourhood of (E, e_0). See [21] for the formal proof.

Theorem 5. *Let (E, e_0) be the error system in (5) and let $\mathcal{M} = \{\vec{0}\}$. Then, $\mathcal{M}$ is asymptotically stable in the sense of Lyapunov w.r.t. $\mathcal{L}(E, e_0)$ if and only if in a sufficiently small neighbourhood $S(\mathcal{M}, r)$, there exist a functional V_k fulfilling:*

- *i. For all sufficiently small $c_1 > 0$ it is possible to find a $c_2 > 0$, such that when $V_k > c_2$ it holds that $\rho(e_k, \mathcal{M}) > c_1$ for $e_k \in S(\mathcal{M}, r)$;*
- *ii. For any $c_4 > 0$ as small as required, it is possible to find a $c_3 > 0$ so small such that when $\rho(e_k, \mathcal{M}) < c_3$ it holds that $V_k \le c_4$ for $e_k \in S(\mathcal{M}, r)$;*
- *iii. V_k is a non-increasing function for all $k > 0$, as long as $e_0 \overset{\tau_k}{\rightarrow} e_k$ with $e_k \in S(\mathcal{M}, r)$ for which exist an infinite trajectory γ such that $e_0 \overset{\tau_k}{\rightarrow} e_k \overset{\gamma}{\rightarrow} \cdots$;*
- *iv. Moreover, $V_k \to 0$ as $k \to \infty$.*

The election of the functional V_k is the key element in the observer design for the stability testing.

This paper proposes a functional that is based in the number of transitions sequences that the term ℓ_k, shown in Figure 2 as part of the observer, is tracking at every k. Before the formal analysis, some intuitive ideas are given in the following example.

Example 1. *Consider that the initial state of the OSS in Figure 1 is $M_0\{p_2^1\}$ and that its respective output is $y_0\{C\}$. The net includes dashed tokens and output signals after a slash that will be used in a later example. With a simple examination, it is easy to see that the markings $M_0\{p_2^1\}$, $M_0\{p_2^2\}$, $M_0\{p_2^3\}$, $M_0\{p_2^4\}$ and $M_0\{p_2^5\}$ all produce the same output $y_0\{C\}$. Let $[\hat{M}]_0 = M_0\{p_2^1\} + M_0\{p_2^2\} + M_0\{p_2^3\} + M_0\{p_2^4\} + M_0\{p_2^5\}$ be the observer estimation. Thus, at $k = 0$, the distance of the observer estimation and the system state is four, i.e. $e_0 = 4$. Now, suppose that t_3^1 is fired at $k = 1$. The output is now $y_1\{D\}$. Notice that $\varphi_B(t_3^1) = \varphi_B(t_3^2) = \varphi_B(t_3^3) = \varphi_B(t_3^4) = \varphi_B(t_3^5)$. Indeed, all of these transitions are enabled at $[\hat{M}]_0$. Therefore, at $k = 1$, the observer computes the possible executed sequences as $\sigma_1 = t_3^1$, $\sigma_2 = t_3^2$, $\sigma_3 = t_3^3$, $\sigma_4 = t_3^4$ and $\sigma_5 = t_3^5$. Let's denote the set of these sequences as $\{\hat{\sigma}\}_1$. Corresponding to $\{\hat{\sigma}\}_1$, the observer computes the possible reached markings as those with one token in the places with subscript three, i.e. from $M_1\{p_3^1\}$ to $M_1\{p_3^5\}$. Let $[\hat{M}]_1 = M_1\{p_3^1\} + \cdots + M_1\{p_3^5\}$ be the summation of these markings. At this point, the observer does not improve the state estimations, i.e. the error remains the same. This is an important characteristic of the SD of an OSS, that is, it is possible that the estimations do not improve at every event execution. In this case, $e_1 = 4 = e_0$. However, the estimations must improve in a finite number of events until they match to the system. Indeed, suppose that t_4^1 is fired, at $k = 2$ and the output changes to $y_2 = \{F\}$. Notice that $\varphi_B(t_4^1) = \varphi_B(t_4^2) = \varphi_B(t_4^3) = \varphi_B(t_4^4) = \varphi_B(t_4^5)$. By using this new information, the observer is able to update the set of possible executed sequences to*

$\{\hat{\sigma}\}_2 = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\}$, where $\sigma_1 = t_3^1 t_4^1$, $\sigma_2 = t_3^2 t_4^2$, $\sigma_3 = t_3^3 t_4^3$, $\sigma_4 = t_3^4 t_4^4$, $\sigma_5 = t_3^5 t_4^5$, $\sigma_6 = t_3^6 t_4^6$, $\sigma_7 = t_3^7 t_4^7$, and $\sigma_8 = t_3^8 t_4^8$. At this point, the observer is able to discard σ_9, since $\varphi_B(t_4^9) \neq \varphi_B(t_4^1)$.

Hence, the possible reached marking $[\hat{M}]_2$, is that with a token on places with subscript four, from $M_2\{p_4^1\}$ to $M_2\{p_3^8\}$. Notice that now, $|\{\hat{\sigma}\}_2| < |\{\hat{\sigma}\}_1|$, and accordingly, $e_2 \le e_1$.

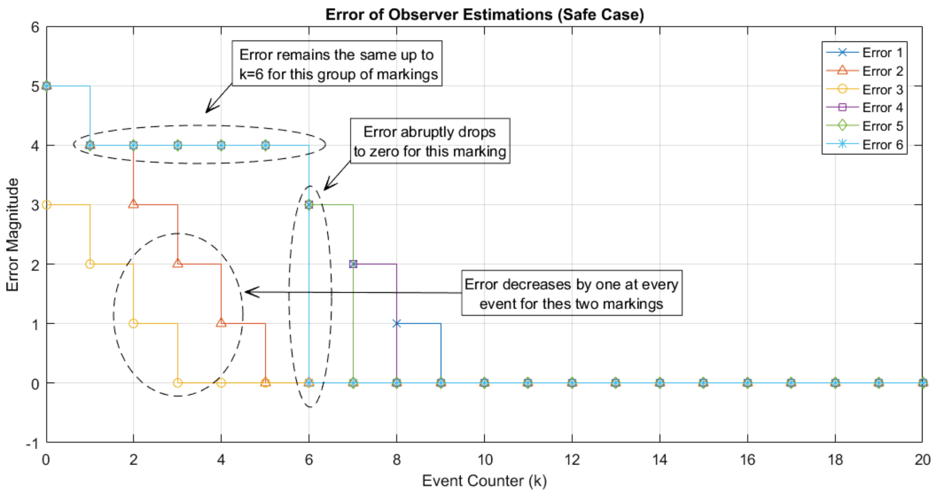

Figure 3. Error in the Observer estimations in a safe well-formed OSS.

The next section proposes a functional term based on the number of transition sequences tracked by the observer, which satisfies the Lyapunov stability criterion.

4. Observer Stability Analysis

Generalizing the idea in the example of the previous section, let y_k be the $k-th$ system output and $\{\hat{\sigma}\}_k$ be the possible sequences executed by the system. Notice that the number of possible transition sequences that the observer may track does not necessarily decrease at every transition firing, as stated. However, this paper shows that for any evolution of the system of length k, with a corresponding set of possible fired transition sequences $\{\hat{\sigma}\}_k$, there exist a finite integer d such that $|\{\hat{\sigma}\}_{k+d}| < |\{\hat{\sigma}\}_k|$. To this aim, consider the following definition for the corrective term of the scheme in Figure 2.

Definition 4. *Let (B, M_0, φ) be a SD and well-formed OSS, where M_0 is probably unknown. Let y_0 be the initial output of the system. Let $\hat{M}_0 = \varphi' y_0$ be the initial estimation for the observer by (5). Let $[\hat{M}]_0 = \hat{M}_0 = [\hat{M}]_k$ be the estimations of the initial and current markings. Define ℓ_k as follows:*

(first step) For $k = 1$:

i. Let y_1 be the first output of the system and let $\Delta y_1 = y_1 - y_0$. Let $\{\hat{T}\}_1 = \{t_j \in T: \varphi_B(t_j) = \Delta y_1\}$ be the first set of single transitions that have been probably fired;

ii. Let $\{\hat{\sigma}\}_1 = \cup \sigma_j$ where $\sigma_j = t_j$ for $t_j \in \{\hat{T}\}_1$, such that $\left[[\hat{M}]_0\right\rangle \sigma_j$;

iii. Let $\ell_1 = -(\varphi' y_0) + \sum B^+ \sigma_j : \sigma_j \in \{\hat{T}\}_1$;

iv. Optionally, update the internal variables $[\hat{M}]_0 = [\hat{M}]_0 - \varphi' y_0 + \sum B^- \sigma_j : \sigma_j \in \{\hat{T}\}_1$ and $[\hat{M}]_k = [\hat{M}]_k - \varphi' y_0 + \sum B^+ \sigma_j : \sigma_j \in \{\hat{T}\}_1$;

(iteration) For $k > 1$:

i. Let y_k be the $k-th$ system output and let $\Delta y_k = y_k - y_{k-1}$. Let $\{\hat{T}\}_k = \{t_j \in T: \varphi_B(t_j) = \Delta y_k\}$;

ii. If $\left|\{\hat{T}\}_k\right| = 1$ then $\{\hat{\sigma}\}_k = \tau t_j$, for $\{\tau\} = \{\hat{\sigma}\}_{k-1}$ and $t_j \in \{\hat{T}\}_k$;

iii. Otherwise, if $\left|\{\hat{T}\}_k\right| > 1$, let $\{\hat{\sigma}\}_k = \cup \tau t_j$ for $\tau = \sigma t_i \in \{\hat{\sigma}\}_{k-1}$ and $t_j \in \{\hat{T}\}_k$, such that $\left[[\hat{M}]_{k-1}\right\rangle t_j$;

iv. Let $\overline{\{\hat{\sigma}\}}_k = \{\hat{\sigma}\}_{k-1} \setminus \{\hat{\sigma}\}_k$, i.e., the set of those elements in $\{\hat{\sigma}\}_{k-1}$ not used in $\{\hat{\sigma}\}_k$;

v. Let $\ell_k^+ = \sum B^+ \vec{t}_i$ and $\ell_k^- = \sum B^- \vec{t}_i$ for $\sigma t_i \in \{\hat{\sigma}\}_k$;

vi. Let $\overline{\ell_k^-} = \sum B^- \vec{t}_j$ for $t_j \sigma \in \overline{\{\hat{\sigma}\}}_k$;

vii. Let $\ell_k = \ell_k^+ - (\ell_k^- + \overline{\ell_k^-})$ be the $k-th$ correction element;

viii. Optionally, update $[\hat{M}]_0 = [\hat{M}]_0 - \overline{\ell(y_k)^-}$ and $[\hat{M}]_k = [\hat{M}]_k + \ell_k$.

Clearly, the element $\{\hat{\sigma}\}_k$ track the transition sequences that the system has been probably fired at every $k > 0$. Notice that in the construction of $\{\hat{\sigma}\}_k$ some sequences in $\{\hat{\sigma}\}_{k-1}$ may be discarded. Thus, if $\{\hat{\sigma}\}_k$ is a singleton (point *ii* in the iteration stage), then the transitions sequence is completely known. Besides, three parts comprise the corrective term, in such a way that the observer estimations approach to the system state, and at the same time, the initial state could be progressively updated. The last part is not required; however, it is included since its computation is direct from the definition of the corrective term.

Let $V_k \coloneqq |\{\hat{\sigma}\}_k| - 1$ be the functional for the Lyapunov stability analysis. That is, the number of sequences in $\{\hat{\sigma}\}_k$ minus one. Thus, if $|\{\hat{\sigma}\}_k| = 1$ then $V_k = 0$. This make sense because under such a condition, the error in the sequence estimation is zero. Intuitively, $\{\hat{\sigma}\}_k$ tends to be a singleton as the events occurrence increase. However, it could be the case that the size of $\{\hat{\sigma}\}_k$ does not decrease at each transition firing. Moreover, if the net is safe, the knowledge of the transition sequence leads to a zero error in the estimated state, as established in [1].

Additionally, as highlighted, if the system state is known from the beginning, then the tracking sequence is unique. Thus, trivially $|\{\hat{\sigma}\}_0| = 1$ and $V_0 = |\{\hat{\sigma}\}_0| - 1 = 0$, as expected. Furthermore, $\rho(\vec{0}, \{\vec{0}\}) = 0$ for every $k \geq 0$, and $\{\vec{0}\}$ is a closed invariant set of the error system (E, e_0) represented by (5), as previously analysed. Thus, the next theorem shows that V_k fulfils the Lyapunov stability requirements.

Theorem 6. *Let (B, M_0, φ) be a well-formed OSS, where M_0 is probably unknown. Let $(B, \widehat{M}_0, \varphi)$ be the observer defined by (4), where the initial output is y_0, and the corresponding initial estimation is $\widehat{M}_0 = \varphi' y_0$. Let (E, e_0) be the error system in (5), where the initial error is $e_0 = (\widehat{M}_0 - M_0)$. Let $r = \|e_0\| - 1 = \rho(\widehat{M}_0, M_0) - 1$ and let $\mathcal{M} = \{\vec{0}\}$. Then V_k satisfies the Theorem 5 in the vicinity $S(\mathcal{M}, r)$.*

Proof:

i. *Without loose of generality, let $c_1 > 0$ be sufficiently small such that $\exists e_k \in S(\mathcal{M}, r)$ where $V_k > c_2$ with $c_2 \geq c_1 + 1$. Then, it must be the case that $\rho(e_k, \mathcal{M}) > c_1$. On the contrary, suppose that $\rho(e_k, \mathcal{M}) \leq c_1$. Then, $\rho(e_k, \mathcal{M}) \leq c_2 - 1$, since $c_1 \leq c_2 - 1$. Moreover, as $\mathcal{M} = \{\vec{0}\}$, then $\rho(e_k, \mathcal{M}) = \|e_k\|$. Thus, $\|e_k\| \leq c_2 - 1$, or $\|e_k\| + 1 \leq c_2$. But $\|e_k\| = \|\widehat{M}_k - M_k\|$ for every $k \geq 0$. Hence $\|\widehat{M}_k - M_k\| + 1 \leq c_2$ or $\|\widehat{M}_k - M_k\| < c_2$. By construction $(\widehat{M}_k - M_k) \geq 0$ for every $k \geq 0$. Thus, it immediately follows that the number of entries in $(\widehat{M}_k - M_k)$, which are different from zero, is strictly less than c_2. On the other hand, Then, $\rho(e_k, \mathcal{M}) \leq c_2 - 1$, since $c_1 \leq c_2 - 1$. Moreover, as $\mathcal{M} = \{\vec{0}\}$, then $\rho(e_k, \mathcal{M}) = \|e_k\|$. Thus, $\|e_k\| \leq c_2 - 1$, or $\|e_k\| + 1 \leq c_2$. But $\|e_k\| = \|\widehat{M}_k - M_k\|$ for every $k \geq 0$. Hence $\|\widehat{M}_k - M_k\| + 1 \leq c_2$ or $\|\widehat{M}_k - M_k\| < c_2$. By construction $(\widehat{M}_k - M_k) \geq 0$ for every $k \geq 0$. Thus, it immediately follows that the number of entries in $(\widehat{M}_k - M_k)$, which are different from zero, is strictly less than c_2. On the other hand, the computation of $\widehat{M}_k$ depends on $\widehat{M}_{k-1}$ and ℓ_k. Furthermore, the terms ℓ_k^+ and ℓ_k^- are computed from $\{\hat{\sigma}\}_k$ as in Definition 4, where the term $\overline{\ell_k^-}$ computed from $\overline{\{\hat{\sigma}\}_k}$, is used to delete some markings from $\widehat{M}_k$ corresponding to those wrongly computed sequences in previous steps. It is clear that, the current transition sequence executed by the net is always contained in $\{\hat{\sigma}\}_k$, for every $k \geq 0$. Then, it must hold that $|\{\hat{\sigma}\}_k| \leq \|\widehat{M}_k - M_k\|$, for both cases, either if the net is SD (for safe nets) or ED (for non-safe net). So, $|\{\hat{\sigma}\}_k| < \|\widehat{M}_k - M_k\| + 1 \leq c_2$. But, $V_k \coloneqq |\{\hat{\sigma}\}_k| - 1$. Hence, $V_k < c_2$, which is a contradiction. Thus, necessarily $\rho(e_k, \mathcal{M}) > c_1$.*

ii. *Without loose of generality, let $c_4 > 0$ be sufficiently small such that there exists $e_k \in S(\mathcal{M}, r)$ with $\rho(e_k, \mathcal{M}) < c_3$ where $0 < c_3 \leq (c_4 - 1)$. Then, it should be the case that $V_k \leq c_4$. On the contrary, suppose that $V_k > c_4$. Since $V_k \coloneqq |\{\hat{\sigma}\}_k| - 1$ and $c_4 \geq c_3 + 1$, then $|\{\hat{\sigma}\}_k| > c_3 + 2$. Moreover, since ℓ_k^+ is computed from every element in $\{\hat{\sigma}\}_k$, then $\|\ell_k^+\| > c_3 + 2$. Thus,$> c_3 + 2$, since from Definition 4, $\widehat{M}_{k-1} \geq (\ell_k^- + \overline{\ell_k^-})$ for every $k \geq 0$. But, $\ell_k^+ + \widehat{M}_{k-1} - (\ell_k^- + \overline{\ell_k^-}) = \widehat{M}_k$. Thus, $\|\widehat{M}_k\| > c_3 + 2$. So, $\|\widehat{M}_k - M_k\| > c_3 + 1$. But,$\|\widehat{M}_k - M_k\| =$*

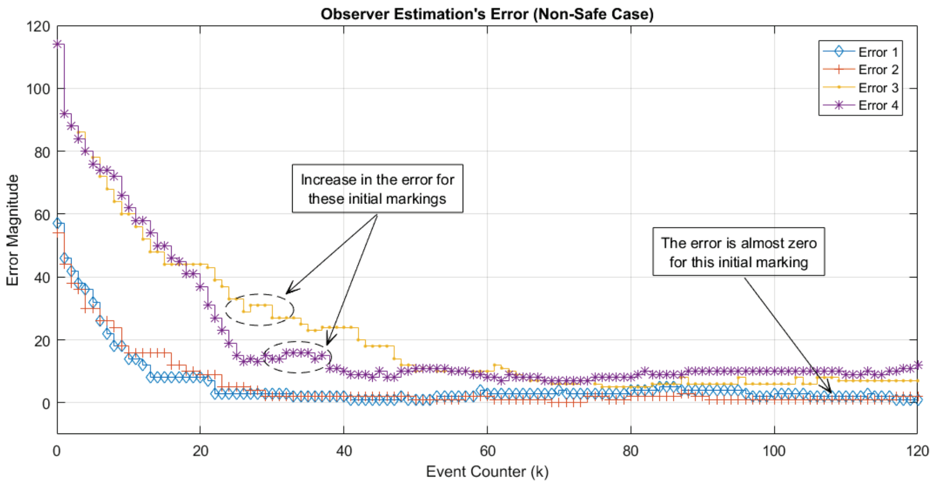

Figure 4. Error in the Observer estimations in a non-safe well-formed OSS.

$\rho(e_k, \mathcal{M})$. Thus, $\rho(e_k, \mathcal{M}) > c_3 + 1$, which is a contradiction. Consequently, it must be the case that $V_k \leq c_4$.

iii. *In order to show that the functional V_k is a non-increasing function of k, notice that by point (ii) in the iteration stage of Definition 4, if $V_k = 0$ for some k, i.e. $|\{\hat{\sigma}\}_{k-1}| = 1$, then $V_{k+l} = 0$ for every $l > 0$. Thus, by contradiction, consider without loss of generality that $V_{k+1} > V_k > 0$, for some k. Then $|\{\hat{\sigma}\}_{k+1}| > |\{\hat{\sigma}\}_k| \geq 2$. But, by Definition 4, $\{\hat{\sigma}\}_{k+1} = \bigcup \tau t_j$ for $\tau = \sigma t_i \in \{\hat{\sigma}\}_k$ and $t_j \in \{\hat{T}\}_{k+1}$, such that $\left\langle\left[\widehat{M}\right]_k\right\rangle t_j$. Since, by contradiction, it holds that $|\{\hat{\sigma}\}_{k+1}| > |\{\hat{\sigma}\}_k| \geq 2$, then it must exist at least a pair of transition sequences, say $\beta t_u, \beta t_v \in \{\hat{\sigma}\}_{k+1}$, constructed from the single sequence $\beta = \sigma t_i \in \{\hat{\sigma}\}_k$ such that $t_u, t_v \in \{\hat{T}\}_{k+1}$, i.e. $\varphi_B(t_u) = \varphi_B(t_v) = \Delta y_k$. On one hand, if the net is non-safe, then $\varphi_B(t_i) = \varphi_B(t_j)$ is a contradiction to the ED of the OSS. On the other hand, if the net is safe, then $\left\langle\left[\widehat{M}\right]_k\right\rangle t_u$ and $\left\langle\left[\widehat{M}\right]_k\right\rangle t_v$. It implies that $\diamond\, t_u =\diamond\, t_v$, which is a contradiction to the SD. Thus, necessarily $V_{k+1} \leq V_k$, for every $k > 0$.*

iv. *In order to show that $V_k \to 0$ as $k \to \infty$, let $V_k > 0$ for some k. Then $|\{\hat{\sigma}\}_k| > 1$, which implies that there exist at least two sequences in $\{\hat{\sigma}\}_k$, say $\sigma_1, \sigma_2 \in \{\hat{\sigma}\}_k$, where $\sigma_1, \sigma_2 \in \mathcal{L}(B, M_0)$ and $\varphi(\sigma_1) = \varphi(\sigma_2)$, which on the one hand, it contradicts the ED if the net is non-safe. On the other hand, if the net is safe, then it should exist an integer $l < \infty$, for which $|\{\hat{\sigma}\}_{k+l}| < |\{\hat{\sigma}\}_k|$. On the contrary, suppose that $|\{\hat{\sigma}\}_{k+l}| \geq |\{\hat{\sigma}\}_k| > 1$ for every $l > 0$. Thus, it must exist at least two transition sequences that could be constructed from σ_1 and σ_2, say*

v. *$\hat{\sigma}_1 = \sigma_1 t_i t_{i+1} t_{i+2} \dots t_{i+l-2} t_{i+l-1} t_{i+l} \dots$, $\hat{\sigma}_2 = \sigma_2 t_i' t_{i+1}' t_{i+2}' \dots t_{i+l-2}' t_{i+l-1}' t_{i+l}' \dots$ such that $\hat{\sigma}_1, \hat{\sigma}_2 \in \{\hat{\sigma}\}_{k+1}$ for every $0 < i \leq l$, where $\varphi(\hat{\sigma}_1) = \varphi(\hat{\sigma}_2)$. Now, let $\{t_i, t_i'\}$, $\{t_{i+1}, t_{i+1}'\}$, $\{t_{i+2}, t_{i+2}'\}, \dots$, $\{t_{i+l-2}, t_{i+l-2}'\}$, $\{t_{i+l-1}, t_{i+l-1}'\}$, $\{t_{i+l}, t_{i+l}'\}$, $\cdots$, etc., be the sequence formed by pairing the corresponding transitions in $\hat{\sigma}_1$ and $\hat{\sigma}_2$, that follows from σ_1 and σ_2, respectively. Then, it holds that $t_i \diamond\!=\!\diamond\, t_{i+1}$, $t_{i+1} \diamond\!=\!\diamond\, t_{i+2}$, …, $t_{i+l-2} \diamond\!=\!\diamond\, t_{i+l-1}$, $t_{i+l-1} \diamond\!=\!\diamond\, t_{i+l}, \dots$, etc., and also $t_i' \diamond\!=\!\diamond\, t_{i+1}'$, $t_{i+1}' \diamond\!=\!\diamond\, t_{i+2}'$, …, $t_{i+l-2}' \diamond\!=\!\diamond\, t_{i+l-1}'$, $t_{i+l-1}' \diamond\!=\!\diamond\, t_{i+l}'$,…, etc. Since $\varphi(\hat{\sigma}_1) = \varphi(\hat{\sigma}_2)$, then in E_B, at least, it holds that $\{t_{i+1}, t_{i+1}'\} \in E_B(i, i')$, $\{t_{i+2}, t_{i+2}'\} \in E_B(i+1, i'+1), \dots$, $\{t_{i+l-1}, t_{i+l-1}'\} \in E_B(i+l-2, i'+l-2)$, $\{t_{i+l}, t_{i+l}'\} \in E_B(i+l-1, i'+l-1), \dots$, etc. As the number of transitions in the net is finite, then at least one element in the previous list of entries in E_B must appear twice. Let say that $\{t_i, t_i'\} \in E_B(i+l, i'+l)$. This conforms a cyclic dependency implying that $\Delta E_B^S = \emptyset$, which directly contradicts the SD of the OSS. Thus, $V_k \to 0$ as $k \to \infty$.* ∎

Notice that the requirement $c_2 \geq c_1 + 1$ with $c_1 > 0$ implies that $r = \|e_0\| - 1 \geq 2$. Additionally, the requirement $0 < c_3 \leq (c_4 - 1)$ with $c_4 > 0$ implies that $c_4 \geq 2$. Thus, it must exist at least two transition sequences that the observer has to track. However, this observer scheme can also be useful in the case of a non-safe net, where $|\{\hat{\sigma}\}_k| = 1, \forall k \geq 0$, as highlighted in the following section.

Illustrative Example

Consider again the OSS in Figure 1, which shows a system with five processes represented by the loops in the model. The places and transitions include superscripts, which represent the loop to which they belong. For example, t_3^1 is the third transition of the first loop. Similarly, p_1^4 is the first place of the fourth loop. Some sensors include a slash. The sensor before the slash is used when the net is considered safe, and that one after the slash is used when the net is considered non-safe. Thus, for p_3^1, D is used when the net is safe, and A_1 when the net is non-safe. Also, some places include grey-filled tokens used as initial markings when the net is non-safe.

The framework in [2] has been used for the simulation of the observer scheme in Figure 2. The corrective term ℓ_k was programmed as in Definition 4. The graphs in Figure 3 correspond to different simulation processes with safe markings. In this plot, the error remains at four up to $k = 6$, for the lines of Error 1, Error 4, Error 5 and Error 6. These errors correspond to the initial conditions $M_0\{p_2^1\}$, $M_0\{p_2^2\}$, $M_0\{p_2^3\}$ and $M_0\{p_2^4\}$, respectively. The line of Error 1 corresponds to the observability constant of the net, i.e. it is the longest convergence error. Indeed, this error reaches zero at $k = 9$, which agrees with the sequences $\varphi(\sigma_1) = \varphi(\sigma_3)$ where $\sigma_1 = t_3^1 t_4^1 t_5^1 t_6^1 t_7^1 t_8^1 t_9^1$ and $\sigma_3 = t_3^2 t_4^2 t_5^2 t_6^2 t_7^2 t_8^2 t_9^2$. These sequences are easily constructed by chaining consecutive entries in the non-empty entries in Table 1. That is, σ_1 and σ_3, correspond to $E_B^s(t_3^1, t_3^2)$, $E_B^s(t_4^1, t_4^2)$, $E_B^s(t_5^1, t_5^2)$, $E_B^s(t_6^1, t_6^2)$, $E_B^s(t_7^1, t_7^2)$, $E_B^s(t_8^1, t_8^2)$, and $E_B^s(t_9^1, t_9^2)$, which are non-empty entries in upper left section of Table 1.

Indeed, the table E_B^s provides information about the loops of a net and their relation to other loops. Thus, for example, the longest sequence of non-empty entries in E_B^s corresponds to the observability convergence constant of the sequences observer. Moreover, if some sequence is too long that it is unacceptable for a specific application, a detailed examination of E_B^s may provide suitable information about the best place to add a new sensor. However, the optimal sensor placement for an OSS is out of the scope of this work.

In the same plot, Error 6 abruptly decreases from 4 to 0 at $k = 6$. Notice that this error corresponds to the initial condition $M_0\{p_2^4\}$. Finally, Errors 4 and 5 decreases from 2 to 0 at $k = 8$, and from 3 to 0 at $k = 7$, respectively. The region of the asymptotic stability of the net is any safe marking, i.e., $\mathcal{M}_a = \{M \in \mathbb{N}^{41}: 0 \leq \|M\| \leq 1\}$. It is not hard to conclude that the size of $\mathcal{M}_a$ is 41.

The graphs in Figure 4 correspond to simulations of the same net, but with non-safe markings. Errors 1 and 2 are from two consecutive simulation processes of the initial marking $M_0\{p_1^1, p_2^1, p_4^1, p_1^2, p_1^3, p_1^4, p_1^5\}$. The difference shown in the evolution of the errors is due to a random firing of the transitions in the system block in Figure 2, when all the transitions are allowed to be fired, i.e. when $u_k = \vec{1}, \forall k \geq 0$. See [2] for details about the firing of the transitions and other configuration options of the simulation framework.

Errors 3 and 4 correspond to two simulation processes with the initial condition $M_0\{2p_1^1, 6p_2^1, 6p_4^1, 2p_1^2, 2p_1^3, 2p_1^4, 2p_1^5\}$. In this case, the initial error is higher than the former due to a greater number of tokens. The error drops as the number of events increases. Notice that, in spite of the transition sequence is known from $k = 1$, the error oscillates, i.e. it increases and decreases randomly. Depending on suitable transition firings in the net, the error may approach to zero. Moreover, it is possible to show that for non-safe markings, the error in (5) is stable. However, a further analysis of this topic is out of the scope of this work.

5. Conclusions

This paper presents the design of observers for DES, which are modelled with OPN. The focus is on a subclass of models called S-Nets. An scheme for tracking the transition sequences executed by the system is proposed, where the feedback for the observer is the system output. The observer is provided with a corrective term ℓ_k to update its estimations. This corrective element is tracking for every possible transition sequences executed by the system. Thus, the observer error decreases as the number of transition sequences tracked by ℓ_k decreases. A Lyapunov stability criterion for characterizing the observer scheme has been used. It shows that the technique developed in this work produces asymptotically stable observers for the case of a safe OSS such that its sequence-detectability property is verified. Thus, in this case, the initial and current state of the system could be perfectly reconstructed with the proposed scheme. When the net is non-safe, the herein introduced scheme produces estimations such that, in some cases, the error may oscillate. An application example illustrates the advantages of the developed techniques.

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