

An Efficient Testing for the Detection of Trajectories in Discrete-Event Systems Modelled by S-Nets

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Abstract: This paper addresses the problem of the detection of sequences of event executed in a Discrete-Event System modelled by Petri Nets. The nets are equipped with output symbols that an external observer is allowed to detect. To provide efficient solutions, the focus of this work is a subclass of nets called S-Systems. The construction of the Sequence-Detectability table leads to a necessary and sufficient condition for the characterization of the sequence detection in the case of safe nets. The safeness requirement is relaxed and its implication on the sequence detection is analyzed. Moreover, the utility of the sequence detectability in the analysis of the observability of the net is studied. An example illustrates the concepts and main results of this paper.

Keywords: Discrete-Event Systems, System Trajectories, Sequence-Detection, Petri Nets, Observability, S-systems.

1. Introduction

The detection of the trajectories in a system is important for several purposes. For example, in Discrete-Event Systems (DES) some control schemes and observer designs use the sequence of events as feedback, while several techniques for the failure detection employ the system trajectories to establish a suitable recovery point [13], [16], [18], [19], [22].

Several approaches reported in the literature dealing with the formal analysis of DES have studied the system trajectories as part of other problems. For instance, the design of controllers, observers, trackers and detectors for DES have all been studied within the automaton field [10], [15], [19], [20], [21].

Petri nets (PN's) have also been used for the analysis of almost the same problems as in the automaton framework. For example, the modeling of manufacturing systems, the supervisory control, state feedback schemes, sequence detection and the observability of PN's and the design of controllers and observers, have been reported in [1], [5], [6], [7], [9], [11], [16].

Previous related work of the authors of this paper is also reported in the literature. In [2], a technique that allows controlling a DES using a supervisor with the feedback provided by an observer of sequences is developed. In [3], preliminary results for addressing the sequence detection of a subclass of PN, called Free-Choice nets, are provided.

This paper extends the previous research of the authors and provides novel results and efficient algorithms addressing the problem of the sequence detection in DES that are modelled by S-Systems, a subclass of PN with well-defined structure, where efficient solutions are derived. Firstly, safe nets are considered. For this case, the construction of the Sequence-Detectability table provides a necessary and sufficient condition for the testing of the sequence-detectability. Secondly, the safeness requirement is relaxed and its implication on the Sequence-Detectability is analyzed. For the non-safe case, the construction of the Event-Detectability table provides a necessary and sufficient condition for the verification of the property. Finally, the relationship among the sequence-detectability and the observability of S-Systems is outlined.

The rest of this paper is organized as follows. Section 2 introduces some basic PN's notions used in this work. Section 3 presents the Sequence-Detectability property and its characterization in well-formed S-Systems. Section 4 outlines how the sequence-detectability supports solving the observability of the net. Section 5 provides the conclusions of this work and a final section shows the bibliographical references.

2. Petri Nets Preliminaries

This work uses Output Petri nets (OPN's) for representing a DES. An OPN is similar to a PN [13] but it is equipped with an output function that produces vectors of output

signals. For more details of OPN's see [3]. Formally, an OPN is a tuple (B, M_0, φ) with the following properties:

- B stands for a standard Petri Net Structure (PNS) $\{P, T, I, O\}$, where P is a set of places, T is a set of transitions; $I \subseteq P \times T$ and $O \subseteq P \times T$ are the flow relations;
- $M_0 \in (\mathbb{N}^+)^m$ is a special vector known as the initial marking, where $m = |P|$;
- The linear function $\varphi: (\mathbb{N}^+)^m \rightarrow (\mathbb{N}^+)^q$ that relates a marking vector to output signals, where q is the number of sensors.

The output function is represented a matrix $\varphi_{[q \times m]}$, where the i -th row vector $\varphi(i, :)$ represents the places associated to the i -th output signal. That is, if $p_j \in P$ is associated with the i -th sensor, then $\varphi(i, j) = 1$, otherwise $\varphi(i, j) = 0$. An external observer is restricted to detect the output of the function φ rather than the event itself executed in the net. Pictorially, circles represent the places and rectangles the transitions, while capital letters, written close to the places, represent the output signals of the net. With a slight abuse of notation, let $I(p_i, t_j)$ denotes the arc from the place p_i to the transition t_j . Thus, if this arc exists in a model, then $I(p_i, t_j) = 1$, otherwise $I(p_i, t_j) = 0$. Similarly, let $O(p_i, t_j)$ denotes the arc from the transition t_j to the place p_i . Then, $O(p_i, t_j) = 1$ means that the arc exists, and $O(p_i, t_j) = 0$ otherwise. The matrices $B^-(i, j) := I(p_i, t_j)$ and $B^+(i, j) := O(p_i, t_j)$ capture the structure of these flow functions. Thus, $B^-(i, j)$ is the flow relation from the place p_i to the transition t_i while $B^+(i, j)$ is the flow relation from the transition t_j to the place p_i . Based on these matrices, the *incidence matrix* is $B := B^+ - B^-$. Notice that $B(i, j) = 0$ is possible while $B^-(i, j) \neq 0$ and $B^+(i, j) \neq 0$. Thus, $B^-(i, j) = B^+(i, j)$. This is called a *self-loop*. The detection of self-loops is useless in the context of DES. Accordingly, this work considers no self-loops in the analyzed OPN.

The *pre-set* and *post-set* of $t_j \in T$, are $\diamond t_j := \{p_i \in P : B^-(i, j) > 0\}$ and, respectively $t_j \diamond := \{p_i \in P : B^+(i, j) > 0\}$. Likewise, for $p_i \in P$ are $\diamond p_i := \{t_j \in T : B^-(i, j) > 0\}$ and

$p_i \diamond := \{t_j \in T : B^+(i, j) > 0\}$. The \diamond operator is extended in a natural way for both sets, of places or of transitions. An OPN is said to be *connected* if, $\forall u, v \in P \cup T$, exists a path from u to v and from v to u , in the sense of the automata theory [7], where the arcs are defined by B^- and B^+ .

The state, or marking, of an OPN is $M(k) \in (\mathbb{N}^+)^m$, with $m = |P|$. The marking represents the number of tokens on each place at time k . For convenience, this paper represents $M(k)$ by M_k , and it uses the notation $M_k(p_i)$, $p_i \in P$, for representing the number of tokens in p_i at time k . Moreover, given a marking $M_k(p_i) = x$, for $p_i \in P$ and $x \in \mathbb{N}^+$, it is represented as $M_k\{x p_i\}$. Thus, $M_k\{2 p_i, p_j\}$ means two tokens in p_i and one in p_j . Notice that curly brackets replace parentheses. The initial marking $M_0 \in (\mathbb{N}^+)^m$ is a primary distribution of tokens in each place, and is used for representing the initial condition of a DES. Thus, $M_0(p_i)$ is the initial number of tokens in p_i . The marking M_0 may enable the firing of transitions. A transition $t_i \in T$ is *enabled* at M_0 , denoted $[M_0]t_i$, if $M_0(p_j) \geq B^-(i, j)$, $\forall p_j \in P$. Given M_k , the notation $[M_k]$ represents the set of all its enabled transitions. The state equation of an OPN is as follows:

$$M_{k+1} = M_k + B \vec{u}_k, y_k = \varphi(M_k) \quad (1)$$

where M_k represents the net's state at time k ; the *Parikh vector* \vec{u}_k represents the firing of one or more transitions enabled at M_k , and B is the incidence matrix, as defined before. The vector M_{k+1} is the state reached by the net and, y_k is the k -th system output. Notice that $\varphi(M_k) := \varphi M_k$ since φ is linear. This fact is used interchangeably where no confusion arises. As with the markings, $y_k\{2A, G\}$ means that the sensor A is measuring two tokens and G one token, at the time k .

If $[M_k]t_j$ and t_j is fired, then by using (1) $M_{k+1} = M_k + B \vec{t}_j = M_k + B(:, t_j)$. In this computation, $\vec{u}_k = \vec{t}_j$ is the Parikh vector with a one in the j -th position and zero anywhere else. This evolution is denoted as $M_k \xrightarrow{t_j} M_{k+1}$, to emphasize the fact that from M_k , t_j is

fired reaching M_{k+1} . The reachability set, denoted by $R(B, M_0)$, is recursively defined as the union of all markings reached by the firing of enabled transitions, where by definition $M_0 \in R(B, M_0)$. A net is *safe* if $\forall M_k \in R(B, M_0)$, it holds that $M_k(p_i) \leq 1, \forall p_i \in P$, and *non-safe* otherwise. The compact representation of any marking M_k in a safe net is always of the form $M_k\{p_i, \dots, p_j\}$ for the set of places marked with a token at M_k .

A transitions sequence is a string $\sigma = t_i t_j t_k \dots t_l$ such that $M_0 \xrightarrow{t_i} M_1 \xrightarrow{t_j} M_2 \xrightarrow{t_k} \dots M_l \xrightarrow{t_l} M_s$. The length of σ , denoted $|\sigma|$, is the number of its transitions. An infinite number of transitions is possible in σ , in such a case $|\sigma| = \infty$. This trajectory, or sequence, is denoted by $M_0 \xrightarrow{\sigma} M_s$. If $M_k \xrightarrow{\sigma} M_s$ for some M_k and σ , then the notation $[M_k]\sigma$ means that M_k enables the firing of the whole sequence σ . Formally, the *Parikh vector* $\vec{\sigma} \in (\mathbb{N}^+)^n$, is a map for every transition in the set T to its number of occurrences in σ . Thus, if $\sigma = t_i t_j t_i$, then $\vec{\sigma}$ is a n-vector, $n = |T|$, with $\vec{\sigma}(i) = 2, \vec{\sigma}(j) = 1$, and zero anywhere else. As in the automata theory [7], given a sequence σ , σ^* denotes its Kleen closure that extends in a natural way for sets. The firing language of (B, M_0, φ) is

$$L(B, M_0) := (\sigma \in T^* : \sigma = t_i t_j t_k \dots t_l)$$

such that $M_0 \xrightarrow{t_i} M_1 \xrightarrow{t_j} M_2 \xrightarrow{t_k} \dots M_r \xrightarrow{t_l} M_s$. Similarly, the middle-language is $\overline{L(B, M_0)} := (\beta \in T^* | \exists \alpha, \gamma : \alpha \beta \gamma \in L(B, M_0))$, where α and γ could be empty. Clearly, $L(B, M_0) \subseteq \overline{L(B, M_0)}$.

The *output word* associated to $\sigma \in \overline{L(B, M_0)}$ is $\varphi(\sigma) := \varphi(M_k) \varphi(M_{k+1}) \dots \varphi(M_{k+r}) \varphi(M_{k+s})$

whit $M_k \xrightarrow{t_i} M_{k+1} \xrightarrow{t_j} M_{k+2} \xrightarrow{t_k} \dots M_{k+r} \xrightarrow{t_l} M_{k+s}$ and $\sigma = t_i t_j t_k \dots t_l$, where $k \geq 0$ and $|\sigma| = s$. Usually, a real DES has a reduced number of sensors, so it is possible that markings $M_k \neq M'_k$ appear alike, i.e. $\varphi M_k = \varphi M'_k$, to an external observer. Therefore, $(M_k - M'_k) \in \ker \varphi$. Thus, it is possible that for an initial marking M_0 , there may be another one, say M'_0 , with $\varphi M_0 = \varphi M'_0$. Thus, by

using the output information, it may not be direct to determine the initial state, the final state, or even the sequence of transitions executed by a DES. Indeed, by (1), $M_{k+1} = M_k + B \vec{u}_k$. Then, $\varphi M_{k+1} = \varphi M_k + \varphi B \vec{u}_k$. The vector $\varphi B \vec{u}_k$, denoted by $\varphi_B(\vec{u}_k)$, is interpreted as the change, or increment, in the system sensors due to u_k , whose Parikh vector is \vec{u}_k . For example, if $\varphi_B(\vec{u}_k) = \{-A, B\}$, it means that the firing of \vec{u}_k turns off the sensor A, while it turns on the sensor B.

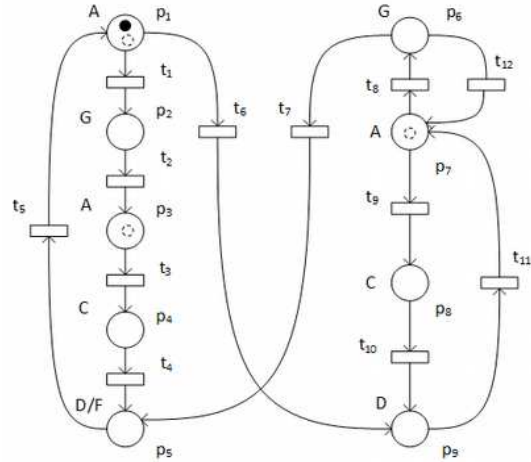


Figure 1. A well-formed OSS.

However, if $\ker \varphi \neq \emptyset$ then it may exist another sequence exists, say u_s , with $\vec{u}_s \neq \vec{u}_k$, such that $\varphi_B(\vec{u}_s) = \varphi_B(\vec{u}_k)$. Notice that it is possible that $\varphi_B(\vec{t}_i) = 0$, while $B \vec{t}_i \neq 0$. Such a transition t_i is known as *silent*. The silent transitions lead to a resilience in the detection of the trajectories as in [14]. However, this work is focused on a constant detection of the trajectories and does not consider silent transitions. More notions of PN's are in [4], [13].

3. Sequence Detection Analysis

Roughly speaking, the Sequence Detection problem deals with the determination of the transition sequence executed by an OPN model.

Definition 1. An OPN (B, M_0, φ) , where M_0 may be unknown, is *Sequence-Detectable (SD)* if there exists an integer, $k_s < \infty$, such that for any $\sigma \in L(B, M_0)$ with $|\sigma| \geq k_s$, where $\sigma = t_i t_j t_k \dots t_l$ and $M_0 \xrightarrow{\sigma} M_s$, the information of the output word $\varphi(\sigma)$ and the structure of (B, M_0, φ) allows to uniquely determine σ .

Testing the SD is a complex task in the general case, since it may require the examination of all $L(B, M_0)$. Fortunately, efficient algorithms are derived for a set of OPN's called S-Systems [4].

Definition 2. An Output S-System (OSS) is an OPN where $\forall t_i \in T : (\diamond t_i) = 1 = (t_i \diamond)$.

The transitions of an OSS have only one input and one output. The simplicity of the structure confers to the matrix B useful properties for an efficient testing of the SD property. A well-formed OSS is strongly connected (see Pag. 28, Theorem 2.25 in [4]). Additionally, these nets are conservative, in the sense that its total number of tokens remains the same for any evolution (see Pag. 43, Theorem 3.6 in [4]). Where no confusion arises, the analyzed OSS are considered to be well-formed, as in Figure 1.

3.1 Sequence detectability in safe OSS

The following example illustrates the concepts developed in this paper, providing intuitive ideas about the sequence detection in an OSS.

Example 1. Consider the well-formed OSS in Figure 1. The net is live and safe since the initial marking puts exactly one token on p_1 . Thus $\sum M_k(p_i) = 1, \forall p_i \in P$ for every marking $M_k \in R(B, M_0)$. The sensor of p_5 is considered to be D and the initial marking $M\{p_1\}$. The F signal for p_5 and the dashed tokens in p_1, p_3 and p_7 , are considered in further examples. Notice that $\varphi M\{p_1\} = \varphi M\{p_3\} = \varphi M\{p_7\}$, since p_1, p_3, p_7 have the same sensor. Thus, an external

observer is unable to decide the initial condition of the OSS by only using $Y_0 = \varphi M\{p_1\}$. Moreover, $M\{p_1\}$ enables the firing of both, t_1 and t_6 . If t_6 is fired, then an external observer can immediately detect its execution since $\varphi_B(\vec{t}_6) = \{-A, D\}$ is different from any other firing. However, if t_1 is fired then its execution is confused with that of t_8 , i.e. $\varphi_B(\vec{t}_1) = \varphi_B(\vec{t}_8)$. Moreover, notice that t_8 is indeed enabled at $M\{p_7\}$. So, by the solely information provided by a single firing, it is impossible to decide which of the transitions, either t_1 or t_8 , have fired. Besides, it is simple to verify that the firing of t_2 is confused with that of t_{12} , and t_3 with t_9 , as well. Indeed, $\sigma_1 = t_1 t_2 t_3 t_4 t_5$ and $\sigma_2 = t_8 t_{12} t_9 t_{10} t_{11}$ are confusing to each other. This is a pair of transition circuits whose firing is confusing, i.e.,

$$\varphi(t_1 t_2 t_3 t_4 t_5) = \varphi(t_8 t_{12} t_9 t_{10} t_{11}).$$

The circuits analyzed in the previous example, are closely related to the SD of a safe OSS. In [3], the authors showed that an intersection of vector spaces suffices for testing the property. This work extends this idea and proposes a full condition for testing the property.

Definition 3. Let (B, M_0, φ) be a safe OSS, with $T = \{t_1, \dots, t_n\}$ is its set of transitions. The Event-Detectability table E_B (Table 1) is a square arrangement of size $[n-1 \times n-1]$, with columns representing the transitions from t_1 to t_{n-1} and rows representing the transitions from t_2 to t_n . The entries of E_B are as follows. For $2 \leq i \leq n; 1 \leq j \leq n-1; j < i$:

Table 1. The Event-Detectability table E_B .

t_2	{ }										
t_3	{ }	{ }									
t_4	{ }	{ }	{ }								
t_5	{ }	{ }	{ }	{ }							
t_6	{ }	{ }	{ }	{ }	{ }						
t_7	{ }	{ }	{ }	{ }	{ }	{ }					
t_8	{ t_{12}, t_2 { t_7, t_2 }	{ }	{ }	{ }	{ }	{ }	{ }				
t_9	{ }	{ }	{ t_{10}, t_4 }	{ }	{ }	{ }	{ }	{ }			
t_{10}	{ }	{ }	{ }	{ t_{11}, t_5 }	{ }	{ }	{ }	{ }	{ }		
t_{11}	{ }	{ }	{ }	{ }	{ t_{12}, t_1 { t_{12}, t_6 { t_8, t_1 { t_8, t_6 }	{ }	{ }	{ }	{ }	{ }	
t_{12}	{ }	{ t_9, t_3 , { t_8, t_3 }	{ }	{ }	{ }	{ }	{ }	{ }	{ }	{ }	{ }
	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}

$$\begin{aligned}
& \text{if } \varphi_B(\vec{t}_i) \neq \varphi_B(\vec{t}_j), \\
& \text{then } E_B(t_i, t_j) = \emptyset, \text{ otherwise,} \\
& E_B(t_i, t_j) = \cup \{t_u, t_v\} \\
& \forall t_u \in (t_i \diamond) \diamond, \forall t_v \in (t_j \diamond) \diamond
\end{aligned} \tag{2}$$

The elements of E_B are pairs in $T \times T$. It is easily noticed that the shape of E_B is a kind of lower-triangular. If the firing of t_i is different from that of t_j , then $E_B(t_i, t_j) = \emptyset$. Otherwise, it is filled with the union of pairs in the Cartesian product $((t_i \diamond) \diamond) \times ((t_j \diamond) \diamond)$. Notice that if the column and row in E_B corresponding to t_i are empty, i.e. $E_B(:, t_i) = \emptyset$ and $E_B(t_i, :) = \emptyset$, then the firing of t_i is different from any other in the net.

Example 2. Consider the Table 1, showing E_B of the OSS in Figure 1. Since $\varphi_B(\vec{t}_1) = \varphi_B(\vec{t}_8)$, the corresponding entry $E_B(t_8, t_1)$ includes the pairs (t_{12}, t_2) and (t_7, t_2) . Similarly, $E_B(t_9, t_3)$, $E_B(t_{10}, t_4)$, $E_B(t_{11}, t_5)$ and $E_B(t_{12}, t_2)$ are non-empty, meaning that its firing is confused to some others. Notice that the entry $E_B(t_{11}, t_5)$ includes four pairs, (t_8, t_1) , (t_8, t_6) , (t_{12}, t_1) and (t_{12}, t_6) , since $(t_{11} \diamond) \diamond = \{t_8, t_{12}\}$ and $(t_5 \diamond) \diamond = \{t_1, t_6\}$. On the other hand, the rows and columns corresponding to t_6 and t_7 are empty. It means that the firing of any of those transitions could be detected by examining E_B .

If a column or row in E_B is nonempty, then an external observer cannot detect the firing of the corresponding transition by solely considering the information in E_B . However, the information of more firings of transitions

contained in the Sequence-Detectability table could be used.

Definition 4. Let E_B be the Event-Detectability table of the OSS (B, M_0, φ) . Consider the Sequence-Detectability table E_B^s (Table 2) obtained from E_B by removing (t_u, t_v) if and only if $E_B(t_u, t_v) = \emptyset$, from any other entry where it appears, say $E_B(t_i, t_j)$, whenever t_i and t_j share no place, no input nor output, and recursively removing the new entries that become empty. Formally,

$$E_B^s(t_i, t_j) = E_B(t_i, t_j) \setminus (t_u, t_v) \text{ iff } E_B(t_u, t_v) = \emptyset$$

and both, $\diamond t_i \cap \diamond t_j = \emptyset$ and $t_i \diamond \cap t_j \diamond = \emptyset$, and recursively, repeat the procedure with the new empty entries that may appear.

As a first step, the definition of E_B^s removes a pair, from any entry where it appears, if its corresponding entry is already empty. Moreover, if new empty entries appear, the recursive step allows its propagation to the entire table E_B^s . Notice the restrictions $\diamond t_i \cap \diamond t_j = \emptyset$ and $t_i \diamond \cap t_j \diamond = \emptyset$ at the first point in the Definition 4. By the one hand, if $\diamond t_i = p_i = \diamond t_j$, then trivially t_i is enabled in an OSS if and only if t_j is enabled as well. Thus, every time that either t_i or t_j is fired, the executed sequence, say σt_i or σt_j becomes ambiguous, since $\varphi_B(\vec{t}_i) = \varphi_B(\vec{t}_j)$. Moreover, since the net is a connected OSS, such sequence σ always exists (see Pag. 182, Theorem 8.11 in [4]). This directly leads to the non-sequence-detectability of the OSS. On the other hand, if $t_i \diamond = p_j = t_j \diamond$ then any transitions sequence, say $t_i \sigma$, enabled by an

Table 2. The Sequence-Detectability table E_B^s .

t_2	{}										
t_3	{}	{}									
t_4	{}	{}	{}								
t_5	{}	{}	{}	{}							
t_6	{}	{}	{}	{}	{}						
t_7	{}	{}	{}	{}	{}	{}					
t_8	$\{t_{12}, t_2\}$	{}	{}	{}	{}	{}	{}				
t_9	{}	{}	$\{t_{10}, t_4\}$	{}	{}	{}	{}	{}			
t_{10}	{}	{}	{}	$\{t_{11}, t_5\}$	{}	{}	{}	{}	{}		
t_{11}	{}	{}	{}	{}	$\{t_8, t_1\}$	{}	{}	{}	{}	{}	
t_{12}	{}	$\{t_9, t_3\}$	{}	{}	{}	{}	{}	{}	{}	{}	{}
	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}

initial marking $M_0\{p_i\}$ for $p_i = \diamond t_i$, could never be distinguished from $t_j \sigma$.

At a first glance, it looks like there is a similarity between the construction method of E_B^s and minimization of automata method in the Myhill–Nerode theorem (see Chapter 3 in [7]). A further analysis is planned as future work.

Example 3. Table 2 shows E_B^s for the OSS in Figure 1. Notice that (t_7, t_2) has been removed from $E_B^s(t_8, t_1)$, since $E_B^s(t_7, t_2) = \emptyset$. Similarly, (t_8, t_3) from $E_B^s(t_{12}, t_2)$, as well as (t_{12}, t_1) , (t_{12}, t_6) and (t_8, t_6) from $E_B^s(t_{11}, t_5)$. However, (t_{12}, t_2) cannot be removed from $E_B^s(t_8, t_1)$ since $E_B^s(t_{12}, t_2) \neq \emptyset$. But, for removing (t_9, t_3) from $E_B^s(t_{12}, t_2)$ it is required removing (t_{10}, t_4) from $E_B^s(t_9, t_3)$. Though, for removing (t_{10}, t_4) , it is required take away (t_{11}, t_5) from $E_B^s(t_{10}, t_4)$. Finally, for removing (t_{11}, t_5) from $E_B^s(t_{10}, t_4)$ it is required take out (t_8, t_1) from $E_B^s(t_{11}, t_5)$, which is the cyclic dependency discussed in the Example 1.

Let $\Phi = \{t_i, t_j\}, \{t_k, t_l\}, \dots, \{t_u, t_v\}$ be the sequence of entries as in the previous example.

Based on Φ , it is easily constructed $\sigma = (\sigma_1, \sigma_2)$ where $\sigma_1 = t_i t_k \dots t_u$ and $\sigma_2 = t_j t_l \dots t_v$ are the concatenation of consecutive transitions in the pairs of Φ . By the definition of E_B^s , σ_1 and σ_2 are allowed to fire since $t_i \diamond = \diamond t_k$, $t_j \diamond = \diamond t_l$, $t_u \diamond = \diamond t_v$, etc. Clearly $\varphi(\sigma_1) = \varphi(\sigma_2)$. Moreover, σ_1 and σ_2 are T-invariants (see Pag. 45, Lemma 3.9 in [4]). Let ΔE_B^s denote the set of circuits, as Φ , in E_B^s . Then, the next theorem holds.

Theorem 1. A safe OSS (B, M_0, φ) is SD if and only if $\Delta E_B^s = \emptyset$.

Proof: (Necessity) Supposes that (B, M_0, φ) is SD but $\Delta E_B^s \neq \emptyset$. Let $\Phi \in \Delta E_B^s$. Without loss of generality, let say that $\sigma = (\sigma_1, \sigma_2)$ is constructed from Φ as before, and that $\sigma_1 = t_i t_k t_u \dots t_w t_y$ and $\sigma_2 = t_j t_l t_v \dots t_x t_z$. Thus, in E_B^s must exist $E_B^s(t_i, t_j)$, $E_B^s(t_k, t_l)$, $E_B^s(t_u, t_v)$, ..., $E_B^s(t_w, t_x)$, $E_B^s(t_y, t_z)$, $E_B^s(t_i, t_j)$ such that $(t_k, t_l) \in E_B^s(t_i, t_j)$, $(t_u, t_v) \in E_B^s(t_k, t_l)$, ..., $(t_y, t_z) \in E_B^s(t_w, t_x)$, and $(t_i, t_j) \in E_B^s(t_y, t_z)$. By Definition 4, it

holds that $\varphi_B(\vec{t}_i) = \varphi_B(\vec{t}_j)$, $\varphi_B(\vec{t}_k) = \varphi_B(\vec{t}_l)$, $\varphi_B(\vec{t}_u) = \varphi_B(\vec{t}_v)$, ..., $\varphi_B(\vec{t}_w) = \varphi_B(\vec{t}_x)$, $\varphi_B(\vec{t}_y) = \varphi_B(\vec{t}_z)$. Moreover, $t_i \diamond = \diamond t_k$, $t_j \diamond = \diamond t_l$, $t_k \diamond = \diamond t_u$, $t_l \diamond = \diamond t_v$, ..., $t_w \diamond = \diamond t_y$, $t_x \diamond = \diamond t_z$, $t_y \diamond = \diamond t_i$, and $t_z \diamond = \diamond t_j$. Let $\{p_i\} = \diamond t_i$ and $\{p_j\} = \diamond t_j$. Since (B, M_0, φ) is live, it is place live, as well (see Pag. 26, Proposition 2.17 in [4]). Then, clearly $M\{p_i\}$ and $M'\{p_j\}$ are valid markings of the net. Thus, $\sigma_1, \sigma_2 \in L(B, M_0)$ where $\sigma_1 = t_i t_k t_u \dots t_w t_y$ and $\sigma_2 = t_j t_l t_v \dots t_x t_z$, since $[M\{p_i\}]\sigma_1$ and $[M'\{p_j\}]\sigma_2$. Let

$$M\{p_i\} \xrightarrow{t_i} M\{p_k\} \xrightarrow{t_k} M\{p_u\} \xrightarrow{t_u} \dots \\ \dots M\{p_w\} \xrightarrow{t_w} M\{p_y\} \xrightarrow{t_y} M\{p_i\}$$

and

$$M'\{p_j\} \xrightarrow{t_j} M'\{p_l\} \xrightarrow{t_l} M'\{p_v\} \xrightarrow{t_v} \dots \\ \dots M'\{p_x\} \xrightarrow{t_x} M'\{p_z\} \xrightarrow{t_z} M'\{p_j\}$$

be the two sequences of markings due to σ_1 and σ_2 , respectively. Clearly $\varphi M\{p_i\} = \varphi M'\{p_j\}$, $\varphi M\{p_k\} = \varphi M'\{p_l\}$, $\varphi M\{p_u\} = \varphi M'\{p_v\}$, ..., $\varphi M\{p_w\} = \varphi M'\{p_x\}$, $\varphi M\{p_y\} = \varphi M'\{p_z\}$. Thus, $\varphi(\sigma_1) = \varphi(\sigma_2)$. Moreover, since σ_1 and σ_2 are circuits, then $\sigma_1^*, \sigma_2^* \in L(B, M_0)$ (see Pag. 45, Lemma 3.9 in [4]). Thus,

$$\varphi(\sigma_1^*) = \varphi(\sigma_1)\varphi(\sigma_1) \dots = \varphi(\sigma_2)\varphi(\sigma_2) \dots = \varphi(\sigma_2^*)$$

which contradicts the net is SD.

(Sufficiency) Suppose that $\Delta E_B^s = \emptyset$ but on the contrary, the net is non-SD. Then, there exist at least two sequences of infinite length, say $\tau_1 = t_i \dots t_k t_u \dots t_w t_y \dots$ and $\tau_2 = t_j t_l t_v \dots t_x t_z \dots$ where $\tau_1, \tau_2 \in L(B, M_0)$, such that $\tau_1 \neq \tau_2$, and $\varphi(\tau_1) = \varphi(\tau_2)$. Since (B, M_0, φ) is safe, then any initial marking that puts one token on any place of the net, is a home marking (see Pag. 169, Proposition 8.2 in [4]). This turns the selection of τ_1 and τ_2 independent of any initial marking. It immediately follows that $\varphi_B(\vec{t}_i) = \varphi_B(\vec{t}_j)$, ..., $\varphi_B(\vec{t}_k) = \varphi_B(\vec{t}_l)$, $\varphi_B(\vec{t}_u) = \varphi_B(\vec{t}_v)$, ..., $\varphi_B(\vec{t}_w) = \varphi_B(\vec{t}_x)$, $\varphi_B(\vec{t}_y) = \varphi_B(\vec{t}_z)$, ..., and so on, since $\varphi(\tau_1) = \varphi(\tau_2)$. Now, consider the sequence $\Phi = (t_i, t_j)$, ..., (t_k, t_l) , (t_u, t_v) , ..., (t_w, t_x) , (t_y, t_z) , (t_k, t_l) ..., etc., formed by pairing the transitions that are confused to each other

in τ_1 and τ_2 , correspondingly. Since Φ is of infinite length, then at least one of its elements must appear twice, as $|T \times T| < \infty$. Let say, without loss of generality, that the first element repeating in Φ is $\{t_k, t_l\}$. Notice that, in τ_1 , the firing of t_k enables that of t_u , since the net is safe. Also, in τ_2 , the firing of t_l enables that of t_v . Indeed, this holds for any two consecutive transitions in τ_1 and in τ_2 . Therefore, in τ_1 it holds that $t_u \in (t_k \diamond) \diamond$, ..., $t_y \in (t_w \diamond) \diamond$, $t_k \in (t_y \diamond) \diamond$, and so on. Similarly, in τ_2 it holds that $t_v \in (t_l \diamond) \diamond$, ..., $t_z \in (t_x \diamond) \diamond$, $t_l \in (t_z \diamond) \diamond$, and so on. Thus, in the construction of E_B , it immediately follows that, at least $(t_u, t_v) \in E_B(t_k, t_l)$, ..., $(t_y, t_z) \in E_B(t_w, t_x)$, $(t_k, t_l) \in E_B(t_y, t_z)$, ..., and so on. Notice that the last element in this sequence emphasizes the fact that the pair (t_k, t_l) belongs to the entry $E_B(t_y, t_z)$. This is consistent with the previous assumption in the sense that (t_k, t_l) is the first element appeared twice in Φ . Now, consider the construction of E_B^s . Clearly, the recursion step cannot remove (t_u, t_v) from $E_B^s(t_k, t_l)$ nor (t_y, t_z) from $E_B^s(t_w, t_x)$. Moreover, the same apply to any of the intermediate elements in the above sequence of elements of E_B . Indeed, it finally holds that (t_k, t_l) cannot be removed from $E_B^s(t_y, t_z)$. But, $(t_y, t_z) \in E_B^s(t_w, t_x)$. This conforms a cyclic dependency, and $\Delta E_B^s \neq \emptyset$.

The next proposition goes further and shows that any non-empty entry in E_B^s leads to a circuit.

Proposition 1. Let E_B^s be the SD table of the OSS (B, M_0, φ) . Then, for any nonempty entry $E_B^s(t_g, t_h)$ there exist a pair of circuits, say σ_g and σ_h , such that $\varphi(\sigma_g) = \varphi(\sigma_h)$.

Proof: Let $E_B^s(t_g, t_h) \neq \emptyset$ be any non-empty entry in E_B^s . Then, by construction, at least $E_B^s(t_g, t_h) \ni (t_i, t_j)$ for some t_i, t_j that fulfil $\varphi_B(\vec{t}_g) = \varphi_B(\vec{t}_h)$ and $\varphi_B(\vec{t}_i) = \varphi_B(\vec{t}_j)$. By supposition both, $\diamond t_g \neq \diamond t_h$ and $t_g \diamond \neq t_h \diamond$. Then, necessarily $E_B^s(t_i, t_j) \neq \emptyset$. Otherwise, if $E_B^s(t_i, t_j) = \emptyset$, then the recursive step in Definition 4 must delete (t_i, t_j) from $E_B^s(t_g, t_h)$, which contradicts the fact that $E_B^s(t_g, t_h) \neq \emptyset$. Thus, let $E_B^s(t_i, t_j) \ni (t_k, t_l)$, for $t_k, t_l \in T$. Then, $\varphi_B(\vec{t}_k) = \varphi_B(\vec{t}_l)$. Again,

since $\diamond t_i \neq \diamond t_j$ and $t_i \diamond \neq t_j \diamond$, then it requires that $E_B^s(t_k, t_l) \neq \emptyset$. Let say that $E_B^s(t_k, t_l) \ni (t_u, t_v)$, with $\varphi_B(\vec{t}_u) = \varphi_B(\vec{t}_v)$. Otherwise, it easily leads a contradiction as before. Moreover, it should be that $E_B^s(t_u, t_v) \neq \emptyset$. By using this reasoning, without loss of generality, let say that the sequence of entries continues as $E_B^s(t_u, t_v)$, ..., $E_B^s(t_w, t_x)$, $E_B^s(t_y, t_z)$, ..., etc. Furthermore, since the net is connected, for every t_i there is always other $t_j \in (t_i \diamond) \diamond$. Thus, this sequence of entries in E_B^s shall continue, and since the number of the entries in this table is finite, then at least one of its elements should be repeated. Let say that (t_k, t_l) is the first pair that repeating twice, i.e., $E_B^s(t_y, t_z) \ni (t_k, t_l)$. Clearly, this selection does not cause loss of generality in the result. Now, let $\sigma_g = t_k t_u \dots t_w t_y$ and $\sigma_h = t_l t_v \dots t_x t_z$ be two transition sequences built from this sequence of entries in E_B^s . Then, clearly $\varphi_B(\vec{t}_k) = \varphi_B(\vec{t}_l)$, $\varphi_B(\vec{t}_u) = \varphi_B(\vec{t}_v)$, ..., $\varphi_B(\vec{t}_w) = \varphi_B(\vec{t}_x)$, and $\varphi_B(\vec{t}_y) = \varphi_B(\vec{t}_z)$, i.e., $\varphi(\sigma_g) = \varphi(\sigma_h)$. Thus, the sequence $E_B^s(t_k, t_l)$, $E_B^s(t_u, t_v)$, ..., $E_B^s(t_w, t_x)$, $E_B^s(t_y, t_z)$, $E_B^s(t_k, t_l)$ is a circular chain of entries in E_B^s where, by Definition 4 it holds that $t_k \diamond = \diamond t_u$, $t_l \diamond = \diamond t_v$, ..., $t_w \diamond = \diamond t_y$ and $t_x \diamond = \diamond t_z$. Hence, σ_g and σ_h are circuits of transitions in (B, M_0, φ) as required.

From Theorem 3 and Proposition 1, the following corollary holds.

Corollary 1. Let (B, M_0, φ) be a safe OSS. Then, $\Delta E_B^s = \emptyset$ iff every entry in E_B^s is empty.

3.2 Sequence detectability in non-safe OSS

Suppose that the net in Figure 1 is able to hold more than one token. Thus, the marking $M_0\{p_1, p_7\}$ is a suitable initial condition of the OSS. It is easy to verify that $[M_0]t_1$, $[M_0]t_8$ and, as discussed earlier, $\varphi_B(\vec{t}_1) = \varphi_B(\vec{t}_8)$. Now, suppose that the entire sequence $\sigma = t_1 t_2 t_3 t_4 t_5$ is fired. Notice that σ is enabled at $M_0\{p_1, p_7\}$. Clearly, the marking reached by the firing of σ is a new $M_0\{p_1, p_7\}$, since $B\vec{\sigma} = \vec{0}$. Thus, the firing of σt_1 confuses with that of σt_8 , i.e.,

$$\varphi(\sigma t_1) = \varphi(\sigma) \varphi_B(\vec{t}_1) = \varphi(\sigma) \varphi_B(\vec{t}_8) = \varphi(\sigma t_8).$$

Since σ may execute forever, then

$$\begin{aligned} \varphi(\sigma \cdots \sigma t_1 \cdots) &= \varphi(\sigma) \cdots \varphi(\sigma) \varphi_B(\vec{t}_1) \cdots = \\ &= \varphi(\sigma) \cdots \varphi(\sigma) \varphi_B(\vec{t}_8) \cdots = \varphi(\sigma \cdots \sigma t_8 \cdots). \end{aligned}$$

It directly implies that the net is not SD. Based in this intuitive idea, the next theorem characterizes the SD in a non-safe OSS.

Theorem 2. Let (B, M_0, φ) be a non-safe OSS. Then, the net is SD if and only if E_B is empty.

Proof: (Necessity) Suppose that (B, M_0, φ) is SD but E_B is non-empty. Then, there exist, at least, a pair of transitions, say $t_i, t_j \in T$, such that $\varphi_B(\vec{t}_i) = \varphi_B(\vec{t}_j)$. Since the OSS is live, then it is also place-live (see Pag. 26, Proposition 2.17 in [4]). Moreover, since the net is non-safe, by using an enough number of tokens, it is possible to construct the marking $M_{ij} \geq M\{p_i\} + M\{p_j\}$, for $p_i \in \diamond t_i$ and $p_j \in \diamond t_j$. Notice that $\exists \sigma \in L(B, M_0)$ such that $M_0 \xrightarrow{\sigma} M_{ij}$ and accordingly $M_{ij} \in R(B, M_0)$. It is clear that $[M_{ij}]t_i$, and $[M_{ij}]t_j$. Thus, it is impossible to distinguish the execution of σt_i from σt_j , since $\varphi(\sigma t_i) = \varphi(\sigma) \varphi_B(\vec{t}_i) = \varphi(\sigma) \varphi_B(\vec{t}_j) = \varphi(\sigma t_j)$, as by assumption it holds that $\varphi_B(\vec{t}_i) = \varphi_B(\vec{t}_j)$. Since every marking in a well-formed OSS is a home marking (see Pag. 169, Proposition 8.2 in [4]), then for each σt_i and σt_j , it should exist other sequences, say $\gamma_1, \gamma_2 \in L(B, M_0)$, such that both $\sigma t_i \gamma_1$ and $\sigma t_j \gamma_2$ reach M_{ij} anew. Hence, there is a confusion again about the firing of $\sigma t_i \gamma_1 t_i$ and $\sigma t_j \gamma_2 t_j$, i.e.,

$$\begin{aligned} \varphi(\sigma t_i \gamma_1 t_i) &= \varphi(\sigma t_i \gamma_1) \varphi_B(t_i) = \\ &= \varphi(\sigma t_i \gamma_1) \varphi_B(t_j) = \varphi(\sigma t_j \gamma_2 t_j). \end{aligned}$$

Applying the same reasoning, it easily leads to the construction of a pair of transition sequences of infinite length that confuses to each other, which directly contradicts the SD of the net.

(Sufficiency) Suppose that E_B is empty. Then, $\varphi_B(\vec{t}_i) \neq \varphi_B(\vec{t}_j)$ for every pair of transitions $t_i, t_j \in T: t_i \neq t_j$. Then, the firing of every transition in the system is distinguished from each other. Accordingly, the transition sequence executed by the net from any M_0 , is determined at every execution step by an inspection of E_B .

The next section uses the results of the sequence detectability for sketching a solution of the observability problem in OSS models.

4. Observability Analysis

The observability is a property that allows discovering the state of a DES. This work uses a quite intuitive definition in the OPN field [3].

Definition 5. An OPN (B, M_0, φ) , where M_0 may be unknown, is *Observable* if there exists an integer $k < \infty$, such that for any $\sigma \in L(B, M_0)$ with $|\sigma| \geq k$, where $\sigma = t_i t_j t_k \dots t_l$ and $M_0 \xrightarrow{t_i} M_1 \xrightarrow{t_j} M_2 \xrightarrow{t_k} \dots M_r \xrightarrow{t_l} M_s$, the information of the output word $\varphi(\sigma)$ and the structure of (B, M_0, φ) , allows to uniquely determine the markings M_0 and M_s .

Based on (1), this work proposes the Firing-Vector-Detectability and Marking-Detectability properties for testing the observability.

Definition 6. An OPN (B, M_0, φ) , where M_0 may be unknown, is *Firing-Vector-Detectable (FVD)* if there exists an integer $k_F < \infty$ such that $\forall \sigma \in L(B, M_0): |\sigma| \geq k_F$, $\sigma = t_i t_j t_k \dots t_l$ with $M_0 \xrightarrow{t_i} M_1 \xrightarrow{t_j} M_2 \xrightarrow{t_k} \dots M_r \xrightarrow{t_l} M_s$, the output word $\varphi(\sigma)$ and the structure (B, M_0, φ) , allows to uniquely determine the Parikh vector $\vec{\sigma}$.

Definition 7. An OPN (B, M_0, φ) , where M_0 may be unknown, is *Marking-Detectable (MD)* if there exists an integer $k_M < \infty$, such that $\forall \sigma \in L(B, M_0): |\sigma| \geq k_M$, $\sigma = t_i t_j t_k \dots t_l$ with $M_0 \xrightarrow{t_i} M_1 \xrightarrow{t_j} M_2 \xrightarrow{t_k} \dots M_r \xrightarrow{t_l} M_s$, the output word $\varphi(\sigma)$ and the structure of (B, M_0, φ) , allows to uniquely determine M_k reached by σ .

By the FVD and MD, next proposition is direct.

Proposition 2. An OPN (B, M_0, φ) that is FVD and MD is *Observable*.

Proof: Let (B, M_0, φ) be FVD, with constant k_F , and MD, with constant k_M . Let $k = \max\{k_F, k_M\}$ and let $\sigma \in L(B, M_0): |\sigma| \geq k$. Thus, $\vec{\sigma}$ and M_k , reached by the firing of σ , are known, as well as $M_k, \vec{\sigma}$ and B .

Hence, the initial marking is computed as $M_0 = M_k - B \vec{\sigma}$ by using the state equation (1).

The next subsection uses the SD for testing the FVD and MD of an OSS.

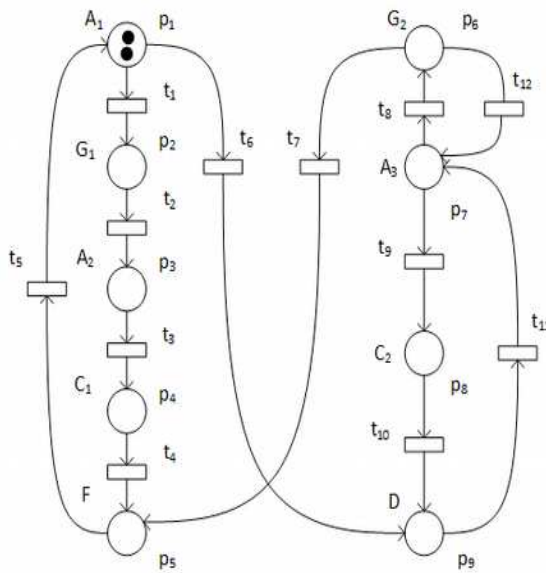


Figure 2. Extra token in p_1 and the new sensor.

4.1 Observability in safe OSS

Clearly, the determination of the Parikh vector from the transition sequence is direct. The next proposition shows that the current marking is also computed easily when the OSS is safe.

Proposition 3. *A well-formed and safe OSS (B, M_0, φ) that is SD, is also MD.*

Proof: *Supposes that (B, M_0, φ) is SD, with convergence constant k_s and let $\sigma \in L(B, M_0)$ such that $|\sigma| \geq k_s$. Then σ is known. Let say that $\sigma = \alpha t_j$ for some $\alpha \in L(B, M_0)$ and $t_j \in T$. Without loss of generality, let say that $t_j \diamond = \{p_j\}$. Since the net is safe, the marking reached by the firing of $\sigma = \alpha t_j$ is clearly $M\{p_j\}$.*

Thus, the following corollary is direct and establishes the observability of a safe OSS.

Corollary 2. *If a well-formed and safe OSS is SD, then it is observable.*

The following example illustrates the result.

Example 4. *Consider again the OSS in Figure 1. Suppose that the sensor of p_5 has been changed from D to F. By this change, the firings of t_4 and t_5 are now different from any other firing in the net. Thus, $E_B^s(t_{10}, t_4)$ and $E_B^s(t_{11}, t_5)$ in the Table 2 must be empty. Therefore, $E_B^s(t_9, t_3)$ becomes empty and hence $E_B^s(t_{12}, t_2)$, as well. Finally, $E_B^s(t_8, t_1)$*

ends up empty and accordingly, $\Delta E_B^s = \emptyset$. So, the net is now SD.

Notice that the initial marking $M\{p_1\}$ enables the firing of t_1 as well as that of t_6 . Suppose that t_1 is fired. Then, an external observer is only able to compute $\sigma_1 = t_1$ and $\sigma_2 = t_8$ as possible executed sequences at $k=1$, since $\varphi_B(t_1) = \varphi_B(t_8)$. Thus, the possible reached markings are $M\{t_1 \diamond\} = M\{p_2\}$ and $M\{t_8 \diamond\} = M\{p_6\}$, respectively. Now, suppose that t_2 is fired. Since $\varphi_B(t_2) = \varphi_B(t_{12})$, then the possible executed sequences, at $k=2$, are $\sigma_1 = t_1 t_2$ and $\sigma_2 = t_8 t_{12}$. Accordingly, the probable reached markings are $M\{t_2 \diamond\} = M\{p_3\}$ and $M\{t_{12} \diamond\} = M\{p_7\}$, correspondingly. It is easy to verify that for $k=3$, the possible executed sequences are $\sigma_1 = t_1 t_2 t_3$ and $\sigma_2 = t_8 t_{12} t_9$, by assuming that t_3 is fired. The probable reached markings are $M\{t_3 \diamond\} = M\{p_4\}$ and $M\{t_9 \diamond\} = M\{p_8\}$, respectively. However, at $k=4$ once t_4 is fired, the sequence $\sigma_1 = t_1 t_2 t_3 t_4$ is distinguished from $\sigma_2 = t_8 t_{12} t_9 t_{10}$. Thus, the observer knows that the sequence executed by the net is σ_1 and it is able to compute its Parikh vector $\vec{\sigma}_1$. Moreover, the marking reached by the firing of σ_1 is $M\{t_4 \diamond\} = M\{p_5\}$. Finally, the initial marking is $M_0 = M\{p_5\} - B\vec{\sigma}_1 = M\{p_1\}$.

4.2 Observability in non-safe OSS

The Theorem 2 states that the detection of the transition sequence in a safe OSS allows for detecting its marking. However, it is not the case in non-safe OSS, as detailed in the next example.

Example 5. *Consider the markings $M_0\{p_1, p_3\}$, $M_0\{p_1, p_7\}$ and $M_0\{2p_1\}$, as possible initial conditions of the OSS in Figure 1. Those markings are valid since the net is supposed to be non-safe in this case. Clearly, $\varphi M_0\{p_1, p_3\} = \varphi M_0\{p_1, p_7\} = \varphi M_0\{2p_1\}$. Let $\sigma = t_1 t_2 t_3 t_4 t_5$. It is easy verified that $[M_0\{p_1, p_3\}] \sigma$ and consequently $\sigma \in L(B, M_0)$. Notice that $B\vec{\sigma} = \vec{0}$. Thus, the marking reached at every firing of $\sigma^* = (t_1 t_2 t_3 t_4 t_5)(t_1 t_2 t_3 t_4 t_5) \dots$, is ambiguous, since always will exist at least three possible reached markings, either $M_0\{p_1, p_3\}$, $M_0\{p_1, p_7\}$ or $M_0\{2p_1\}$.*

Based on the previous idea, the next theorem provides characterizes the MD in non-safe OSS.

Theorem 3. A non-safe OSS (B, M_0, φ) is MD if and only if $\ker \varphi = \emptyset$.

Proof: (Necessity) Suppose that (B, M_0, φ) is MD but, on the contrary, $\ker \varphi \neq \emptyset$. Then, there exist at least, a linear combination of the columns in φ such that

$$\alpha_i \varphi(:, i) + \dots + \alpha_j \varphi(:, j) + \dots + \alpha_k \varphi(:, k) + \dots + \alpha_l \varphi(:, l) = \vec{0},$$

where not all the α 's coefficients are zero. Since by definition φ is non-negative, it must exist a re-arrangement of this linear combination where the elements at both sides of the equality have positive signs. Without loss of generality let

$$\alpha_i \varphi(:, i) + \dots + \alpha_j \varphi(:, j) = \alpha_k \varphi(:, k) + \dots + \alpha_l \varphi(:, l)$$

be such an arrangement. Then, by using an enough number of tokens, it is possible the construction of markings $M_0\{\alpha_i p_i + \dots + \alpha_j p_j\}$ and $M'_0\{\alpha_k p_k + \dots + \alpha_l p_l\}$ as initial conditions of the net. It is easy to note that

$$\begin{aligned} \varphi(2M_0\{\alpha_i p_i + \dots + \alpha_j p_j\}) &= \\ &= \varphi(M_0\{\alpha_i p_i + \dots + \alpha_j p_j\}) + \\ &+ M'_0\{\alpha_k p_k + \dots + \alpha_l p_l\} \end{aligned}$$

i.e., these markings are confusing to each other. Because the net is well-formed then, it is connected and covered by circuits (see Pag. 106, Theorem 5.18 in [4]). Thus, it must exist a circuit of minimum size including the set of places $\{p_i, \dots, p_j\}$ which are marked at $M_0\{\alpha_i p_i + \dots + \alpha_j p_j\}$. Let $\gamma_{\{p_i, \dots, p_j\}} = \gamma p_i t_i \dots p_j t_j \delta$ be such a circuit, where $\diamond t_i = \{p_i\}, \dots, \diamond t_j = \{p_j\}$ for the transitions t_i, \dots, t_j in $\gamma_{\{p_i, \dots, p_j\}}$. Let $\sigma_{\{p_i, \dots, p_j\}} = t_i \dots t_j$ be the ordered sequence formed by the transitions in $\gamma_{\{p_i, \dots, p_j\}}$. Clearly, $\sigma_{\{p_i, \dots, p_j\}} \in L(B, M_0)$, since the marking $M_0\{\alpha_i p_i + \dots + \alpha_j p_j\}$ is enabling its firing. Since $\sigma_{\{p_i, \dots, p_j\}}$ is a T-invariant (see Pag. 45, Lemma 3.9 in [4]) then, $(\sigma_{\{p_i, \dots, p_j\}})^* \in L(B, M_0)$. By monotonicity, it is clear that both markings, $M_0\{\alpha_i p_i + \dots + \alpha_j p_j\} + M'_0\{\alpha_k p_k + \dots + \alpha_l p_l\}$ and $2M_0\{\alpha_i p_i + \dots + \alpha_j p_j\}$, enables the firing of $\sigma_{\{p_i, \dots, p_j\}}$. Consequently, there is an ambiguity about the marking reached by the firing of the transition sequence $\sigma_{\{p_i, \dots, p_j\}}$. That is,

$$\begin{aligned} \varphi(M_0\{\alpha_i p_i + \dots + \alpha_j p_j\} + \\ + M'_0\{\alpha_k p_k + \dots + \alpha_l p_l\}) + B\vec{\sigma}_{\{p_i, \dots, p_j\}} &= \\ = \varphi(2M_0\{\alpha_i p_i + \dots + \alpha_j p_j\}) + \varphi(B\vec{\sigma}_{\{p_i, \dots, p_j\}}) &= \\ = \varphi(2M_0\{\alpha_i p_i + \dots + \alpha_j p_j\}) + B\vec{\sigma}_{\{p_i, \dots, p_j\}} \end{aligned}$$

Moreover, since $B\vec{\sigma}_{\{p_i, \dots, p_j\}} = \vec{0}$, then, $(\sigma_{\{p_i, \dots, p_j\}})^*$ is always enabled. A contradiction to the MD of the net.

(Sufficiency) Suppose that $\ker \varphi = \emptyset$. Then φ is invertible. Thus, for any initial output of the system y_0 there exists a unique marking M_0 such that $M_0 = \varphi^{-1} y_0$. Moreover, for every y_k it is possible to compute $M_k = \varphi^{-1} y_k$, as required.

Clearly, the Theorem 3 implies the Theorem 2. Thus, the following corollary holds.

Corollary 3. Let (B, M_0, φ) be a non-safe OSS. Then, (B, M_0, φ) is observable iff $\ker \varphi = \emptyset$.

The following example illustrates the result.

Example 6. Consider the Figure 2. The Theorem 2 requires E_B to be empty. Thus, subscripts have been added to B_1, B_2 and C_1, C_2 . It is easy to note that E_B is empty now. Accordingly, an observer is able to track any transitions sequence in the net. However, the extra token in p_1 is yet undiscoverable, since an observer is unable to decide in which of the places, either, p_1, p_3 nor p_7 , the extra token remains. The Theorem 3 requires that φ be of full range. Thus, subscripts have been added for sensor A , as well. It is easy to verify that now $\ker \varphi = \emptyset$. Notice that the Theorem 3 could be achieved with more efficient solutions, however, the optimal sensor placement is part of the future work.

If, for a particular problem the only interest is the Sequence-Detectability of a DES then, the Theorem 2 is the unique requirement, and accordingly, the Theorem 3 could be avoided.

5. Conclusions

This paper addressed the analysis of the sequence detection in DES modelled by PN's. The work is focused on well-formed OSS. The characterization of the property was firstly developed for safe OSS. Then, the safeness

requirement was relaxed, and its implications in the sequence detection analyzed. The utility of the Sequence-Detectability in the testing of the observability of the OSS was analyzed. An example developed through the paper was used as an illustration of the main results. The future work includes the analysis of the relationship between the Myhill–Nerode theorem and the method for obtaining the Sequence-Detectability table proposed in this paper.

REFERENCES

1. CAMPOS-RODRIGUEZ, R., M. ALCARAZ-MEJIA, **A Matlab/Simulink Framework for the Design of Controllers and Observers for Discrete-Event Systems**, Journal of Electronics and Electrical Engineering, vol. 99(3), 2010, pp. 63-68.
2. CAMPOS-RODRIGUEZ, R., M. ALCARAZ-MEJIA, J. MIRELES-GARCIA, **Supervisory Control of Discrete Event Systems Using Observers**, Proc. of the IEEE Mediterranean Conference on Control and Automation, vol. 1, 2007, pp. 1-7.
3. CAMPOS-RODRIGUEZ, R., A. RAMIREZ-TREVINO, E. LOPEZ-MELLADO, **Observability Analysis of Free-Choice Petri Net Models**, Proceedings of IEEE/SMC Conference on Systems of Systems Engineering vol. 6, 2006, pp. 77-82.
4. DESEL, J., J. ESPARZA, **Free Choice Petri Nets**, Cambridge Univ. Press, 1995.
5. GIUA, A., C. SEATZU, **Observability of Place / Transition Nets**, IEEE Transactions on Automatic Control, vol. 47(9), 2002, pp. 1424-1437.
6. GIUA, A., C. SEATZU, F. BASILE, **Observer-based State-feedback Control of Timed Petri Nets with Deadlock Recovery**, IEEE Transaction on Automatic Control, vol. 49(1), 2004, pp. 17-29.
7. HERNANDEZ-MARTINEZ, E. G., E. S. PUGA-VELAZQUEZ, S. A. FOYO-VALDES, J. A. MEDA-CAMPANA, **Modeling Framework for Automated Manufacturing Systems Based on Petri Nets and ISA Standards**, Studies in Informatics and Control, vol. 22 (2), 2013, pp. 163-174.
8. HOPCROFT, J. E., R. MOTWANI, J. D. ULLMAN, **Introduction to Automata Theory, Languages, and Computation**, Addison-Wesley, 2001.
9. KAMMOUN, M. A., N. REZG, Z. ACHOUR, S. REZIG, **State Space Search for Safe Time Petri Nets Based on Binary Decision Diagrams Tools: Application to Air Traffic Flow Management Problem**, Studies in Informatics and Control, vol. 25(1), 2016, pp. 39-50.
10. KUMAR, R., M. A. SHAYMAN, **Formulae Relating Controllability Observability and Co-observability**, Automatica, 1998, pp. 211-215.
11. MINCA, E., **A New Approach for the Control Optimization of an Assembly/Disassembly Mechatronics Line Served by an Autonomous Robotic System**, Studies in Informatics and Control, vol. 23 (1), 2014, pp. 13-22.
12. MOODY, J. O., P. J. ANTSAKLIS, **Petri Net Supervisors for DES with Uncontrollable and Unobservable Transitions**, IEEE Transactions on Automatic Control, vol. 45(3), 2000, pp. 462-476.
13. MURATA, T., **Petri Nets: Properties, Analysis and Applications**, Proceedings of the IEEE, vol. 77(4), 1989, pp. 541-580.
14. OZVEREN, C. M., A. S. WILLSKY, **Observability of Discrete Event Dynamic Systems**, IEEE Transactions on Automatic Control, vol. 35(7), 1990, pp. 797-806.
15. OZVEREN, C. M., A. S. WILLSKY, **Tracking and Restrictability in Discrete-event Dynamic Systems**, SIAM Journal of Control and Optimisation, vol. 30(6), 1992, pp. 1423-1446.
16. RIVERA-RANGEL, I., A. RAMÍREZ-TREVIÑO, L. AGUIRRE-SALAS, J. RUIZ-LEON, **Geometrical Characterization of Observability in Interpreted Petri Nets**, Kybernetika, vol. 41(5), 2005, pp. 553-574.
17. SHU, S., F. LIN, H. YING, **Detectability of Discrete Event Systems**, IEEE

- Transactions on Automatic Control, vol. 52(12), 2007, pp. 2356-2359.
18. SHU, S., F. LIN, **Detectability for Discrete Event Systems with Dynamic Event Observation**, System Control Letters, vol. 59(1), 2010, pp. 9-17.
 19. SHU, S., F. LIN, **I-Detectability of Discrete-Event Systems**, IEEE Transactions of Automatic Science and Engineering, vol. 10(1), 2013, pp. 187-196.
 20. TAKAI, S., T. USHIO, S. KODAMA, **Static-state Feedback Control of Discrete-event Systems under Partial Observation**, IEEE Transaction on Automatic Control, vol. 40(11), 1995, pp. 1950-1954.
 21. WONG, K. C., W. M. WONHAM, **On the Computation of Observers in Discrete-Event Systems**, Discrete Event Dynamic Systems, vol. 14(1), 2004, pp. 55-107.