# Scheduling of Local Robust Control Laws for Nonlinear Systems

**Pedro Teppa[1], Jacques Bernussou[2], Germain García[2,3], Hael Khansah[2]**

[1] Departamento de Procesos y Sistemas, Universidad Simón Bolívar, Sartenejas, Baruta, Edo. Miranda, Apdo. Postal 89000, Venezuela.

[2] LAAS – CNRS, Toulouse, France.

[3] INSA, Toulouse, France

**Abstract:** This paper presents a systematic approach for nonlinear control design by using the gain scheduling technique to insure the transition of a nonlinear dynamical process from an actual operating condition to a desired one. The nonlinear system is represented in neighborhoods of equilibrium points by a family of polytopic uncertain linear systems. The nonlinear equations of the system are imbedded in the interior of an inclusion polyhedron. A robust control law is built so as to insure asymptotic stability to a given equilibrium within a maximal ellipsoidal region contained in the interior of the polyhedron. Given a pre-specified equilibrium curve connecting the initial and final points, it is shown how to fix a sequence of equilibrium points together with the local associated ellipsoids covering the curve and enabling a convergent control sequence. State feedback and output feedback for the local robust control synthesis are considered, together with local performance criteria. A simple numerical experiment is provided to illustrate both the effectiveness of the synthesis and the performance achieved.

**Keywords:** Gain Scheduling, Transition Control, Polytopic Uncertainty, Quadratic Stability, Linear Matrix Inequalities (LMI).

**Pedro Teppa** received the B.S. degree in electrical engineering at Metropolitana University, Caracas, Venezuela in 1989. The M.S. degrees at Simón Bolívar University, Caracas, Venezuela in electronic engineering and mathematics, in 1994 and 1998, respectively, and the Ph.D. in control systems at Université Paul Sabatier of Toulouse, France in 2003. He is currently an Associate Professor in the Process and System Department at Simón Bolívar University. His research interest include optimal, robust, fuzzy and nonlinear control

**Jacques Bernussou** graduated from the Engineering School ENSEEIHT in Toulouse and got his Doctorat d'Etat at Université Paul Sabatier of Toulouse in 1974. He is now Directeur de Recherche in LAAS-CNRS and member of the research group "Methods and Algorithms in Control". His field of interest is linked with optimal, robust and non linear control with applications in the aerospace domain

**Germain Garcia** graduated from the Engineering School INSA in Toulouse (France) and got his "Habilitation à diriger des Recherches" at Université Paul Sabatier in Toulouse in 1997. He is now Professor at INSA de Toulouse in the Electrical Engineering and Computer Sciences Department and member of the research group "Methods and Algorithms in Control" in LAAS-CNRS. His field of interest is related to robust control, constrained control with applications in the aeronautical domain.

**Hael Khansah** was born in Syria in 1977. He was a student in the department of Electronic Engineering at the University of Damascus from 1995-2000. He is preparing the Ph.D. in robust control in the Laboratoire d'Analyse et d'Architecture des Systemes (LAAS-CNRS) in Toulouse (France).

## 1. Introduction

The control of nonlinear dynamical systems over a wide range of operating conditions has attracted a great deal of attention in recent years. Fixed controllers are definitely not adequate to control these kind of systems. An adaptive controller or a varying controller structure is needed to meet the control demands at multiple operating points. The common approach to solve a complex problem is to use the strategy of divide and conquer. This strategy leads to multiple models or multiple controllers approaches to deal with nonlinear systems. Some of the approaches used frequently are: multiple models adaptive control [1, 25], supervisory control [18], the Quasi-LPV approach [6, 20] and gain scheduling [19, 20, 2]. Multiple models adaptive control is a model-based form of gain scheduled. It uses a set of paired models and controllers. The model/controller pairs are fixed. As a result, any model based controller design strategy can be applied. It also uses a weighting function to decide which model or combination of models fits the data best. Then a corresponding controller or combination of controllers is used to send an input to the plant. In supervisory control, a supervisor selects the best feedback controller from a set of linear controllers and switches into feedback to cause the output of the process to track a set point. The supervisor decides which controller to put in feedback with the process by comparing performance signals generated by the candidate controllers. The Quasi-LPV approach is based on the possibility of rewriting the plants equations in a form where nonlinear terms can be hidden with newly defined, time-varying parameters that are then included in the scheduling variable. Since nonlinear terms involve the state, this means that some state variables must be relabeled as parameters in various part of the model.

Because a practical unified theory for nonlinear control does not currently exist, engineers must often resort to techniques not based on a completely rigorous theory, but demonstrated to work in practical situations. Gain scheduling is among the most familiar of these techniques. It is an effective way of controlling systems whose dynamics change with the operating conditions. It is normally used in the control of nonlinear plants where the relationship between the plant dynamics and operating conditions is known, and for which a single linear-time invariant model is insufficient. In its standard form, this procedure is executed by linearizing the systems dynamics about a family of equilibrium points, also called operating points and applying linear control theory at each design point to derive local linear control laws. The local performance of gain-scheduled controllers is usually good, which makes them attractive for practical applications. Unfortunately, gain scheduling doesn't provide guarantees on the stability and performance of the closed-loop systems at operating conditions between the scheduling points. Although, there is a long history of gain scheduling in applications [13, 20, 22], the theoretical treatments of gain scheduling as a worthy design methodology for nonlinear control are rare until the 1990s [19, 20, 21]. As mentioned, a major limitation of linearization gain scheduling is that stability properties of the closed loop system can be guaranteed only in a neighborhood of the equilibrium manifold. A notion of extended local linear equivalence has been proposed to address this problem [14]. Along these lines, we find the idea of stability-based switching. In [15, 16], it's considered scheduling the controller transition based on stability properties of the linear points designs. The idea is to select operating conditions and to perform linear point designs so that each operating condition is in the capture region of a neighboring operating condition. In this manner, one can assure stable transitions among operating conditions by appropriately switching controllers.

In this work, we develop an alternative gain scheduling procedure that exhibits some of the performance properties of standard gain scheduling, but provides stability guarantees between the scheduling points [23]. In particular, we consider the use of the gain scheduling approach to the problem of stable guaranteed transition from an actual operating point to a desired one by using a pre-specified path in the equilibrium manifold of the nonlinear system connecting the operating points. A polytopic uncertain linear system is used to describe the nonlinear system in a polyhedral region around an equilibrium point. A Lyapunov function for the uncertain linear system is used to estimate ellipsoidal stability regions of maximal volume of the nonlinear system in the interior of the polyhedron. Repeating the procedure on the pre-specified path in the equilibrium manifold connecting the two operating conditions, we can cover the path by a set of ellipsoids included in the polyhedrons. The algorithm developed assure closed loop stability of the nonlinear system and local performance (pole assignment, $H_2$ or $H_\infty$ norm minimization) by using results from quadratic stability [2] and convex optimization, mainly linear matrix inequalities (LMI) [3]. The paper is organized as follows: Section 2 is devoted to give some preliminary results. In section 3 the stable transition problem is stated and a transition algorithm is formulated to solve such a problem. Section 4 concerns the local robust control laws synthesis and some insights are given for the practical problem of output scheduling. Finally, a simple numerical example is presented in section 5.

# 2. Some Previous Results

The algorithm developed in this paper applies to nonlinear systems with input-affine models of the form

$$\Sigma : \dot{x}(t) = F\big(x(t)\big) + G\big(x(t)\big)u(t) = f(x(t), u(t)), \quad t \geq 0 . \tag{1}$$

Where $x(t) \in X \subseteq \Re^n$ is the state vector, $X$ is an open set, $u(t) \in U \subseteq \Re^m$ is the control input, $U$ is the set of all admissible controls such that $u(.) \in U$ is a measurable function. The operating space of the system (1) is $\Psi = X \text{ x } U$ with operating vector $\psi = (x, u) \in \Psi$. The second form in (1) will be used for convenience. We assume that the nonlinear function $f$ is Lipschitz defined in a neighborhood of the equilibrium points of the nonlinear system.

## 2.1. Gain Scheduling Formulation

Let $Z$ be the scheduling space, i.e, the set of scheduling variables $z(t) \in Z \subseteq \Re^s$. The scheduling space $Z$ is a connected compact set. In many cases there will exist a function $s : \Psi \to Z$ that projects the operating vector $\psi$ onto a lower dimensional scheduling space $Z$, thus $z = s(\psi)$ with $\dim(z) < \dim(\psi)$. It is assumed that there exists an equilibrium manifold [20] which is parameterized through the scheduling variable, $z(t) \in Z$. That is, there exist continuous functions, $x^0 : Z \to X$ and $u^0 : Z \to U$ such that

$$f\left(x^0(z), u^0(z)\right) = 0 \ , \tag{2}$$

for all $z(t) \in Z$ . The scheduling variable can be a function of the state, input, and exogenous signals. Although the scheduling variable is a function of time, in the gain scheduled controller implementation, it is viewed as a parameter in the design process. The Jacobian linearization of the nonlinear plant around the equilibrium $\left(x^0(z), u^0(z)\right)$ is written as

$$\Sigma(z) : \dot{x}(t) = A(z)\left(x(t) - x^0(z)\right) + B(z)\left(u(t) - u^0(z)\right), \tag{3}$$

where

$$A(z) = \left.\frac{\partial f}{\partial x}\right|_{\left(x^0(z), u^0(z)\right)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\left(x^0(z), u^0(z)\right)} . \tag{4}$$

For each $z$, that is $z(t)\big|_{t=t_j} = z_j$, the nonlinear system $\Sigma$ is represented locally by the linear time invariant system $\Sigma\left(z_j\right)$

$$\Sigma(z_j) : \dot{x}(t) = f_L\left(x(t), u(t)\right) = A(z_j)\left(x(t) - x_{eq}^{(j)}\right) + B(z_j)\left(u(t) - u_{eq}^{(j)}\right), \tag{5}$$

where $\left(x_{eq}^{(j)}, u_{eq}^{(j)}\right) = \left(x^0(z_j), u^0(z_j)\right)$ and $f_L$ denote the local approximation of the nonlinear function $f$ (1). Based on the plant equilibrium manifold and linearization, a linear time invariant controller $C_L^{(j)}$ is designed for selected values of $z_j$. The global gain scheduled controller $C$ is implemented by combining these linear controllers $C = \bigcup_j C_L^{(j)}$ using real time measurements of $z(t)$ to vary the controller's parameters.

## 2.2. Geometric Tools

In this section we describe the geometric tools that will be used, the polyhedrons [17] used in the local approximation of the nonlinear equations of the process around the equilibrium points; and the ellipsoids [3], that allow the estimation of asymptotic stability regions in the neighborhood of the equilibrium points.

**Definition 1** *A polyhedron $P$ is the intersection of a finite number of half spaces*

$$P \hat{=} \left\{x \in \Re^n : a_i^T x \le b_i, i = 1, \cdots, p\right\}$$
$$= \left\{x \in \Re^n : Ax \le b\right\}, \tag{6}$$

where $a_i^T$ is the i-th row of the matrix $A$ and $\le$ is meant component wise.

**Definition 2** An <u>ellipsoid</u> $E$ centered at $x_c \in \Re^n$ can be represented by the expression

$$E \hat{=} \left\{x \in \Re^n : (x - x_c)^T S^{-1} (x - x_c) \le 1, S = S^T > 0\right\}. \tag{7}$$

The maximum volume ellipsoid $E$ contained in the polyhedron $P$ is given by [3, 24]

$$\max Trace(S)$$
$$\begin{cases} S = S^T > 0 \\ c_i = b_i - a_i^T x_c > 0,\ a_i^T S a_i - c_i^2 \le 0,\ \mathrm{i} = 1, \cdots, \mathrm{p} \end{cases}. \tag{8}$$

It's a classical convex optimization problem that will be used forward in the robust control laws design.

## 2.3. Quadratic Stability

Consider a polytopic uncertain linear system described $\forall t \ge 0$ by the equations

$$\dot{x}(t) = Ax(t), \quad A \in D_A, \tag{9}$$

where $D_A$ is a polytopic uncertain domain given by

$$D_A := \left\{ A \in \Re^{n \times n} : A = \sum_{i=1}^{N_A} \alpha_i A_i, \alpha_i \ge 0, \sum_{i=1}^{N_A} \alpha_i = 1 \right\}. \tag{10}$$

We first study stability of the system (9-10), that is, we ask whether all trajectories of the system (9-10) converge to zero as $t \to \infty$. A sufficient condition for this is the existence of a quadratic function $V(\xi) = \xi^T Q \xi,\ Q > 0$ that decreases along every nonzero trajectory of the system. If there exists such a $Q$, we say that the system (9-10) is quadratically stable and we call $V$ a quadratic Lyapunov function. Since $d/dt(V(x)) = x^T \left(A^T Q + QA\right)x$, a necessary and sufficient condition for quadratic stability of (9-10) is

$$Q > 0, \quad A_i^T Q + QA_i < 0, \quad i = 1, \cdots, N_A. \tag{11}$$

or the dual condition

$$S = Q^{-1} > 0, \quad SA_i^T + A_i S < 0, \quad i = 1, \cdots, N_A. \tag{12}$$

Quadratic stability can also be interpreted in terms of invariant ellipsoids [3]. Let $E$ denote the ellipsoid centered at $x_c$ (7). The ellipsoid $E$ is said to be invariant for the system (9-10) if for every trajectory of the system, $x(t_0) \in E$ implies $x(t) \in E,\ \forall t \ge t_0$. This is the case if and only if the matrix $S$ satisfies (12).

## 2.4. Local Approximation of the Nonlinear System

We assume that, for any inclusion polyhedron $P^{(j)}$ represented as in the equation (6) associated to the equilibrium point $\left(x_{eq}^{(j)}, u_{eq}^{(j)}\right)$ of the system (1), is possible to construct a set

$$\left\{\left(A_{i_A}^{(j)}, B_{i_B}^{(j)}\right); i_A = 1,\cdots,N_A, i_B = 1,\cdots,N_B, j = 1,\cdots,N\right\} \ \forall\ \psi = (x,u) \in P^{(j)}. \qquad (13)$$

So that the model (1) satisfies the inclusion condition:

$$f(x,u) \in \subset_0 \left\{ \begin{matrix} A_{i_A}^{(j)}\left(x - x_{eq}^{(j)}\right) + B_{i_B}^{(j)}\left(u - u_{eq}^{(j)}\right), \\ i_A = 1,\cdots,N_A, i_B = 1,\cdots,N_B, j = 1,\cdots,N \end{matrix} \right\} \ \forall\ \psi = (x,u) \in P^{(j)}. \qquad (14)$$

Where $\subset_0$ denotes the convex hull. Under this condition, all trajectories of the nonlinear system (1) corresponding to input trajectories $u(t)$ for which the operating vector $\psi = (x,u) \in P^{(j)}$ for all $t \geq 0$ are also trajectories of the linear differential inclusion (14) [3].

$$x(t) \in \subset_0 \left\{ \begin{matrix} A_{i_A}^{(j)}\left(x - x_{eq}^{(j)}\right) + B_{i_B}^{(j)}\left(u - u_{eq}^{(j)}\right), \\ i_A = 1,\cdots,N_A, i_B = 1,\cdots,N_B, j = 1,\cdots,N \end{matrix} \right\}. \qquad (15)$$

Thus the evolution of (1) is captured by an uncertain linear model generated by considering all possible linear combinations of a given set of linear models. In order to compute the inclusion polyhedron $P^{(j)}$, we define a domain of validity from an admissible error $\varepsilon \in \Re^n$ between the nonlinear system (1) and its linear approximation calculated in the *j*-th equilibrium point $\left(x_{eq}^{(j)}, u_{eq}^{(j)}\right)$. Fixing an approximating error leads to a polyhedron into the state space. We show the case of a non linearity depending on one variable in the figure (1). We have the following result.

**Proposition 1** Giving an approximating error $\varepsilon \in \Re^n$ between the nonlinear system (1) and its linear approximation calculated in the j-th equilibrium point $\left(x_{eq}^{(j)}, u_{eq}^{(j)}\right)$ (5), then there is an inclusion polyhedron $P^{(j)} = \left\{x \in \Re^n : a_i^T x \leq b_i; i = 1,\cdots,2n\right\}$ that contains the non linearities of the system (1). □

**Proof** Let the error vector $\varepsilon = [\varepsilon_1,\cdots,\varepsilon_n]^T \in \Re^n$ then we have for every component in equations (1) and (5)

$$\left|f_i - f_{Li}\right| \leq \varepsilon_i, \qquad (16)$$

The equation (16) leads to a polyhedron into the space state X, such that

$$\left|x_i(t) - x_{i\ \mathrm{eq}}^{(j)}\right| \leq \delta_i^{(j)}; i = 1,\cdots,n, j = 1,\cdots,N. \qquad (17)$$

Choosing

$$\delta^{(j)} \in \Re = min\left\{\delta_1^{(j)},\cdots,\delta_n^{(j)}\right\}. \qquad (18)$$

The polyhedron $P^{(j)}$ will be symmetric with respect the j-th equilibrium point. We write

$$P^{(j)} = \left\{x \in \Re^n : a_i^T x \le b_i, i = 1,\cdots,2n\right\}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ -1 & 0 & \cdots & 0 \\ 0 & -1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \\ x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \le \begin{bmatrix} \delta^{(j)} + x_{1\,\mathrm{eq}}^{(j)} \\ \vdots \\ \vdots \\ \delta^{(j)} + x_{n\,\mathrm{eq}}^{(j)} \\ \delta^{(j)} - x_{1\,\mathrm{eq}}^{(j)} \\ \vdots \\ \vdots \\ \delta^{(j)} - x_{n\,\mathrm{eq}}^{(j)} \end{bmatrix}. \quad (19)$$

■

**Remark 1** If there are linear components in the nonlinear vector function (1), we make $\varepsilon_i = 0$.

**Remark 2** We have adopted a numerical procedure in order to find (19). We define a mesh in the state space and we impose the condition (16) then using (18) the symmetric polyhedron (19) is obtained.

Considering every possible combination generated by the extreme values taken by the non linearities of the system $\Sigma$ (1) in the interior of $P^{(j)}$, we can construct an uncertain linear system $\Sigma_I^{(j)}$ such that

$$\Sigma_I^{(j)} : \dot{x}(t) = A_i^{(j)}\left(x(t) - x_{eq}^{(j)}\right) + B_i^{(j)}\left(u(t) - u_{eq}^{(j)}\right), \quad (20)$$

where the matrices $\left\{A_i^{(j)}, B_i^{(j)}\right\}$ belong to the polytopic uncertain domains $\left\{D_A^{(j)}, D_B^{(j)}\right\}$ such that

$$D_A^{(j)} = \left\{ A_i^{(j)} = A(z_j)\Big|_{x_{i\,eq}^{(j)} \pm \delta^{(j)}} : \sum_{i=1}^{N_A} \alpha_i A_i^{(j)}, \sum_{i=1}^{N_A} \alpha_i = 1, \alpha_i \ge 0 \right\}, \quad (21)$$

$$D_B^{(j)} = \left\{ B_i^{(j)} = B(z_j)\Big|_{x_{i\,eq}^{(j)} \pm \delta^{(j)}} : \sum_{i=1}^{N_B} \mu_i B_i^{(j)}, \sum_{i=1}^{N_B} \mu_i = 1, \mu_i \ge 0 \right\}. \quad (22)$$

The notation $(\cdot)\big|_{x_{ieq}^{(j)} \pm \delta^{(j)}}$ must be interpreted as the evaluation of the corresponding Jacobian matrix in every possible combination around the j-th equilibrium point. However, for a practical system, finding the polytopic uncertain domains may be a difficult task especially in the case when the non linearities have several arguments. Some works have been recently devoted to the approximation of nonlinear systems by means of piecewise affine models and can be usefully used to extend the proposed approach for nonlinear system control [11, 12].

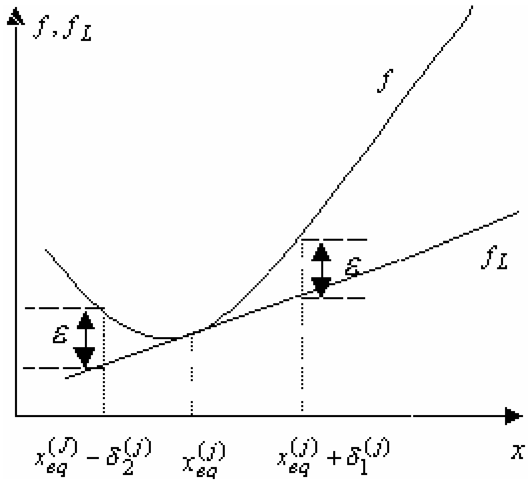


**Figure 1.** Local approximation of the nonlinear system

# 3. Transititon Algorithm Formulation

The guaranteed transition from an actual operating point to a desired one is an interesting control engineering problem. In this section, we develop a transition algorithm to switch local robust controllers. This algorithm insures the stable transition between operating points and allows for the local definition of performance indexes (pole assignment, $H_2$ or $H_\infty$ norm minimization) in the neighborhood of the equilibrium points supplied by the gain scheduling technique.

## 3.1. Statement of the Problem

We are interested in finding a scheduled control that guarantees a suitable transition between an initial operating point and a final operating point. To assure a stable transition too, we are searching for an intermediary equilibrium points set, such that, every intermediary equilibrium point $i+1$ is supposed to be in the interior of the asymptotic stability region of the point $i$. The problem is to find that set of equilibrium points, and the associated control laws designed to insure, for instance, a suitable fast transient response.

**Problem 1** *Find a state feedback global control law* $u(.) \in U \subseteq \Re^m$ *that guarantees the transition of the nonlinear system* $\Sigma$ (1) *from the initial operating point* $\psi_0 = (x_0, u_0) \in \Psi$ *to the final operating point* $\psi_f = (x_f, u_f) \in \Psi$ *assuring the closed loop stability of* $\Sigma$ *and certain local control criteria (pole assignment,* $H_2$ *or* $H_\infty$ *norm minimization).*

## 3.2. Transition Algorithm

The following assumptions are made:

i. There are functions $x^0 : Z \to X$ and $u^0 : Z \to U$ such that $f\left(x^0(z), u^0(z)\right) = 0$ for all $z(t) \in Z$ . That is, there is a family of equilibrium points parameterized by the scheduling variable z written as $\Pi = \left\{\psi = (x,u) \in \Psi : f\left(x^0(z), u^0(z)\right) = 0, \forall z \in Z\right\}$

ii. There is a path $C_z$ in $\Pi$ depending on the scheduling variable z connecting the points $x_0$ and

$x_f$. This means that it is possible to specify a path $C_\rho$ parameterized now by the scalar $0 \le \rho \le 1$, such that $C_\rho \big|_{\rho=0} \equiv x_0$ and $C_\rho \big|_{\rho=1} \equiv x_f$.

iii. The pairs $\left(A(z_j), B(z_j)\right), j = 1, \cdots, N$ are assumed to be stabilisable.

After the scalar parameterization we can write the equilibrium point set as

$$\Pi = \begin{cases} \psi = (x,u) \in \Psi : f\left(x^0(\rho), u^0(\rho)\right) = 0, \ 0 \le \rho \le 1, \\ \psi_0 = \left(x^0(0), u^0(0)\right), \ \psi_f = \left(x^0(1), u^0(1)\right) \end{cases} \Bigg\}.$$

Under the previous assumptions the state transition from an actual operating condition $x_0$ to a desired operating condition $x_f$ may be established using a pre-specified path in the state space as is stated in the following algorithm.

1. Choose the initial equilibrium point as $\left(x_{eq}^{(1)}, u_{eq}^{(1)}\right) = \left(x_f, u_f\right)$. Do $j = 1, \rho^{(j)} = 1$. (The control law is implemented off line).

2. Fix the approximation error $\varepsilon = [\varepsilon_1, \cdots, \varepsilon_n]^T \in \Re^n$ such that $\left|f_i - f_{Li}\right| \le \varepsilon_i ; \forall i = 1, \cdots, n$.

3. Find the *j*-th inclusion polyhedron $P^{(j)}$ (19) around the equilibrium point $\left(x_{eq}^{(j)}, u_{eq}^{(j)}\right)$.

4. $\forall \, x(t) \in P^{(j)}$ represents the nonlinear system $\Sigma$ (1) by the polytopic uncertain linear system $\Sigma_I^{(j)}$ (20-22)

5. Built a state feedback robust control law

$$u(t)^{(j)} = K^{(j)}\left(x(t) - x_{eq}^{(j)}\right) + u_{eq}^{(j)}, \tag{23}$$

that maximizes the volume of the ellipsoid

$$E = \left\{x \in \Re^n : (x - x_c)^T S^{-1}(x - x_c) \le 1, S = S^T > 0\right\}, \tag{24}$$

contained in the inclusion polyhedron $P^{(j)}$, by the resolution of the optimization problem (to be developed in section 4)

$$\begin{aligned} &\text{maximize} \quad volume(E^{(j)}) \\ &\textit{such that} \quad \begin{cases} E^{(j)} \subseteq P^{(j)} \\ \text{Stability Constraints} & \forall j \\ \text{Performance Constraints} \end{cases} \end{aligned}. \tag{25}$$

6. Compute the following equilibrium point $\left(x_{eq}^{(j+1)}, u_{eq}^{(j+1)}\right)$ by the resolution of the optimization problem (by dichotomy)

$$\text{minimize } \rho$$

$$\text{such that} \begin{cases} x^0(\rho) \in E^{(j)} & \forall j \\ 0 < \rho < 1 \end{cases}. \quad (26)$$

The objective is to find the point such that $x_{eq}^{(j+1)} \in interior\left(E^{(j)}\right)$ insuring that the intersection among the successive ellipsoids is not empty. The above optimization problem will provide the minimum number of commutations from the initial to the final point. The resulting policy may not be robust in the sense that $x_{eq}^{(j+1)}$ is almost on the boundary of the attraction domain of $x_{eq}^{(j)}$ so that to insure convergence the switch has to be fired in a very small neighborhood of $x_{eq}^{(j+1)}$ which may result in a very long response time. A way to avoid this drawback is to introduce a parameter $\beta$ and make

$$\rho^{(j+1)} = \rho_{min} + \beta\left(\rho^{(j)} - \rho_{min}\right), \ \beta \in (0,1),$$

yielding the following equilibrium point as

$$\left(x_{eq}^{(j+1)}, u_{eq}^{(j+1)}\right) = \left(x^o(\rho^{(j+1)}), u^o(\rho^{(j+1)})\right).$$

which is forced to belong to the interior of the attraction domain as deeply as $\beta$ is increased.

7. If the initial point $x_0 \in E^{(j)}$ Then $\left(x_{eq}^{(j+1)}, u_{eq}^{(j+1)}\right) = \left(x_0, u_0\right)$ and return to step 3

8. If $x_{eq}^{(j+1)} = x_0$ Then the design is complete and the global control law is given by (27) else return to step 3

$$u_{Global}(x) = \begin{cases} u(t)^{(N)} = K^{(N)}\left(x(t) - x_{eq}^{(N)}\right) + u_{eq}^{(N)}, \ x \in E^{(N)} - \bigcup_{j=1}^{j=N-1} E^{(j)} \\ \vdots \\ u(t)^{(2)} = K^{(2)}\left(x(t) - x_{eq}^{(2)}\right) + u_{eq}^{(2)}, \ x \in E^{(2)} - E^{(1)} \\ u(t)^{(1)} = K^{(1)}\left(x(t) - x_{eq}^{(1)}\right) + u_{eq}^{(1)}, \ x \in E^{(1)} \end{cases}. \quad (27)$$

## 3.3. Transition Study $(\mathbf{x_0} \to \mathbf{x_f})$

In the following, we study the transition between two operating conditions of a dynamical nonlinear system $\Sigma$. We denote a trajectory of (1) as $s(t, x(t_0), u)$ and the trajectory converging to $x_{eq}$ as $s(t, x(t_0), u) \to x_{eq}$.

**Lemma 1** *Consider a set* $\Gamma \in \Re^n$ *and the equilibrium point* $x_{eq} \in \Gamma$. *If the trajectory* $s(t, x(t_0), u) \to x_{eq}$ *then* $s(t_1, x(t_0), u) \in \Gamma$ *for some* $t_1 > t_0 \in \Re$.

In the following we show that the global robust control law (27) supplied by the transition algorithm assures the nonlinear system transition from the initial equilibrium point $x_0$ to the final equilibrium point $x_f$ .

**Proposition 2** *If the initial condition belongs to the N-th invariant ellipsoid centered at* $x_{eq}^{(N)}$ *denoted as* $x(t_0) = x_0 \in E^{(N)}\left(x_{eq}^{(N)}\right)$ *then the trajectory* $s(t, x(t_0) = x_0, u_{Global}(x)) \rightarrow x_f$ *for* $u_{Global}(x)$ *given by equation* (27). □

**Proof** Without loosing any generality, we assume the number of controllers equals to $N = 3$. If $x_0 \in E^{(3)}\left(x_{eq}^{(3)}\right)$ and $x_0 \notin \left\{E^{(2)}\left(x_{eq}^{(2)}\right), E^{(1)}\left(x_{eq}^{(1)}\right)\right\}$ the algorithm shows that only controller $u(t)^{(3)} = K^{(3)}\left(x(t) - x_{eq}^{(3)}\right) + u_{eq}^{(3)}$ is on, by the lemma 1, $s\left(t, x_0, u(t)^{(3)}\right) \rightarrow x_{eq}^{(3)}$. As $x_0 \notin \left\{E^{(2)}\left(x_{eq}^{(2)}\right), E^{(1)}\left(x_{eq}^{(1)}\right)\right\}$ there is not convergence to $x_{eq}^{(2)}$ neither to $x_{eq}^{(1)}$. By construction, we can find some instant $t_1 > t_0 \in \Re$ and a controller $u(t)^{(2)} = K^{(2)}\left(x(t) - x_{eq}^{(2)}\right) + u_{eq}^{(2)}$ such that $s\left(t_1, x_0, u(t)^{(3)}\right) = x(t_1) \in E^{(2)}\left(x_{eq}^{(2)}\right)$ and the trajectory $s\left(t_1, x(t_1), u(t)^{(2)}\right) \rightarrow x_{eq}^{(2)}$. Repeating this reasoning we can find some instant $t_2 > t_1 \in \Re$ and controller $u(t)^{(1)} = K^{(1)}\left(x(t) - x_{eq}^{(1)}\right) + u_{eq}^{(1)}$ such that $s\left(t_2, x(t_1), u(t)^{(2)}\right) = x(t_2) \in E^{(1)}\left(x_{eq}^{(1)} = x_f\right)$ and $s\left(t_2, x(t_2), u(t)^{(1)}\right) \rightarrow x_{eq}^{(1)} = x_f$. That implies the transition from $x_0$ to $x_f$. ■

# 4. Robust Control Synthesis

Based on the nonlinear system equilibrium manifold and linearization, a linear time invariant controller $C_L^{(j)}$ is designed for selected values of the scheduling variable. The global gain scheduled controller $C$ is implemented by combining these linear controllers $C = \bigcup_j C_L^{(j)}$ using real time measurements of the scheduling variable to modify the control parameters. The local controller $C_L^{(j)}$ will be a robust one and will be designed for the *j*-th polytopic uncertain linear system $\Sigma_I^{(j)}$ (20-22). In the following, we will drop the indices related to the linear local models to unclutter displayed equations, though the indices are retained for emphasis when discussing particular variables.

## 4.1. Scheduled State Feedback Control

We consider in this section the case of state feedback synthesis as in equation (23). The state feedback robust control law that assigns the poles of the polytopic uncertain linear system $\Sigma_I^{(j)}$ (20-22) in the convex region $R_c^{(j)}(\alpha_c, \rho_c, \theta_c)$ (see figure 2) and that maximizes the ellipsoid volume $E^{(j)}$ (24) contained in the inclusion polyhedron $P^{(j)}$ (19) is given by the following theorem, which itself, combines the results from [5] and those from section 2.

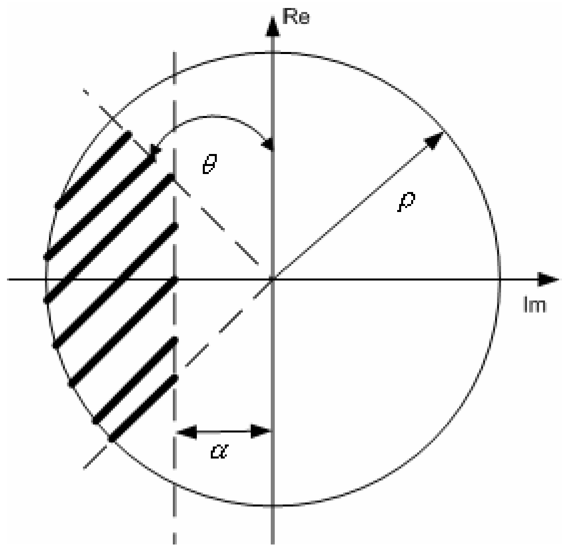


**Figure 2.** Convex region $R_c^{(j)}(\alpha_c, \rho_c, \theta_c)$

**Theorem 1** Let the symmetric positive definite matrix $S^0 \in \Re^{n \times n}$ and the matrix $R^0 \in \Re^{m \times n}$ be solutions of the optimization problem

$$\underset{S,R}{Maximize}\, Trace(S)$$

$$S = S^T > 0$$

$$\left(c_i = b_i - a_i^T x_c > 0\right),\ a_i^T S a_i - c_i^2 \le 0$$

$$A_{i_A} S + S A_{i_A}^T + B_{i_B} R + R^T B_{i_B}^T + 2\alpha S < 0$$

$$\begin{bmatrix} -\rho S & * \\ A_{i_A} S + B_{i_B} R & -\rho S \end{bmatrix} < 0$$

$$\begin{bmatrix} \cos\theta(A_{i_A} S + S A_{i_A}^T + B_{i_B} R + R^T B_{i_B}^T) & * \\ \sin\theta(S A_{i_A}^T - A_{i_A} S + R^T B_{i_B}^T - B_{i_B} R\ ) & \cos\theta(A_{i_A} S + S A_{i_A}^T + B_{i_B} R + R^T B_{i_B}^T) \end{bmatrix} < 0$$

Then the invariant stability domain

$$E^{(j)} = \left\{ x \in \Re^n : (x - x_{eq}^{(j)})^T \left(S^0\right)^{-1} (x - x_{eq}^{(j)}) \le 1 \right\},$$

corresponds to the maximal volume ellipsoid contained in the inclusion polyhedron $P^{(j)}$ (19), and the state feedback robust control law calculated for the j-th equilibrium point $\left(x_{eq}^{(j)}, u_{eq}^{(j)}\right)$ of the nonlinear dynamical system $\Sigma$ (1)

$$u(t)^{(j)} = K^{(j)}\left(x(t) - x_{eq}^{(j)}\right) + u_{eq}^{(j)} = R^0\left(S^0\right)^{-1}\left(x(t) - x_{eq}^{(j)}\right) + u_{eq}^{(j)}$$

assigns the poles of the polytopic uncertain linear system $\Sigma_I^{(j)}$ (20-22) in the convex region $R_c^{(j)}(\alpha_c, \rho_c, \theta_c)$; $j = 1, \cdots, N$, $i = 1, \cdots, 2n$, $i_A = 1, \cdots, N_A$, $i_B = 1, \cdots, N_B$. (The * denotes a Transpose Block Matrix □

**Remark 3** Theorem 1 requires $(2n + 3N_A N_B + 1)N$ LMIs and $n[(n+1)/2 + m]N$ unknowns.

In the quadratic concept, it is easy to add any LMI constraints such as $H_2$ or $H_\infty$ norms bounds [23]. The idea of ellipsoid volume maximization, that is, the local attraction domains of the equilibrium scheduling points, is in the spirit of having a small number of switches between the local robust control laws, for a given transition $x_0 \to x_f$. The synthesis problem definition in Theorem 1 would imply local fast response performances, a necessary condition to get good enough global time response.

## 4.2. Scheduled Output Feedback Control

In the case of scheduled state feedback control, the switching policy implies the guaranteed transition from the initial state to the final one. The extension of the approach to the case of partial information about the state in real time is much more involved. Two propositions for scheduled output control are discussed below: the first one concerns scheduled control with observers, the second is based on dynamic output control.

### 4.2.1. Scheduled observer based controllers

Following the principle of scheduled state feedback in the context of partial state information, the first natural idea to come is to think about state estimation. For control purpose, the will to estimate an unmeasurable linear combination of the state variables is clearly justified, in the case of linear models without uncertainty by the well known separation principle. In our case (nonlinear system described as piecewise uncertain linear one) no such principle directly holds. However, the $H_2$ robust filtering problem [8, 9] may be of some use for the proposal of a scheduled output control strategy. Let us consider the following linear time invariant system

$$\begin{cases} \dot{x}(t) = Ax(t) + B_w w(t) \\ y(t) = Cx(t) + D_w w(t) \end{cases}. \tag{28}$$

Where $x \in \Re^n$ is the state, $w \in \Re^m$ is the zero mean white noise input with identity power spectrum density matrix and $y \in \Re^q$ is the measured output. It is assumed that

All matrix dimensions are known

Matrix $M \in \Re^{(n+q)x(n+m)}$ defined as

$$M = \begin{bmatrix} A & B_w \\ C & D_w \end{bmatrix}, \tag{29}$$

is unknown but it belongs to a given convex bounded polyhedral domain $D_M$. Hence, from [7] each uncertain matrix of this set can be written as an unknown convex combination of $N$ given extreme matrices $M_1, M_2, \cdots, M_N$, that is, $M \in D_M$ if and only if

$$M = \sum_{i=1}^{N} \lambda_i M_i \ , \tag{30}$$

holds for some $\lambda_1 \geq 0, \cdots, \lambda_N \geq 0$ such that $\lambda_1 + \cdots + \lambda_N = 1$.

The problem to be dealt is to design a full estimate $\hat{x}$ of $x$ given by $\hat{x} = \Im . y$ where $\Im$ is a linear, finite dimensional, and time-invariant operator producing at any time the estimation error $e := x - \hat{x}$, such that, the controlled transfer function from the noise input $w$ to the estimation error $e$ satisfies the following:

**Problem 2** *Let a guaranteed estimation performance index* $\gamma(\Im)$ *such that*

$$\sup_{M \in D_M} \left\| T_M(s) \right\|_2^2 \leq \gamma(\Im), \tag{31}$$

and given $\mu > 0$ denote $\Omega$ the set of all filters $(\Im \in \Omega)$ such that *(31)* holds for $\gamma(\Im) = \mu$. Among all feasible filters, find the optimal one that minimizes $\gamma(\Im)$ over $\Omega$.

*In the above problem, the feasible set* $\Omega$ *is used to impose a particular class of linear, finite dimensional, and causal operators. In the present case, we consider* $\Omega$ *as the set of all linear time-invariant operators with state space realization of the form*

$$\frac{d}{dt}\hat{x}(t) = A_f \hat{x}(t) + B_f y(t), \tag{32}$$

where the matrices $A_f \in \Re^{n \times n}$, $B_f \in \Re^{n \times q}$ are to be determined. In other words $\Omega$ is taken as the set of all strictly proper linear time-invariant operators of order $n$. It is assumed that the initial conditions of system (28) and the filter (32) are the same (to be discussed later on). Connecting the filter to the system (28) we can write the transfer function $T_M(s)$ as

$$T_M(s) = \tilde{C}\left(sI - \tilde{A}\right)^{-1} \tilde{B}, \tag{33}$$

where matrices $\tilde{A}, \tilde{B}$ and $\tilde{C}$ of compatible dimensions are given by

$$\tilde{A} := \begin{bmatrix} A & 0 \\ B_f C & A_f \end{bmatrix}, \ \tilde{B} := \begin{bmatrix} B_w \\ B_f D_w \end{bmatrix}, \ \tilde{C} := \begin{bmatrix} I & -I \end{bmatrix}. \tag{34}$$

The next theorem gives a solution, expressed in terms of LMI, to the $H_2$ robust filtering problem (problem 2).

***Theorem 2*** *[8, 9]* Assume that $M \in D_M$ is fixed but arbitrary and the scalar $\mu > 0$ is given. Then, the transfer function $T_M(s)$ satisfy the inequality $\left\| T_M(s) \right\|_2^2 < \mu$ for all $M \in D_M$ if and only if the filter transfer function is expressed as

$$T_f(s) = I\left[ sI - U^{-1} Q \left(U^T Z\right)^{-1} \right]^{-1} \left(V^{-1} F\right), \tag{35}$$

where the pair $(U,V)$ satisfy the relation $XY + UV^T = I$ and the matrices $W = W^T$, $Z = Z^T$, $Y = Y^T$, Q, F and G are the solutions of the following convex problem expressed in terms of LMI

$$\underset{W,Z,Y,Q,F,G}{Trace}(W) \le \mu , \tag{36}$$

$$\begin{bmatrix} Z & Z & I - G^T \\ Z & Y & I \\ I - G & I & W \end{bmatrix} > 0 , \tag{37}$$

$$\begin{bmatrix} ZA + A^T Z & ZA + A^T Y + C^T F^T + Q^T & ZB_w \\ A^T Z + YA + FC + Q & YA + FC + A^T Y + C^T F^T & YB_w + FD_w \\ B_w^T Z & B_w^T Y + D_w^T F^T & -I \end{bmatrix} < 0 . \tag{38}$$

□

The scheduled observer based controllers policy will be based on the belonging of the state estimate $\hat{x}(t),\ t \ge 0$ to the ellipsoid $\hat{E}^{(j)}$ given by the equation

$$\hat{E}^{(j)} = \left\{ \hat{x} \in \Re^n : (\hat{x} - x_{eq}^{(j)})^T \left(S^0\right)^{-1} (\hat{x} - x_{eq}^{(j)}) \le 1 \right\}$$

where the symmetric positive definite matrix $S^0 \in \Re^{n \times n}$ results from the Theorem 1. This scheduling policy obviously relies on the state estimate quality, hence, there is some heuristic degree. Another heuristic degree is added by the assumption of identical initial conditions between the system and the filter. Practically, the choice of the equilibrium states for the filter initialization at each switching would reduce the approximation.

#### 4.2.2. Scheduled dynamic output control

A “time open loop” switching policy can be considered. As far as every intermediary equilibrium point is placed in the attraction domain of the following equilibrium point, it’s obvious that if the local transition is made to last for a long time (asymptotic convergence to the equilibrium point) the global convergence would hold but this lead to a very poor response time. The successive transitions and thus the response time can be improved by imposing both a fast dynamic for the local robust controllers and also by taking the equilibrium points to be far enough in the interior of the attraction domain of the successive equilibrium point (parameter $\beta$ in section 3.2).

The linear uncertain system around the *j*-th equilibrium point is described by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} , \tag{39}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^m$ is the control signal, and $y(t) \in \Re^q$ represent the measured output. The polytopic uncertain domain is given by

$$\widetilde{\mathrm{A}} \hat{=} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \in D_{\widetilde{\mathrm{A}}} = \left\{ \widetilde{\mathrm{A}} : \widetilde{\mathrm{A}} = \sum_{i_{\widetilde{A}}=1}^{N_{\widetilde{A}}} \lambda_i \widetilde{\mathrm{A}}_i, \lambda_i \geq 0, \sum_{i_{\widetilde{A}}=1}^{N_{\widetilde{A}}} \lambda_i = 1 \right\}. \quad (40)$$

We consider for the local controller $C_L^{(j)}$ a dynamic output control of the form

$$\begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c y(t) \\ u(t) = C_c x_c(t) \end{cases}, \quad (41)$$

where $x_c$ is the dynamic output control state vector. For a strictly proper dynamic output control, the global dynamic system will be represented $\forall\, t \geq 0$ as

$$\begin{cases} \dot{X}_e(t) = A_{e_{i\widetilde{A}}} X_e(t) \\ z(t) = C_e X_e(t) \end{cases}, \quad (42)$$

with

$$X_e \hat{=} \begin{bmatrix} x \\ x_c \end{bmatrix}, \quad A_{e_{i\widetilde{A}}} \hat{=} \begin{bmatrix} A_{i_{\widetilde{A}}} & B_{i_{\widetilde{A}}} C_c \\ B_c C_{i_{\widetilde{A}}} & A_c \end{bmatrix}, \quad C_e \hat{=} \begin{bmatrix} 0 & C_c \end{bmatrix}.$$

We propose a dynamic output control strictly proper in order to quadratically stabilize the system (39-40) with a positive Lyapunov decay rate $\eta$, while maximizing the stability ellipsoid volume

$$E = \left\{ X_e \in \Re^{2n} : \left( X_e - X_{eq}^{(j)} \right)^T S^{-1} \left( X_e - X_{eq}^{(j)} \right) \leq 1, S \in \Re^{2n \times 2n}, S = S^T > 0 \right\}, \quad (43)$$

contained in the inclusion polyhedron

$$P = \left\{ X_e \in \Re^{2n} : a_i^T X_e \leq b_i, i = 1, \cdots 2(2n) \right\}. \quad (44)$$

The inclusion polyhedron must be written as (19). The polyhedron size is calculated as it was proposed early, hence, by fixing an approximation error $\varepsilon \in \Re^n$ between the nonlinear system and its linear approximation. The equilibrium points associated to the primal state variables are obtained by solving the algebraic system equations

$$f\left( x_{eq}, u_{eq} \right) = 0$$

and the equilibrium points associated to the dual state variables (dynamic output control) are chosen to be zero. The dual state variables are equally approximated by a polyhedral region in order to have a compact domain, and so, a well posed optimization problem. The following result shows the way to assure quadratic stability with a positive Lyapunov decay rate $\eta$ for the system (39-40) while maximizing the stability ellipsoid volume $E$ (43) contained in the inclusion polyhedron $P$ (44).

***Theorem 3*** Let the symmetric positive definite matrices $X^0, Y^0 \in \Re^{n \times n}$ and the matrices $M^0 \in \Re^{n \times n}, L^0 \in \Re^{m \times n}$ and $F^0 \in \Re^{n \times q}$ solutions of the optimization problem

$$\underset{X,Y,M,L,F}{Maximise\ Trace}\left(\begin{bmatrix} X & I \\ I & Y \end{bmatrix}\right), \tag{45}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0, \tag{46}$$

$$\left(c_i = b_i - a_i^T X_{eq} > 0\right),\ a_i^T \begin{bmatrix} X & I \\ I & Y \end{bmatrix} a_i - c_i^2 \le 0, \tag{47}$$

$$\begin{bmatrix} H_{i_{\tilde{A}}}(X,F) & Z_{i_{\tilde{A}}} + M \\ Z_{i_{\tilde{A}}}^T + M^T & G_{i_{\tilde{A}}}(Y,L) \end{bmatrix} < 0, \tag{48}$$

*with*

$$H_{i_{\tilde{A}}}(X,F) = A_{i_{\tilde{A}}}^T X + X A_{i_{\tilde{A}}} + F C_{i_{\tilde{A}}} + C_{i_{\tilde{A}}}^T F^T + 2\eta X,$$

$$G_{i_{\tilde{A}}}(Y,L) = A_{i_{\tilde{A}}} Y + Y A_{i_{\tilde{A}}}^T + B_{i_{\tilde{A}}} L + L^T B_{i_{\tilde{A}}}^T + 2\eta\ Y,$$

$$Z_{i_{\tilde{A}}} = A_{i_{\tilde{A}}} + Y A_{i_{\tilde{A}}}^T X + L^T B_{i_{\tilde{A}}}^T X + Y C_{i_{\tilde{A}}}^T F^T + 2\eta I\ .$$

Then, (43) is the maximum stability volume ellipsoid contained in the inclusion polyhedron (44) and the strictly proper dynamic output control (41) defined for the j-th equilibrium point of the nonlinear system stabilizes quadratically the system (39-40) with a positive Lyapunov decay rate $\eta$, $i_{\tilde{A}} = 1,\cdots,N_{\tilde{A}}$ and $i = 1,\cdots,2(2n)$. □

***Proof*** The closed loop system (42) is asymptotically stable with a positive Lyapunov decay rate equal to $\eta$ if there exists a symmetric positive definite matrix $S \in \Re^{2n \times 2n}$ such that

$$A_{e_{i\tilde{A}}} S + S A_{e_{i\tilde{A}}}^T + 2\eta S < 0, \tag{49}$$

where $S$ et $S^{-1}$ are matrices partitioned into four blocks of dimension n x n such that

$$S \hat{=} \begin{bmatrix} Y & V \\ V^T & \hat{Y} \end{bmatrix},\quad S^{-1} \hat{=} \begin{bmatrix} X & U \\ U^T & \hat{X} \end{bmatrix},$$

where $X = \left(Y - V\hat{Y}^{-1}V^T\right)^{-1} > Y^{-1}$ resulting in the matrix inequality (46). Substituting S in (8) by the partitioned matrix

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix}$$

gives (47). Introducing the new variables

$$L = C_c V^T, F = U B_c, M = V A_c^T U^T, \tag{50}$$

and multiplying *(49)* on the left and right by the matrices $T_e^T$ and $T_e$ where

$$T_e = \begin{bmatrix} X & U \\ I & 0 \end{bmatrix},$$

gives *(48)*. ■

**Remark 4** *Theorem* 3 *requires* $\lfloor 2(2n) + N_{\tilde{A}} + 1 \rfloor N$ *LMIs and* $n[3(n+1)/2 + m + q]N$ *unknowns*.

The previous matrix inequalities are not linear but bilinear. However, the particular structure of the matrix inequalities allows us the use of relaxation algorithms [10] for every uncertain linear model. The quadratic Lyapunov function of the system (42) is

$$V(X_e) = X_e^T S^{(j)} X_e. \tag{51}$$

The positive Lyapunov decay rate means

$$\frac{d}{dt} V(X_e(t)) \leq -2\eta^{(j)} V(X_e(t)), \tag{52}$$

thus

$$V(X_e(t)) \leq e^{-2\eta^{(j)}t} V(X_e(0)). \tag{53}$$

So that the asymptotic stability of the augmented system (42) can be expressed as

$$\|X_e(t)\| \leq \sqrt{\kappa(S^{(j)})}\, e^{-\eta^{(j)}t} \|X_e(0)\|, \tag{54}$$

where $\sqrt{\kappa(S^{(j)})}$ is a Lyapunov matrix depending constant [3]. Doing

$$\sqrt{\kappa(S^{(j)})} = e^{\eta^{(j)}t_0},$$

with $t_0$ the initial operation time, we can evaluate (54) at $t = t_0 + \tau/\eta^{(j)}$, yielding

$$\|X_e(t)\| \leq e^{-\tau} \|X_e(0)\|. \tag{55}$$

In other words, the positive Lyapunov decay rate $\eta^{(j)}$ plays the role of a constant time for the augmented system (42). By changing $\tau$, it will be possible to modulate the neighborhood around an attractive equilibrium point, by fixing a minimum time transition such that the augmented system's state is into the attraction domain of the following equilibrium point. By this way, we'll have a scheduled dynamic output control technique based on a minimum time transition $j \rightarrow j+1$

$$t_j = t_0 + \frac{\tau}{\eta^{(j)}}, \; j = 1, \cdots, N-1. \tag{56}$$

## 5. Numerical Application

The objective of this section is simply to illustrate the implementation of the transition algorithm, Let a mechanical vibrational system [4] represented for the state equations

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_1(t) - \left[(x_1(t))^2 - 1\right] x_2(t) + u(t) \end{cases}. \tag{57}$$

The operating vector is $\psi = [x_1, x_2, u]^T$ and doing $f(x^0, u^0) = 0$ yields

$$x_2 = 0, u = x_1, \tag{58}$$

resulting in the equilibrium manifold

$$\Pi = \left\{ \psi = [x_1, x_2, u]^T \in \Psi : x_2, u \text{ satisfy (58) } \forall x_1 \right\}. \tag{59}$$

We study the feasibility of the approach by considering the state feedback synthesis for the transition $[x_1, x_2]: [-0.5, 0] \rightarrow [0.5, 0]$ in $\Pi$. However, to show the robustness of the algorithm with respect to the initial condition selection, we choose $[x_1, x_2]: [-1, 0.4]$ belonging to the attraction domain of $[-0.5, 0]$

The figures 3, 4 and 5 show the state variables and the control signal, respectively. We see that the transition time is 7 sec. And the control signal presents discontinuities at the instants of commutation. Finally, the figure 6 allows us to see the state trajectory in the interior of the stability invariant ellipsoids and these contained in the inclusion polyhedrons.

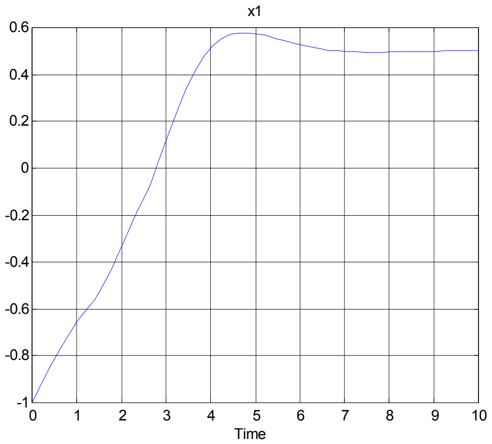


**Figure 3.** Evolution of the state variable $x_1$

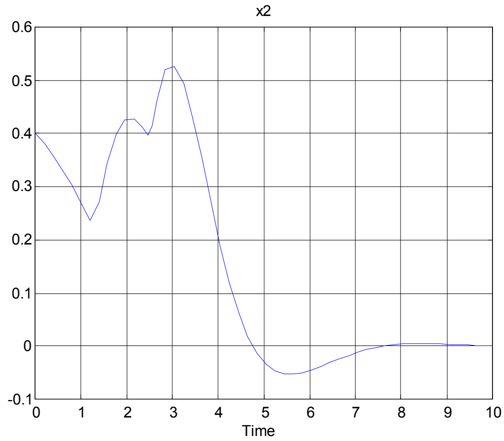


**Figure 4.** Evolution of the state variable $x_2$

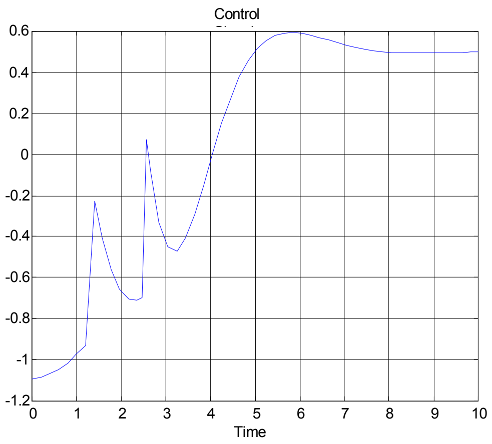


**Figure 5.** Evolution of the control signal

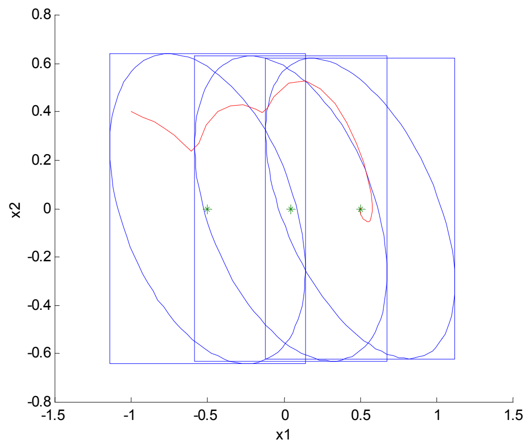


**Figure 6.** State trajectory for the initial condition $[x_1, x_2]: [-1, 0.4]$, inclusion polyhedrons and stability invariant ellipsoids for the considered transition.

## 6. Conclusion

We have developed an algorithm that guarantees the closed loop stable transition of a nonlinear system from an actual operating point to a desired one. A polytopic uncertain linear system is used to describe the nonlinear system in a polyhedral region around an equilibrium point. A Lyapunov function for the uncertain linear system is used to estimate ellipsoidal stability regions of maximal volume of the nonlinear system in the interior of the polyhedron. Repeating the procedure using a pre-specified path in the equilibrium manifold of the nonlinear system connecting the two operating conditions, we can cover the path by a set of ellipsoids included in the polyhedrons that assure the closed loop stability of the nonlinear system. In the case of state feedback, the robust control laws scheduling is executed when the state trajectory reaches the attraction domain of the following equilibrium point. If we can only measure the output, we have two approaches: observer-based controller and dynamic compensation. In the first one, the scheduling policy is based on the belonging of the estimated state to the stability ellipsoid of the next equilibrium point. In dynamic compensation, the scheduling function is of open loop type, where we fix a minimal duration of every individual transition. Current investigation is doing in order to take into account dynamic performance not only in a local way but in a global one.

## Acknowledgement

The authors wish to thank the support of the PCP in France (Programme de Coopération de Post-Graduation: Optimisation des Procédés) and FONACIT in Venezuela (Fondo Nacional de Ciencia, Tecnología e Innovación).

## REFERENCES

1. BEQUETTE, W. and SCHOTT, K., **Control of Chemical Reactors Using Multiple Models Adaptive Control**. IFAC DYCORD, Denmark 1995.

2. BERNUSSOU, J., PERES, P. and GEROMEL, J., **A Linear Programming Oriented Procedure for a Quadratic Stabilization of Uncertain Systems**. Systems and Control Letters, Vol. 13, pp. 65-72, 1989.

3. BOYD, S., EL GHAOUI, L., FERON, E. and BALAKRISHNAN, V., **Linear Matrix Inequalities in System and Control Theory.** Society for industrial and applied mathematics (SIAM), Philadelphia, 1994.

4. CHEN, C. and CHANG, M., **Optimal Design of Fuzzy Sliding-mode Control: A Comparative Study**. Fuzzy Sets and Systems, vol. 93, pp. 37-48, 1998.

5. CHILALI, M. and GAHINET, P., **$H_\infty$ Design with Pole Placement Constraints: An LMI Approach**. IEEE Transactions on Automatic Control, Vol. 41, No. 3, pp. 358-367, 1995.

6. COURTIES, C., **Sur la commande robuste et LPV de systèmes a paramètres lentement variants.** Thèse de doctorat, Université Paul Sabatier, Toulouse, France, 1999.

7. GEROMEL, J., PERES, P. and BERNUSSOU, J., **On a Aonvex Parameter Space Method for Linear Control Design of Uncertain Systems**. SIAM Journal of Control and Optimization, Vol. 29, No. 2, pp. 381-402, 1991.

8. GEROMEL, J., **Optimal Linear Filtering under Parameter Uncertainty**. IEEE Transactions on Signal Processing, Vol. 47, No. 1, pp. 168-175, 1999.

9. GEROMEL, J., BERNUSSOU, J., GARCIA, G. and OLIVEIRA, M., **$H_2$ and $H_\infty$ Robust Filtering for Discrete-time Linear Systems**. In Proceedings of the IEEE Conference on Decision and Control, pp. 632-637, 1998.

10. GEROMEL, J., BERNUSSOU, J. and OLIVEIRA, M., $H_2$ **Norm Optimization with Constrained Dynamic Output Feedback Controllers: Decentralized and Reliable Control**. IEEE Transactions on Automatic Control, Vol. 44, No. 7, pp. 1449-1454, 1999.

11. JULIAN, P., DESAGES, A. and AGAMENNONI, O., **High-level Canonical Piecewise Linear Representation Using a Simplicial Partition**. IEEE Transactions on Circuits and Systems I – Fundamental Theory and Applications, Vol. 46, No. 4, pp. 464-480, 1999.

12. JULIAN, P., DESAGES, A. and D'AMICO, B., **Orthonormal High-level Canonical PWL Functions with Applications to Model Reduction**. IEEE Transactions on Circuits and Systems I – Fundamental Theory and Applications, Vol. 47, No. 5, pp. 702-712, 2000.

13. KALLSTROM, C.G., ASTROM, K. J., THORELL, N. E., ERIKSSON, J. and STEN L. **Adaptive Autopilots for Tankers**. Automatica; Vol. 15, pp. 241-254, 1979.

14. LEITH, D. and LEITHEAD, W., **Gain–scheduling Controller Design: An Analytic Framework Directly Incorporating Non–equilibrium Plant Dynamics**. Int. J. Control, Vol. 70, pp. 249-269, 1998.

15. LEONESSA, A., HADDAD, W. M. and CHELLABOINA, V., **Nonlinear System Stabilization via Stability-based Switching**. Proceedings of the 37th IEEE conference on decision and control, Tampa, FL, pp. 2983-2996, 1988.

16. McCONLEY, M. B., APPLEBY, B. D., DAHLEH, M. A. and FERON, E., **A Computationally Efficient Lyapunov-based Scheduling Procedure for Control of Nonlinear Systems with Stability Guarantees**. IEEE Transactions on Automatic Control, Vol. 45, No. 1, pp. 33-49, 2000.

17. NESTEROV, Y. and NEMIROVSKII, A., **Interior-point Polynomial Methods in Convex Programming**. Society for Industrial and Applied Mathematics SIAM: Philadelphia, 1994.

18. PIVOSO, M. and KOSANOVICH, A., **A Dynamical Supervisor Strategy for Multi-products Processes**. Computers and Chemical Engineering, Vol. 21, pp. 149-154. 1997.

19. RUGH, W., **Analytical Framework for Gain Scheduling**. IEEE Control Systems magazine, Vol. 11, No. 1, pp. 79-84, 1991.

20. RUGH, W. and SHAMMA, J., **Research on Gain Scheduling**. Automatica, Vol. 36, pp. 1401-1425, 2000.

21. SHAMMA, J. S. and ATHANS, M., **Analysis of Nonlinear Gain Scheduled Control Systems**. IEEE Transactions on Automatic Control, Vol. 35, No. 8, pp. 898-907, 1990.

22. STEIN, G., HARTMANN, G. L. and HENDRICK, R. L., **Adaptive Control Laws for F-8 Flight Test**. IEEE Transactions on Automatic Control, Vol. 22, No. 5, pp. 758-767, 1977.

23. TEPPA, P. A**., Commande par séquencements de gains robustes pour systèmes non linéaires**, Thèse de doctorat, Université Paul Sabatier, Toulouse, France, 2003.

24. VANDENBERGHE, L., BOYD, S. and WU, S., **Determinant Maximization with Linear Matrix Inequality Constraints**. SIAM Journal on Matrix Analysis and Applications, Vol. 19, No. 2, pp. 499-533, 1998.

25. YU, C., ROY, R. J., KAUFMAN, H. and BEQUETTE, W., **Multiple-models Adaptive Predictive Control of Mean Arterial Pressure and Cardiac Output**. IEEE Transactions on Biomedical Engineering, Vol. 39, pp. 765-777, 1992.