Asset Allocation Models in Discrete Variable

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Abstract: In the classical portfolio selection theory the value of the assets is considered infinitely divisible. In the real portfolio selection models one should consider only finitely divisible assets. This is because the investors purchase only a finite number of shares or minimum transaction lots. We present several asset allocation models in discrete variable and we make an analysis of the results. Our models are closer to reality but they are more difficult to be solved.

Keywords: portfolio selection, asset allocation, finitely divisible assets, minimum transaction lots, integer programming model

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1. Introduction

The original Markowitz model of portfolio selection has received a widespread theoretical acceptance and it has been the basis for various portfolio selection techniques. The model is known in the literature also, as the mean-variance portfolio selection model. Generally, in the classical mean-variance portfolio selection approach several realistic features are not taken into account.

Among these ”forgotten” aspects, one of particular interest is the not infinite divisibility of the financial asset to select, i.e. the obligation to buy/sell only integer quantities of asset lots which contain a predetermined number of shares.

The portfolio selection problems with discrete constraints (such as buy-in thresholds, cardinality constraints and transaction roundlot restrictions) were studied in the literature mainly in the last decade. Integer programming approaches in the mean-risk models were studied in [3]-[14], [16], [17]. Mansini and Speranza in [9] consider the constraint stating that assets can be traded only in indivisible lots of fixed size. In this case, the problem is formulated in terms of integer-valued variables — as opposed to real-valued ones — that represent, for each asset, the number of purchased lots, instead of the real-valued ones.

Given that assets are normally composed by units, this constraint is certainly meaningful; its practical importance however depends on the ratio between the size of the minimum trading lots and the size of the shares involved in the portfolio. In [4] Corazza and Favaretto study the existence problem for the solutions of a discrete mean–variance portfolio selection model.

Mansini and Speranza in [10] consider a single-period mean-safety portfolio selection problem with transaction costs and integer constraints on the quantities selected for the securities (rounds). They propose an exact approach based on the partition of the initial problem into two sub-problems and the use of a simple local search heuristic to obtain an initial solution.

In [6], [9], [11] and [17] are presented heuristic algorithms for the portfolio selection problem with minimum transaction lots.

In [3] is presented an approach of the mean-variance portfolio selection with the nonlinear mixed-integer programming. The authors proposed an algorithm (which is based on the proposed conditions) for finding a "good" feasible solution and proved its convergence.

In [8] the classic mean-variance framework is extend to a broad class of investment decisions under risk where investors select optimal portfolios of risky assets that include perfectly divisible as well as perfectly indivisible assets. The author develops an algorithm for solving the associated mixed-integer nonlinear program and report on the results of a computational study.

In [5] are examined the effects of applying buy-in thresholds, cardinality constraints and transaction roundlot restrictions to the portfolio selection problem. Such discrete constraints are of practical importance but make the efficient frontier discontinuous. The resulting quadratic mixed-integer (QMIP) problems are NP-hard and therefore computing the entire efficient frontier is computationally challenging. The authors proposed alternative approaches for computing this frontier and provided insight into its discontinuous structure.

A fundamental question in the mathematical finance is how risk should be measured properly. The mean-variance models behave well in the case when the distribution of the random vector of returns is close to a multivariate normal distribution, and not so well in other cases. An important problem is to build portfolio selection models based on risk measures that capture risk adequately. In 1952, Roy [15] suggested that investors are interested in selecting a portfolio so as to maximize the probability of achieving at least a given return. The idea is that if return falls below the threshold there will be some bad consequence. This model of investor behavior is called safety-first. Drawing the efficient frontier in standard deviation - expected return space, the portfolio which maximizes the probability of realized return being greater than the threshold, can be found by identifying the straight line passing through the expected return axis at the threshold that is tangent to efficient frontier. The portfolio at the point of tangency is the desired portfolio.

The development of the theory of stochastic dominance had stimulated the research for asymmetric risk measures (downside risk measures) like: shortfall probability, expectations of loss, semi-variance and lower partial moments. An important class of risk measures considered in the literature is the coherent risk measures [1], [14].

A new tool for financial analysis is the Omega function [2]. If X is a random variable denote by F_X the cumulative distribution function of X. The Omega function associated to X and to the interval $[a, b]$ is defined as follows:

$$\Omega_X(r) = \frac{\int_r^b \left(1 - F_X(x)\right) dx}{\int_a^r F_X(x)\, dx}, \quad r \in [a, b]$$

The Omega function has the advantage that incorporates all the information of the returns. The evolution of the Omega function over the time provides a complete picture of performance and risk.

In order to consider the discrete feature, we build several linear integer programming problems. The present paper continues the ideas from [13]. The risk measure considered in this paper is the lower partial moment of the first order. We propose a formulation of this problem in terms of quantities, i.e. integer numbers of asset lots to buy, instead of starting capital percentages to invest. We give necessary and sufficient conditions for the existence of feasible solution(s) with a sequence of steps to be followed by the investor when he wants to make investment decisions in the presence of minimum transaction lots. A numerical example illustrating the previous points is presented.

2. Mean-risk Models with Minimum Lot Constraints

Denote by h_i the price for the minimum lot of asset *i*. In case that there are no minimum lots, then h_i is the price of a share of asset i. The integer variable x_i represents the number of the minimum lots for the asset *i*. Let a particular portfolio be defined by a vector $x = (x_1, x_2, ..., x_n)^T \in \mathbf{Z}^n$ where the integer variable x_i represents the number of the minimum lots invested in the asset *i*.

Let the assets returns be represented by a vector of random variables $\xi = (\xi_1, \xi_2, ..., \xi_n)^T$ with means $\mu = (\mu_1, \mu_2, ..., \mu_n)^T$. Denote **H**=diag(**h**), that is **H** is the diagonal matrix whose diagonal is equal to vector **h**. Consider the vector **d**$=(d_1, d_2, \ldots, d_n)^T$, $d_i = h_i \mu_i$ for all $1 \le i \le n$. Denote $\bar{x}_i = h_i x_i, 1 \le i \le n$ and consider the vector $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)^T$.

Then $\bar{x}_i$ is the sum invested by the investor in asset *i*.

Denote by t_+ the positive part of the real number *t*, that is:

$$t_+ = \frac{|t| + t}{2} = \max(t, 0).$$

The risk of the investment in portfolio **x** with respect the target τ is defined as the lower partial moment of the first order of the return $\xi^T \bar{\mathbf{x}}$, that is $\mathrm{LPM}_\tau(\mathbf{x}) = E[(\tau - \xi^T \bar{\mathbf{x}})_+]$. Note that the average return is $E(\xi^T \bar{\mathbf{x}}) = \boldsymbol{\mu}^T \bar{\mathbf{x}} = \mathbf{d}^T \mathbf{x}$. Let W be the minimum level of the expected return desired by the investor and M_1 and M_2 be the upper and lower bounds for the investment sum.

Then the purpose of the minimum risk portfolio selection problem with minimum lot constraints is to find that value of **x** that will

$$(P_1)\begin{cases} \text{minimize} \quad E\left[\left(\tau - \xi^{\mathrm{T}} \bar{\mathbf{x}}\right)_+\right] \\ M_1 \le \mathbf{h}^{\mathrm{T}} \mathbf{x} \le M_2,\ \mathbf{d}^{\mathrm{T}} \mathbf{x} \ge W, \mathbf{x} \in \mathbf{Z}^n \text{ and } \mathbf{x} \ge 0 \end{cases}$$

The purpose of the maximum return portfolio selection problem with minimum lot constraints is to find that value of **x** that will

$$(P_2)\begin{cases} \text{maximize} \quad (\mathbf{d}^{\mathrm{T}} \mathbf{x}) \\ M_1 \le \mathbf{h}^{\mathrm{T}} \mathbf{x} \le M_2, E\left[\left(\tau - \xi^{\mathrm{T}} \bar{\mathbf{x}}\right)_+\right] \le r, \mathbf{x} \in \mathbf{Z}^n \text{ and } \mathbf{x} \ge 0 \end{cases}$$

Here *r* is an upper bound for the risk accepted by the investor.

In case the risk aversion of the investor is taken into account then we can formulate the trade-off mean-risk portfolio selection problem:

$$(P_3)\begin{cases} \text{minimize} \quad \left((1-\theta) E\left[\left(\tau - \xi^{\mathrm{T}} \bar{\mathbf{x}}\right)_+\right] - \theta\, \mathbf{d}^T \mathbf{x}\right) \\ M_1 \le \mathbf{h}^{\mathrm{T}} \mathbf{x} \le M_2,\ \mathbf{x} \in \mathbf{Z}^n \text{ and } \mathbf{x} \ge 0 \end{cases}$$

Here $\theta \in [0,1]$ is the coefficient of risk aversion. The case when θ takes small values corresponds to a risk-avoiding investor who is more worried about below average returns than attracted by the above average gains. The case when θ takes great values corresponds to a risk-loving investor who is more attracted about above-average gains than deterred by below-average returns. By varying θ from 0 to 1 all efficient portfolios can be determined.

In the problems (P_1), (P_2) and (P_3) the restriction $\mathbf{x} \ge 0$ can be replaced with the more realistic restriction $0 \le \mathbf{x} \le \mathbf{a}$. The upper bounding constraints exist for legal, personal or institutional reasons. Some components of the vector a may be infinite.

Suppose a previous history of asset returns is available at several moments of time t=1,2,...,m.

Consider a time horizon composed of several moments t = 1,2,…,m. Denote by $\mathbf{R} = (r_{ti})$ the $m \times n$ matrix of historical rates of returns of the assets. r_{ti} is the rate of return at moment t for asset i. Consider the matrix $\overline{\mathbf{R}} = (\bar{r}_{ti})$, $\overline{\mathbf{R}} = \mathbf{R}\,\mathbf{H}$. Note that $\bar{r}_{ti} = r_{ti} h_i$ for every $t \in \{1,2,\dots,m\}$, $i \in \{1,2,\dots,n\}$. In the following we shall use the following estimates:

$c_i = \frac{\sum_{t=1}^{m} r_{ti}}{m}$ is an estimation for the expected return μ_i of asset i,

$\frac{\sum_{t=1}^{m} r_{ti} h_i}{m}$ is an estimation for the entry d_i of the vector $\mathbf{d}$,

$\frac{\sum_{t=1}^{m}\sum_{i=1}^{n} r_{ti} h_i x_i}{m}$ is an estimation for the expected return $E(\xi^T \overline{\mathbf{x}})$ of the portfolio $\mathbf{x}$

$\frac{1}{m}\sum_{t=1}^{m}\left[\tau - \sum_{i=1}^{n} \bar{r}_{ti} x_i\right]_+$ is an estimation for the risk $E\left[(\tau - \xi^T \overline{\mathbf{x}})_+\right]$ of the portfolio $\mathbf{x}$

Denote $\mathbf{c} = (c_1, c_2, \dots, c_n)$. Taking into account the above estimations we shall associate to the models (P_1), (P_2) and (P_3) the following models:

$$(P_1')\begin{cases} \text{minimize} \quad \frac{1}{m}\sum_{t=1}^{m}\left[\tau - \sum_{i=1}^{n} \bar{r}_{ti} x_i\right]_+ \\ M_1 \le \mathbf{h}^T\mathbf{x} \le M_2,\ \mathbf{d}^T\mathbf{x} \ge W, \mathbf{x} \in \mathbf{Z}^n \text{ and } \mathbf{x} \ge 0 \end{cases}$$

$$(P_2')\begin{cases} \text{maximize} \quad (\mathbf{d}^T\mathbf{x}) \\ M_1 \le \mathbf{h}^T\mathbf{x} \le M_2, \\ \frac{1}{m}\sum_{t=1}^{m}\left[\tau - \sum_{i=1}^{n} \bar{r}_{ti} x_i\right]_+ \le r, \\ \mathbf{x} \in \mathbf{Z}^n \text{ and } \mathbf{x} \ge 0 \end{cases}$$

$$(P_3')\begin{cases} \text{minimize } \left((1-\theta)\frac{1}{m}\sum_{t=1}^{m}\left[\tau - \sum_{i=1}^{n} \bar{r}_{ti} x_i\right]_+ - \theta\,\mathbf{d}^T\mathbf{x}\right) \\ M_1 \le \mathbf{h}^T\mathbf{x} \le M_2,\ \mathbf{x} \in \mathbf{Z}^n \text{ and } \mathbf{x} \ge 0 \end{cases}$$

In order to show that the above models are equivalent to mixed-integer linear models we need the following lemma.

Lemma. Let τ be a real number, K be a nonempty set and $f_i : K \to \mathbf{R}$, i = 1,2,3,…,n be a set of n functions. Denote:

$$L(\tau) = \left\{(x, \mathbf{y}) \in K \times \mathbf{R}_+^n : \tau - f_i(x) \le y_i,\ i = 1,2,\dots,n,\ \mathbf{y} = (y_1, y_2, \dots, y_n)\right\}$$

Consider the problems:

$$(Q_1) \quad \min\left\{\sum_{i=1}^{n} (\tau - f_i(x))_+ : x \in K\right\}$$

$$(Q_2) \quad \min\left\{\sum_{i=1}^{n} y_i : (x, \mathbf{y}) \in L(\tau)\right\}$$

and their optimal values

$l_1(\tau) = \min\left\{\sum_{i=1}^{n} (\tau - f_i(x))_+ : x \in K\right\}$ and

$$l_2(\tau) = \min\left\{\sum_{i=1}^{n} y_i : (x, \mathbf{y}) \in L(\tau)\right\}.$$

Suppose that there exist $x_0 \in K$ and $(x_1, \mathbf{y}_1) \in L(\tau)$ such that

$l_1(\tau) = \sum_{i=1}^{n} (\tau - f_i(x_0))_+$ and $l_2(\tau) = \sum_{i=1}^{n} y_{1i}$

where $\mathbf{y}_1 = (y_{11}, y_{12}, \dots, y_{1n})$.

Then the two problems (Q_1) and (Q_2) are equivalent, that is

$$l_1(\tau) = l_2(\tau) = \sum_{i=1}^{n} (\tau - f_i(x_1))_+ .$$

Proof. Let $y_{0i} = (\tau - f_i(x_0))_+$, i = 1,2,…,n and $\mathbf{y}_0 = (y_{01}, y_{02}, \dots, y_{0n})$. Note that

$(x_0, \mathbf{y}_0) \in L(\tau)$, hence

$l_2(\tau) \le \sum_{i=1}^{n} y_{0i} = l_1(\tau)$. From

$$l_1(\tau) = \sum_{i=1}^{n} (\tau - f_i(x_0))_+ \le \sum_{i=1}^{n} (\tau - f_i(x_1))_+ \le \sum_{i=1}^{n} y_{1i} = l_2(\tau)$$

The conclusion of the lemma follows.

From the above lemma it follows that the models $(P_1'), (P_2')$ and (P_3') are equivalent to the following mixed-integer linear models:

$$(P_1'')\begin{cases} \text{minimize} \quad \frac{1}{m}\sum_{t=1}^{m} y_t \\ \tau - \sum_{i=1}^{n} \bar{r}_{ti} x_i \le y_t, \; t = 1, 2, \ldots, m \\ M_1 \le \mathbf{h}^{\mathrm{T}}\mathbf{x} \le M_2, \; \mathbf{d}^{\mathrm{T}}\mathbf{x} \ge W, \\ \mathbf{x} \in \mathbf{Z}^n, \mathbf{y} \in \mathbf{R}^m \text{ and } \mathbf{x}, \mathbf{y} \ge 0 \end{cases}$$

$$(P_2'')\begin{cases} \text{maximize} \quad (\mathbf{d}^{\mathrm{T}}\mathbf{x}) \\ \tau - \sum_{i=1}^{n} \bar{r}_{ti} x_i \le y_t, \; t = 1, 2, \ldots, m \\ M_1 \le \mathbf{h}^{\mathrm{T}}\mathbf{x} \le M_2, \\ \frac{1}{m}\sum_{t=1}^{m} y_t \le r, \\ \mathbf{x} \in \mathbf{Z}^n, \mathbf{y} \in \mathbf{R}^m \text{ and } \mathbf{x}, \mathbf{y} \ge 0 \end{cases}$$

$$(P_3'')\begin{cases} \text{minimize} \left[(1-\theta)\frac{1}{m}\sum_{t=1}^{m} y_t - \theta\, \mathbf{d}^T\mathbf{x} \right] \\ \tau - \sum_{i=1}^{n} \bar{r}_{ti} x_i \le y_t, \; t = 1, 2, \ldots, m \\ M_1 \le \mathbf{h}^{\mathrm{T}}\mathbf{x} \le M_2, \\ \mathbf{x} \in \mathbf{Z}^n, \mathbf{y} \in \mathbf{R}^m \text{ and } \mathbf{x}, \mathbf{y} \ge 0 \end{cases}$$

3. The Structure of the Investment Process for the Minimum Risk Model

The structure of the investment process is presented in the following.

Step 1. The investor chooses a set S of assets where he intends to invest his money and a sum of money M_2 to be invested in the assets.

Step 2. The investor gathers information (historical data) from the stock market and forecasts from financial experts about the chosen set of assets. As a result he obtains a matrix $\mathbf{R} = (r_{ti})$ where r_{ti} is the rate of return at moment t for asset i. He finds also the vector h of minimum transaction lots of the assets from S.

Step 3. Starting from the matrix R and taking into account some statistical hypotheses about the returns vector, the investor computes estimates for the vector μ of mean returns.

Step 4. Starting from M_2 the investor computes

$$M_0 = \max\left\{\mathbf{h}^T\mathbf{x} : \mathbf{h}^T\mathbf{x} \le M_2, \mathbf{x} \in \mathbf{Z}^n, \mathbf{x} \ge 0\right\}$$

Step 5. The investor chooses a lower bound M_1 for the sum $\mathbf{h}^T\mathbf{x}$ invested in the assets from S in the range $[0, M_0]$.

Step 6. The investor computes

$$W_1 = \min\left\{\mathbf{d}^T\mathbf{x} : M_1 \le \mathbf{h}^T\mathbf{x} \le M_2, \mathbf{x} \in \mathbf{Z}^n, \mathbf{x} \ge 0\right\}$$

$$W_2 = \max\left\{\mathbf{d}^T\mathbf{x} : M_1 \le \mathbf{h}^T\mathbf{x} \le M_2, \mathbf{x} \in \mathbf{Z}^n, \mathbf{x} \ge 0\right\}$$

Step 7. The investor compares W_2, that is the greatest mean rate of return to the risk-less rate of return and makes a decision.

In case he decides not to make an investment in the assets from S then he returns at step 1. and chooses another set of assets S'.

In case he decides to make an investment in the assets from S then he goes to the next step.

Step 8. The investor chooses a lower bound W, for the return of his investment, in the range $[W_1, W_2]$.

Step 9. The investor computes

$$\tau_1 = \min\left\{\sum_{i=1}^{n} \bar{r}_{ti} x_i : t = 1, 2, \ldots, m, M_1 \le \mathbf{h}^T\mathbf{x} \le M_2, \mathbf{d}^{\mathrm{T}}\mathbf{x} \ge W, \mathbf{x} \in \mathbf{Z}^n \text{ and } \mathbf{x} \ge 0\right\}$$

$$\tau_2 = \max\left\{\sum_{i=1}^{n} \bar{r}_{ti} x_i : t = 1, 2, \ldots, m, M_1 \le \mathbf{h}^T\mathbf{x} \le M_2, \mathbf{d}^{\mathrm{T}}\mathbf{x} \ge W, \mathbf{x} \in \mathbf{Z}^n \text{ and } \mathbf{x} \ge 0\right\}$$

and chooses τ in the range $[\tau_1, \tau_2]$.

Step 10. The investor solves model (P_1'') and find an optimal portfolio $\mathbf{x}^*$. In case the investor wants to continue his research he can start again from step 1. If this is not the case then the investor stops.

Numerical Results

In the following we shall illustrate how the minimum risk model works. We shall consider n = 7 assets and m = 8 moments of time (the years 2000,2001,…,2007). Take $M_2 = 1000000$ euros. The entries of the matrix $\mathbf{R}=(r_{ti})$ of historical rates of return are displayed in table 1.

The investor chooses M_1=500000 and computes the range for the parameter W. He finds W_1=124380 euros and W_2=450000 euros. Then the investor chooses W=230000 euros and computes the range for the parameter τ. He finds τ_1= 92463 euros, τ_2=659000 euros. We divide the range $[\tau_1, \tau_2]$ in ten equal intervals $[\tau'_i, \tau'_{i+1}]$, i = 1,2,…,10. The composition of the optimal portfolios for the minimum risk problem (P''_1), for various values of parameter τ are displayed in table 2.

Table 1. Matrix R=(r_{ti}) of historical rates of return.

Year	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7
2000	0.30	0.30	0.55	0.36	0.34	0.39	0.30
2001	0.54	0.26	0.44	0.31	0.30	0.26	0.44
2002	0.25	0.43	0.51	0.32	0.34	0.43	0.51
2003	0.20	0.68	0.61	0.52	0.24	0.23	0.61
2004	0.10	0.53	0.61	0.52	0.54	0.53	0.51
2005	0.13	0.44	0.71	0.68	0.66	0.25	0.21
2006	0.17	0.34	0.51	0.48	0.56	0.55	0.51
2007	0.30	0.34	0.50	0.47	0.55	0.56	0.51

Table 2. Composition of the optimal portfolio for various values of parameter τ

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5	Asset 6	Asset 7
τ'_1= 92463	92463	0	0	0	0	0	0
τ'_2=149116,7	86074	0	0	0	1478	0	9347
τ'_3=205770,4	75960	0	0	0	20033	0	0
τ'_4=262424,1	60234	0	0	0	22660	0	9672
τ'_5=319077,8	12809	0	0	0	0	0	67070
τ'_6=375731,5	33548	0	80622	1	21784	0	0
τ'_7=432385,2	23974	0	100000	6	0	36085	8916
τ'_8=489038,9	3154	0	100000	33575	0	0	30870
τ'_9=545692,6	0	0	100000	10376	14063	0	23884
τ'_{10}=602346,3	0	0	100000	100000	1	0	23076
τ'_{11}=659000	0	0	100000	100000	1	0	23076

The prices of the minimum lot of assets, that is the components of the vector $\mathbf{h}$, are $h_1 = 10$ euros, $h_2 = 7$ euros, $h_3 = 5$ euros, $h_4 = 2$ euros, $h_5 = 12$ euros, $h_6 = 4$ euros, $h_7 = 13$ euros.

One can easily see that M_0= 1000000 euros.

4. Conclusions

We have presented several nonlinear integer programming models for the portfolio selection with minimum transaction lots. The risk in the models is measured by the lower partial moment of the first order. We determine all the steps the investor must perform in order to make an optimal investment for the minimum risk model. The composition of the optimal portfolio for various values of parameter τ is computed.

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