Determination of McMillan Degree via Hankel Block Matrices Associated to Taylor Series Expansion

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Abstract: It is shown that the McMillan degree of a strictly proper rational matrix $G(s)$ can be determined by using a sufficiently large Hankel block matrix associated to the Taylor series expansion of $G(s)$ at $s=0$.

Keywords: McMillan degree, strictly proper rational matrix, Hankel block matrix, Taylor parameter matrices.

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1. Introduction

The McMillan degree of a strictly proper $(p\times m)$ rational matrix $G(s)$, $s\in\mathrm{C}$, is the (integer) degree $n\geq 1$ of its (monic) *pole polynomial* $p(s)$. Assuming that in every entry of $G(s)$ the numerator and the denumerator are co-prime, the pole polynomial $p(s)$ is the least common denominator of all the non-identically zero minors of all orders of $G(s)$.

Let the entries of $G(s)$ have real coefficients and let $(\hat{A},\,\hat{B},\,\hat{C})$ be a *realisation of order* r of $G(s)$ such that:

$$G(s)=\hat{C}(I_r s-\hat{A})^{-1}\hat{B},\ \hat{A}\in\mathrm{R}^{r\times r},\ \hat{B}\in\mathrm{R}^{r\times m},\ \hat{C}\in\mathrm{R}^{p\times r}, \tag{1}$$

where I_r is the unit matrix of order r.

From the systemic point of view the McMillan degree $n\,£\;r$ corresponds to the minimum number of independent energy/substance-storing elements requested to represent the input – state – output dynamical transfer of the system described by the strictly proper transfer matrix $G(s)$. In this context, n is, at the same time, the order of any *minimal realisation* (A,B,C) of $G(s)$ for which

$$G(s)=C(I_n s-A)^{-1}B,\ A\in R^{n\times n},\ B\in R^{n\times m},\ C\in R^{p\times n}. \tag{2}$$

Equivalently, the pairs (A,B) and (C,A) are *completely controllable* and *completely observable* respectively, i.e. the controllability matrix C_μ and the observability matrix O_ν, defined respectively by:

$$\mathrm{C}_\mu=\begin{bmatrix}B & AB & \dots & A^{\mu-1}B\end{bmatrix}, \tag{3}$$

$$\mathrm{O}_\nu=\begin{bmatrix}C^* & A^*C^* & \dots & (A^*)^{\nu-1}C^*\end{bmatrix}^*, \tag{4}$$

(superscript * stands for "transposition") satisfy the conditions:

$$\operatorname{rank} \mathrm{C}_{\mu} = \operatorname{rank} \mathrm{C}_{n} = n, \ \forall \mu \geq n, \tag{5}$$

$$\operatorname{rank} \mathrm{O}_{\nu} = \operatorname{rank} \mathrm{O}_{n} = n, \ \forall \nu \geq n \ . \tag{6}$$

By analogy with (3) and (4), the controllability and observability matrices

$$\hat{\mathrm{C}}_{\delta} = \left[\hat{B} \quad \hat{A}\hat{B} \quad \ldots \quad \hat{A}^{\delta-1}\hat{B} \right], \tag{7}$$

$$\hat{\mathrm{O}}_{\gamma} = \left[\hat{C}^{*} \quad \hat{A}^{*}\hat{C}^{*} \quad \ldots \quad (\hat{A}^{*})^{\gamma-1}\hat{C}^{*} \right]^{*} \tag{8}$$

may be respectively defined, for which the following properties hold:

$$\operatorname{rank} \hat{\mathrm{C}}_{\delta} = \operatorname{rank} \hat{\mathrm{C}}_{r} = \operatorname{rank} \mathrm{C}_{n} = n, \ \forall \delta \in \{n, n+1, \ldots.\}, \ \forall r \geq n, \tag{9}$$

$$\operatorname{rank} \hat{\mathrm{O}}_{\gamma} = \operatorname{rank} \hat{\mathrm{O}}_{r} = \operatorname{rank} \mathrm{O}_{n} = n, \ \forall \gamma \in \{n, n+1, \ldots.\}, \ \forall r \geq n. \tag{10}$$

In connection with the realisation problem of $G(s)$ it is well known, [1], [2], [3], that an alternate procedure for the determination of McMillan degree (besides that one of computation of $p(s)$ directly from $G(s)$, which was evoked from the very beginning of this paper) relies on the following *Laurent series expansion* of $G(s)$ at $s = \infty$:

$$G(s) = \sum_{k=1}^{\infty} M_k s^{-k} , \tag{11}$$

where M_k, $k = 1, 2, \ldots,$ are the *Markov parameter matrices* of $G(s)$. The procedure consists in the building of the *Hankel block matrices* $\mathrm{H}_{i,j}^{M}$ whose entries are the Markov parameter matrices (superscript *M* stands for "Markov parameter matrices") from Laurent series (11):

$$\mathrm{H}_{i,j}^{M} = \begin{bmatrix} M_1 & M_2 & \ldots & M_j \\ M_2 & M_3 & \ldots & M_{j+1} \\ \vdots & \vdots & & \vdots \\ M_i & M_{i+1} & \ldots & M_{i+j-1} \end{bmatrix} \in \mathrm{R}^{ip \times jm}, \quad i, j = 1, 2, \ldots , \tag{12}$$

and in applying the following result.

Theorem 1

Let $\alpha, \beta \geq 1$ be the smallest integers such that

$$\operatorname{rank} \mathrm{H}_{\alpha,\beta}^{M} = \operatorname{rank} \mathrm{H}_{\alpha+i,\beta+j}^{M}, \quad i, j = 1, 2, \ldots . \tag{13}$$

Then $\alpha = \beta = n$, where n is the order of the minimal realisation of $G(s)$. □

Let (1) be any realisation of $G(s)$, not necessarily minimal. Taking into account that $(I_r s - \hat{A})^{-1}$ is the sum of the *infinite geometric series* (the series expansion at $s = \infty$):

$$(I_r s - \hat{A})^{-1} = \sum_{k=1}^{\infty} \hat{A}^{k-1} s^{-k} , \tag{14}$$

it follows from (1) and (14) that the Markov parameter matrices of $G(s)$ are expressed by:

$$M_k = \hat{C}\hat{A}^{k-1}\hat{B}, \ k = 1, 2, \ldots . \tag{15}$$

Further, by replacing matrices (15) into block matrices (12) it results that:

$$\mathrm{H}_{i,j}^{M} = \begin{bmatrix} \hat{C}\hat{B} & \hat{C}\hat{A}\hat{B} & \ldots & \hat{C}\hat{A}^{j-1}\hat{B} \\ \hat{C}\hat{A}\hat{B} & \hat{C}\hat{A}^{2}\hat{B} & \ldots & \hat{C}\hat{A}^{j}\hat{B} \\ \vdots & \vdots & & \vdots \\ \hat{C}\hat{A}^{i-1}\hat{B} & \hat{C}\hat{A}^{i}\hat{B} & \ldots & \hat{C}\hat{A}^{i+j-2}\hat{B} \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \\ \vdots \\ \hat{C}\hat{A}^{i-1} \end{bmatrix}}_{\hat{\mathrm{O}}_i} \underbrace{\begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \ldots & \hat{A}^{j-1}\hat{B} \end{bmatrix}}_{\hat{\mathrm{C}}_j} =$$

$$= \hat{\mathrm{O}}_i \hat{\mathrm{C}}_j, \quad i, j = 1, 2, \ldots \, . \tag{16}$$

This result, appropriately used together with (9) and (10), offer the support for the proof of Theorem 1.

In this note, continuing [4] in a new manner, a result similar to Theorem 1, but using Hankel block matrices associated to the Taylor series will be proved and generalised.

2. Main Results

Let $G(s)$ be a strictly proper rational matrix as already assumed and having no pole in $s = 0$. Then the *Taylor series expansion* of $G(s)$ at $s = 0$ has the form:

$$G(s) = \sum_{k=1}^{\infty} T_k s^{k-1}, \tag{17}$$

where T_k, $k = 1, 2, \ldots$, are the *Taylor parameter matrices* which are given by:

$$T_k = \frac{1}{(k-1)!} \left[\frac{\mathrm{d}^{k-1}}{\mathrm{d}s^{k-1}} G(s) \right]_{s=0}, \quad k = 1, 2, \ldots \, . \tag{18}$$

By analogy with (12), the *Hankel block matrices* $\mathrm{H}_{i,j}^{T}$ associated to Taylor series expansion (17) (superscript *T* stands for "Taylor parameter matrices") can be defined as:

$$\mathrm{H}_{i,j}^{T} = \begin{bmatrix} T_1 & T_2 & \ldots & T_j \\ T_2 & T_3 & \ldots & T_{j+1} \\ \vdots & \vdots & & \vdots \\ T_i & T_{i+1} & \ldots & T_{i+j-1} \end{bmatrix} \in \mathrm{R}^{ip \times jm}, \quad i, j = 1, 2, \ldots \, , \tag{19}$$

whose connection with the McMillan degree of $G(s)$ is revealed by the next theorem. It is similar to Theorem 1 and simultaneously generalises a result derived in [5] for $m = p = 1$.

Theorem 2

Assume that $G(s)$ has no pole in $s = 0$. Let $\alpha, \beta \geq 1$ be the smallest integers such that

$$\operatorname{rank} \mathrm{H}_{\alpha,\beta}^{T} = \operatorname{rank} \mathrm{H}_{\alpha+i,\beta+j}^{T}, \quad i, j = 1, 2, \ldots \, . \tag{20}$$

Then $\alpha = \beta = n$, where n is the order of the minimal realisation of $G(s)$.

Proof. Let $\det \hat{A} \neq 0$ according to the fact that $G(s)$ has no pole in $s = 0$. Then, by analogy with (14), the matrix $(I_n s^{-1} - \hat{A}^{-1})^{-1}$ is the sum of the following infinite geometric series (i.e. of the series expansion at $s^{-1} = \infty$ or, equivalently, at $s = 0$):

$$(I_r s^{-1} - \hat{A}^{-1})^{-1} = \sum_{k=1}^{\infty} \hat{A}^{-k+1} s^k \ . \tag{21}$$

On the other hand, by replacing (21) into the right-hand side of the following identity:

$$(I_r s - \hat{A})^{-1} \equiv s^{-1}(-\hat{A}^{-1})(I_r s^{-1} - \hat{A}^{-1})^{-1} \ , \tag{22}$$

it results that the Taylor series expansion of $(I_r s - \hat{A})^{-1}$ at $s = 0$ is given by:

$$(I_r s - \hat{A})^{-1} = \sum_{k=1}^{\infty} (-\hat{A}^{-k}) s^{k-1} \ . \tag{23}$$

Substituting (23) into a realisation of the form (1), not necessarily minimal, and comparing the result with (17), it follows that the Taylor parameter matrices of $G(s)$ have the forms:

$$T_k = -\hat{C}\hat{A}^{-k}\hat{B}, \ \ k = 1, 2, \ldots \ . \tag{24}$$

Using (24), the associated Hankel block matrices (19) may be expressed as follows:

$$\mathrm{H}_{i,j}^{T} = -\begin{bmatrix} \hat{C}\hat{A}^{-1}\hat{B} & \hat{C}\hat{A}^{-2}\hat{B} & \ldots & \hat{C}\hat{A}^{-j}\hat{B} \\ \hat{C}\hat{A}^{-2}\hat{B} & \hat{C}\hat{A}^{-3}\hat{B} & \ldots & \hat{C}\hat{A}^{-(j+1)}\hat{B} \\ \vdots & \vdots & & \vdots \\ \hat{C}\hat{A}^{-i}\hat{B} & \hat{C}\hat{A}^{-(i+1)}\hat{B} & \ldots & \hat{C}\hat{A}^{-(i+j-1)}\hat{B} \end{bmatrix} =$$

$$= -\begin{bmatrix} \hat{C} \\ \hat{C}\hat{A}^{-1} \\ \vdots \\ \hat{C}\hat{A}^{-(i-1)} \end{bmatrix} \hat{A}^{-1} \begin{bmatrix} \hat{B} & \hat{A}^{-1}\hat{B} & \ldots & \hat{A}^{-(j-1)}\hat{B} \end{bmatrix} =$$

$$= -\begin{bmatrix} \hat{C}\hat{A}^{i-1} \\ \hat{C}\hat{A}^{i-2} \\ \vdots \\ \hat{C} \end{bmatrix} \hat{A}^{-(i+j-1)} \begin{bmatrix} \hat{A}^{j-1}\hat{B} & \hat{A}^{j-2}\hat{B} & \ldots & \hat{B} \end{bmatrix} =$$

$$= -\underbrace{\begin{bmatrix} 0 & \ldots & 0 & I_p \\ 0 & \ldots & I_p & 0 \\ \vdots & & \vdots & \vdots \\ I_p & \ldots & 0 & 0 \end{bmatrix}}_{(ip \times ip)} \underbrace{\begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \\ \vdots \\ \hat{C}\hat{A}^{i-1} \end{bmatrix}}_{\hat{\mathrm{O}}_i} \hat{A}^{-(i+j-1)} \underbrace{\begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \ldots & \hat{A}^{j-1}\hat{B} \end{bmatrix}}_{\hat{\mathrm{C}}_j} \underbrace{\begin{bmatrix} 0 & \ldots & 0 & I_m \\ 0 & \ldots & I_m & 0 \\ \vdots & & \vdots & \vdots \\ I_m & \ldots & 0 & 0 \end{bmatrix}}_{(jm \times jm)} . \tag{25}$$

Now, it is a simple matter to see that using (25) and (16) it results:

$$\mathrm{rank}\,\mathrm{H}_{i,j}^{T} = \mathrm{rank}\,\mathrm{H}_{i,j}^{M} = \mathrm{rank}\,\hat{\mathrm{O}}_i \hat{\mathrm{C}}_j, \ \ i, j = 1, 2, \ldots \ . \tag{26}$$

Consequently, according to (26), Theorems 1 and 2 are equivalent. □

By applying a procedure similar to that one used in relation (25), it may be proved that sufficiently large diagonal minors of matrices (19) have the same rank properties as matrices (19). In order to generalise the Theorem 2 in this sense, define the Hankel block matrices:

$$
\mathrm{H}^{T}_{\rho;\,i,\,j} = \begin{bmatrix} T_{\rho+1} & T_{\rho+2} & \cdots & T_{\rho+j} \\ T_{\rho+2} & T_{\rho+3} & \cdots & T_{\rho+j+1} \\ \vdots & \vdots & & \vdots \\ T_{\rho+i} & T_{\rho+i+1} & \cdots & T_{\rho+i+j-1} \end{bmatrix} \in \mathrm{R}^{\,ip \times jm}, \quad \rho = 0,1,2,\ldots, \;\; i, j = 1,2,\ldots \qquad (27)
$$

For these matrices, which are diagonal minors of matrices (19), the following result may be easily proved by appropriate factorisations (as in (25)) operated in (27) with (24).

Theorem 3

Assume that $G(s)$ has no pole in $s = 0$. Let $\alpha,\, \beta \geq 1$ be the smallest integers such that

$$
\operatorname{rank} \mathrm{H}^{T}_{\rho;\,\alpha,\,\beta} = \operatorname{rank} \mathrm{H}^{T}_{\rho;\;\alpha+i,\,\beta+j}, \quad i, j = 1,2,\ldots, \quad \rho = 0,1,2,\ldots \qquad (28)
$$

Then $\alpha = \beta = n$, where n is the order of the minimal realisation of $G(s)$. □

From the point of view of rank computation, it may be more convenient to use a slightly modified Hankel block matrices, namely:

$$
\mathrm{H}^{a^q T}_{\rho;\,i,\,j} = \begin{bmatrix} a^{q} T_{\rho+1} & a^{q+1} T_{\rho+2} & \cdots & a^{q+j-1} T_{\rho+j} \\ a^{q+1} T_{\rho+2} & a^{q+2} T_{\rho+3} & \cdots & a^{q+j} T_{\rho+j+1} \\ \vdots & \vdots & & \vdots \\ a^{q+i-1} T_{\rho+i} & a^{q+i} T_{\rho+i+1} & \cdots & a^{q+i+j-2} T_{\rho+i+j-1} \end{bmatrix},
$$

$$
\rho = 0,1,2,\ldots,\; i, j = 1,2,\ldots, \qquad (29)
$$

where $a \in R \setminus \{0\}$ and $q \in Z$, for which the following result may be proved.

Theorem 4

Assume that $G(s)$ has no pole in $s = 0$. Let $\alpha, \beta \geq 1$ be the smallest integers such that

$$
\operatorname{rank} \mathrm{H}^{a^q T}_{\rho;\,\alpha,\,\beta} = \operatorname{rank} \mathrm{H}^{a^q T}_{\rho;\,\alpha+i,\,\beta+j}, \quad \rho = 0,1,2,\ldots, \quad i, j = 1,2,\ldots \qquad (30)
$$

Then $\alpha = \beta = n$, where n is the order of the minimal realisation of $G(s)$.

Proof. Theorems 4 and 3 are equivalent because:

$$\operatorname{rank} \mathrm{H}_{\rho;\, i,j}^{a^q T} =$$

$$= \operatorname{rank} a^q \underbrace{\begin{bmatrix} I_p & 0 & \dots & I_p \\ 0 & a I_p & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & a^{i-1} I_p \end{bmatrix}}_{(ip \times ip)} \begin{bmatrix} T_{\rho+1} & T_{\rho+2} & \dots & T_{\rho+j} \\ T_{\rho+2} & T_{\rho+3} & \dots & T_{\rho+j+1} \\ \vdots & \vdots & & \vdots \\ T_{\rho+i} & T_{\rho+i+1} & \dots & T_{\rho+i+j-1} \end{bmatrix} \underbrace{\begin{bmatrix} I_m & 0 & \dots & 0 \\ 0 & a I_m & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ I_m & 0 & \dots & a^{j-1} I_m \end{bmatrix}}_{(jm \times jm)} =$$

$$= \operatorname{rank} \mathrm{H}_{\rho;\, i,j}^{T}, \quad \rho = 0,1,2,\dots, \quad i,j = 1,2,\dots \ . \ \square \tag{29}$$

3. Illustrative Example

Given the rational matrix

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+1} \\ \dfrac{1}{s+1} & \dfrac{1}{s+2} \end{bmatrix},$$

compute the McMillan degree by using the Theorem 4.

First, it can be easily seen that $p(s) = (s+1)^2 (s+2)$. This means that $n = 3$. On the other hand, by direct calculation, the Taylor parameter matrices have the forms:

$$T_k = (-1)^{k-1} \begin{bmatrix} 1 & 1 \\ 1 & 2^{-k} \end{bmatrix}, \ k = 1,2,\dots \ .$$

Choosing $\rho = 0,\ a = 2,\ q = 1$, for i,j sufficiently large, the rank of $\mathrm{H}_{0\ \alpha+i,\beta+j}^{2T}$ may be evaluated as follows:

$$\operatorname{rank}\mathrm{H}_{0\ \alpha+i,\beta+j}^{2T} = \operatorname{rank} \left[\begin{array}{cc|cc|cc|cc|c} 2 & 2 & -2^2 & -2^2 & 2^3 & 2^3 & -2^4 & -2^4 & \cdots \\ 2 & 1 & -2^2 & -1 & 2^3 & 1 & -2^4 & -1 & \cdots \\ \hline -2^2 & -2^2 & 2^3 & 2^3 & -2^4 & -2^4 & 2^5 & 2^5 & \cdots \\ -2^2 & -1 & 2^3 & 1 & -2^4 & -1 & 2^5 & 1 & \cdots \\ \hline 2^3 & 2^3 & -2^4 & -2^4 & 2^5 & 2^5 & -2^6 & -2^6 & \cdots \\ 2^3 & 1 & -2^4 & -1 & 2^5 & 1 & -2^6 & -1 & \cdots \\ \hline -2^4 & -2^4 & 2^5 & 2^5 & -2^6 & -2^6 & 2^7 & 2^7 & \cdots \\ -2^4 & -1 & 2^5 & 1 & -2^6 & -1 & 2^7 & 1 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right].$$

Add now the first row multiplied by $2,\ -2^2,\ 2^3,\ -2^4,\dots$ to the rows 3, 5, 7, 9,… . It results:

$$\mathrm{rank}\mathrm{H}^{2T}_{0\ \alpha+i,\beta+j} = \mathrm{rank}\left[\begin{array}{cc:cc:cc:cc:c} 2 & 2 & 0 & -2^2 & 0 & 2^3 & 0 & -2^4 & \cdots \\ 2 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & \cdots \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -2^2 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & \cdots \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2^3 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & \cdots \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -2^4 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & \cdots \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right] =$$

$$= \mathrm{rank}\begin{bmatrix} 2 & 2 & 2^2 & 2^3 & 2^4 & \cdots \\ 2 & 1 & 1 & 1 & 1 & \cdots \\ 2^2 & 1 & 1 & 1 & 1 & \cdots \\ 2^3 & 1 & 1 & 1 & 1 & \cdots \\ 2^4 & 1 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \mathrm{rank}\begin{bmatrix} 2 & 2 & 2 & 6 & 14 & \cdots \\ 2 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 0 & 0 & 0 & 0 & \cdots \\ 14 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The last matrix was obtained form the preceding one by subtracting the 2nd row from the next ones and then by subtracting the 2nd column from the next ones. In the last matrix the 3rd line will be amplified with $-3, -7, -15, \ldots$ and then successively added to the next rows. In a similar manner will be operated with the columns. Finally, it results:

$$\mathrm{rank}\mathrm{H}^{2T}_{0\ \alpha,\beta} = \mathrm{rank}\mathrm{H}^{2T}_{0\ \alpha+i,\beta+j} = \mathrm{rank}\begin{bmatrix} 2 & 2 & 2 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \mathrm{rank}\begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} = 3 \cdot$$

$$\text{for } \alpha = \beta = 3, \quad i, j = 1, 2, \ldots.$$

It follows that $n = \alpha = \beta = 3$.

4. Concluding Remarks

a) Based on Hankel block matrices associated to Taylor series of strictly proper transfer matrix $G(s)$, three theorems dealing with its McMillan degree were derived.

b) Theorem 2, exploiting the rank properties of Hankel block matrices (19) associated to Taylor series expansion of $G(s)$ at $s = 0$, is similar (*mutatis mutandis*) with Theorem 1, which is based on the rank of Hankel block matrices (12) associated to the Laurent series expansion of $G(s)$ at $s = \infty$. Theorem 2 generalizes an earlier result, [4], derived for a scalar transfer function $G(s)$.

c) Theorem 3 generalises Theorem 2 by extending the rank condition of Theorem 2 to any sufficiently large diagonal minors of Hankel block matrices associated to the Taylor series expansion of $G(s)$ at $s = 0$.

d) Theorem 4, which is equivalent to Theorem 3, may be more convenient for numerical manipulations in the cases when the dispersion of the entries of the Taylor parameter matrices is very high.

e) The hypothesis that $G(s)$ has no pole at $s = 0$, which has been assumed in Theorems 2, 3 and 4, does not reduce the generality of the obtained results. If $G(s)$ has η poles at $s = 0$, then Theorems 2, 3 and 4 can be applied for the strictly proper part of the matrix $s^{\eta}G(s)$.

f) By analogy with (29), Hankel block matrices associated to Laurent series can be defined (i.e. using the Markov parameter matrices). Then Theorem 1 can be generalised by results similar (*mutatis mutandis*) to Theorems 3 and 4.

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