Some Classes of Binary Operations in Approximate Reasoning

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Abstract. The aim of this paper is to study some binary operations in approximate reasoning. In the first part we summarize a former practical investigation of the applicability of the implication function based on the nilpotent minimum in that framework [1]. We also present some numerical examples to illustrate the results. In the second part we recall a constructive approach to the axiomatics of generalized modus ponens (GMP) published in [5]. As a consequence, a system of functional equations is obtained. Idempotent as well as non-idempotent conjunctions fulfilling this system are studied. The obtained results support the use of non-commutative and non-associative conjunctions and the corresponding implications in approximate reasoning.

Keywords. Approximate reasoning; generalized modus ponens; nilpotent minimum, and the related implication; conjunctions; R- and S-implications.

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1. Introduction

The compositional rule of inference plays a central role in fuzzy control and in approximate reasoning. It can be used to associate a fuzzy output with a fuzzy input through a relationship (in terms of a fuzzy relation) between them. In this context, mainly the minimum t-norm is used. In order to find further suitable operators, one has to know both the theoretical (mathematical) and the practical (computational) behaviour of the candidates.

The main aim of the present paper is twofold. First, to overview some results related to the usefulness of the implication operator based on the nilpotent minimum (see [3] for the original introduction, and [11] for a practical overview) in the above framework. Since if--then rules play a central role in diverse fuzzy systems (see e.g. [8, 12]), promising new candidates are important from both theoretical and practical points of view.

Nowadays it is needless to emphasize the dominance of t-norms, t-conorms, strong negations and related implications. Their sound theoretical foundation as well as their wide variety have given them almost an exclusive role in different theoretical investigations and practical applications. However, people are inclined to use them also as a matter of routine. The following examples support this statement and advocate the study of enlarged classes of operations for fuzzy sets and reasoning.

1) When one works with binary conjunctions and there is no need to extend them for three or more arguments, associativity is an unnecessarily restrictive condition. The same is valid for commutativity if the two arguments have different semantical backgrounds and it has no sense to interchange one with the other.

2) In GMP, a number of intuitively desirable properties is not possessed using t-norms and implications defined by t-norms.

3) Obviously, properties of conjunctions, disjunctions and negations have to be connected and to be in accordance with those of fuzzy implications. However, if one compares usual axioms for fuzzy implications with properties of R- and S-implications defined by t-norms, t-conorms and strong negations, then it can easily be observed that these two families have `much nicer' properties than it would be axiomatically expected.

4) Triangular norm based R- and S-implications are, in general, different. For continuous t-norms, these can coincide if and only if the underlying t-norm is isomorphic to the Łukasiewicz t-norm. For left-continuous t-norms, equality holds when we use e.g. the nilpotent minimum [3].

These observations, which are very often left out of consideration, have urged us to revise definitions and properties of operations in fuzzy logic. Thus, the second main aim of this paper is to give an overview of a new unifying approach for the investigation of these connectives. The original, more detailed presentation is [4].

2. Compositional Rule of Inference

Let X and Y be two given sets, A a fuzzy subset of X, and R be a fuzzy relation on $X \times Y$. We can associate with A a fuzzy subset B of Y by using R through the so-called compositional rule of inference, expressed by the following formula

$$B(y) = \sup_{x \in X} T(A(x), R(x,y)),$$

see [13]. We use the notation $B = A \circ_T B$.

One can look at this equation from a reverse point of view. Given a fuzzy subset A of X, a fuzzy subset B of Y and a t-norm T, can we find a fuzzy relation R on $X \times Y$ such that

(1) $A \circ_T R = B.$

This equation is called a fuzzy relational equation. The following result can be proved on the solution of equation (eq:cri) (see [7] and further references there).

Theorem 1. Let A, B be fuzzy subsets of X and Y, respectively, and let T be a left-continuous t-norm. The following are equivalent:

(i) There exists a fuzzy relation R on $X \times Y$ which is the solution of the relational equation (1).

(ii) The following fuzzy relation R_T on $X \times Y$ is a solution of (1):

(2) $R_T(A,B)(x,y) = I_T(A(x), B(y)).$

Moreover, if (1) is solvable, then $R_T(A,B)$ is the largest solution of (1).

Logical interpretation of the compositional rule of inference generalizes the classical Modus Ponens (see later in this paper).

In fuzzy control the value $R_T(A,B)(x,y)$ can be interpreted as the truth value of the statement

\`If x is A then y is B ',

which is called an if-then rule. Notice that there is only one input and one output in this case.

In practical problems both the number of inputs and outputs is higher. Nevertheless, we can restrict ourselves to the case when Y is equal to the set of real numbers R. Indeed, in case of higher dimensional output space we can consider several one-dimensional output systems in parallel. Unfortunately, this cannot be done in case of higher dimensional input spaces.

So, let X be an arbitrary input space, let $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ be normal fuzzy subsets of X and R with Borel measurable membership functions, respectively. Let T be a Borel measurable t-norm, and consider the rule base

$$
\begin{array}{l}
\text{If } x \text{ is } A_1 \text{ then } y \text{ is } B_1 \\
\text{If } x \text{ is } A_2 \text{ then } y \text{ is } B_2 \\
\qquad\qquad \vdots \\
\text{If } x \text{ is } A_n \text{ then } y \text{ is } B_n
\end{array}
$$

The first type of fuzzy controllers was defined in [mamdani]. The fuzzy relation R on $X \times \mathbf{R}$ is given by

$$R(x,y) = \max\left\{T(A_1(x), B_1(y)), \ldots T(A_n(x), B_n(y))\right\},$$

and the input-output function $F_M : X \to \mathbf{R}$ of the Mamdani controller is given by

$$F_M(x) = \frac{\int_{\mathbf{R}} R(x,y) \cdot y \, dy}{\int_{\mathbf{R}} R(x,y) \, dy}, \tag{3}$$

provided that $\int_{\mathbf{R}} R(x,y) \, dy > 0$.

Note that the measurability conditions above are usually satisfied in practical situations. In addition, the \`center of gravity' defuzzification method was used in (3).

3. Approximate Reasoning Using Nilpotent Minimum and Related Operations

Fuzzy inference systems generate inference results based on fuzzy if--then rules, expressing pieces of knowledge. Fuzzy implications are mostly used as a way of interpretation of the if--then rules with fuzzy antecedent and/or fuzzy consequent.

Consider a single input -- single output rule in the following form, as the k -th rule in the knowledge base:

\`If x is A_k then y is B_k '

where A_k and B_k are fuzzy subsets of X and Y, respectively.

Fuzzy if--then rules may be interpreted in two ways: as a conjunction of the antecedent and the consequent (Mamdani combination, see the previous section) or in the spirit of the classical logical implication, i.e., as a fuzzy implication. This second interpretation is applied in the present paper.

Suppose we observe that \` $x$ is $A'_k$ ', where $A'_k$ is a fuzzy subset of $X$ . Then, what can we infer from this observation? Something like \` y is B'_k ' with a fuzzy subset B'_k of Y .

Based on the Generalized Modus Ponens, the following equality describes B'_k (in terms of membership functions):

$$B'_k(y) = \sup_{x\in X} T(A'(x), I(A_k(x), B_k(y))), \quad (4)$$

where T is a t-norm and I is a fuzzy implication.

When we have a finite number of rules, two methods of approximate reasoning are mostly used: composition based inference (first aggregate then infer (FATI)) and individual-rule based inference (first infer then aggregate (FITA)).

Taking into account an arbitrary input fuzzy set and using the generalized modus ponens we obtain the output of fuzzy inference in a closed form. In individual-rule-based inference each rule in the fuzzy rule base determines an output fuzzy set and after that an aggregation via intersection or average operation is performed.

1. FATI type:

$$B'(y) = \sup_{x\in X} T\left(A'(x), \bigwedge_{k=1}^{n} I(A_k(x), B_k(y))\right) \quad (5)$$

2. FITA type:

$$B''(y) = \bigwedge_{k=1}^{n}\left[\sup_{x\in X} T\left(A'(x), I(A_k(x), B_k(y))\right)\right] \quad (6)$$

It was noted in [1] that $B' \subseteq B''$ in all cases. If the input fuzzy set A' is a singleton (i.e., crisp) then we have $B' = B''$ for any fuzzy implication.

A. Examples

In [1] we considered the following rule base:

If x is *fuzzyzero* then y is *fuzzyzero*.
If x is *fuzzyhalf* then y is *fuzzyhalf*.
If x is *fuzzyone* then y is *fuzzyone*.

where $fuzzyzero = (-0.5, 0, 0.5)$, $fuzzyhalf = (0, 0.5, 1)$, and $fuzzyone = (0.5, 1, 1.5)$ are triangular fuzzy numbers, just as the input $A' = fuzzyquarter = (0.125, 0.25, 0.75)$. These fuzzy numbers are visualized in Figure 1.

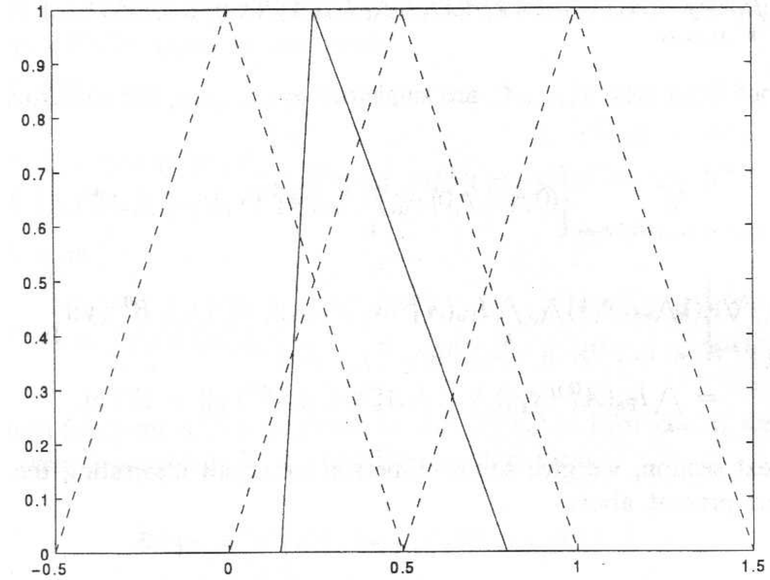

Figure 1: The knowledge base and the input

The aggregation operator is the minimum, the implication I_0 is the one defined from the nilpotent minimum $\wedge_0$. For more details see [3] and [11]. Therefore, we can write $I_0(A_k(x), B_k(y)) = 1$ if $A_k(x) \le B_k(y)$, and otherwise $I_0(A_k(x), B_k(y)) = (1 - A_k(x)) \vee B_k(y)$. This implication is illustrated in Figure 2.

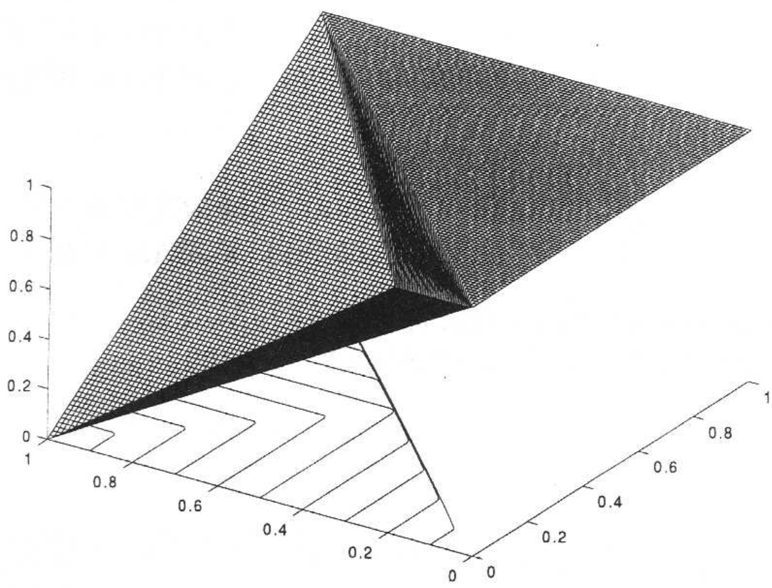

Figure 2: The implication function defined from the nilpotent minimum

The membership function of the output can be seen in Figure 3 when using the reasoning scheme FATI, while output of FITA is illustrated in Figure 4.

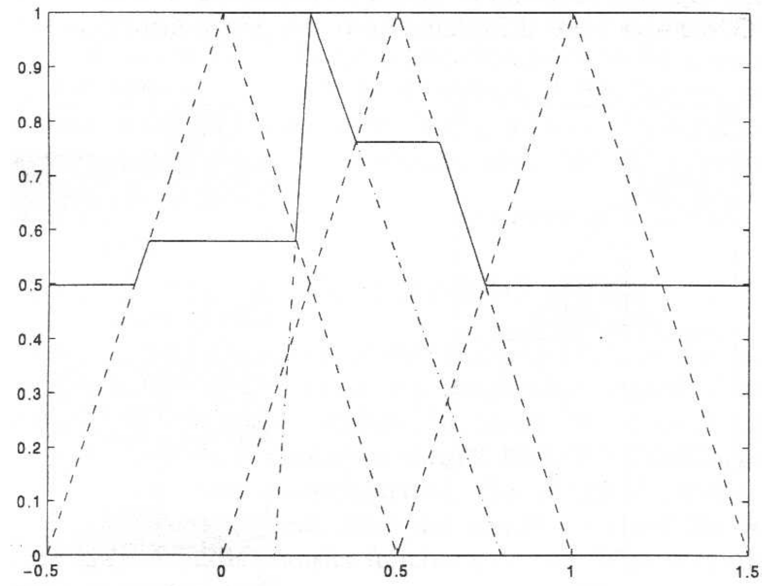

Figure 3: The output using FATI

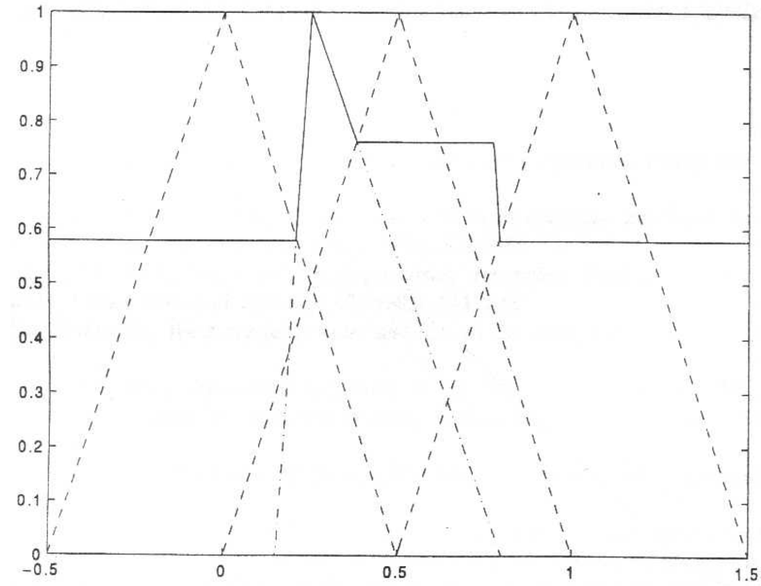

Figure 4: The output using FITA

For the particular case of singleton inputs (being important in constructing fuzzy controllers) the Fuzzy Logic Toolbox of Matlab was applied, with some extended special procedures. The inference system can be seen in Figure 5.

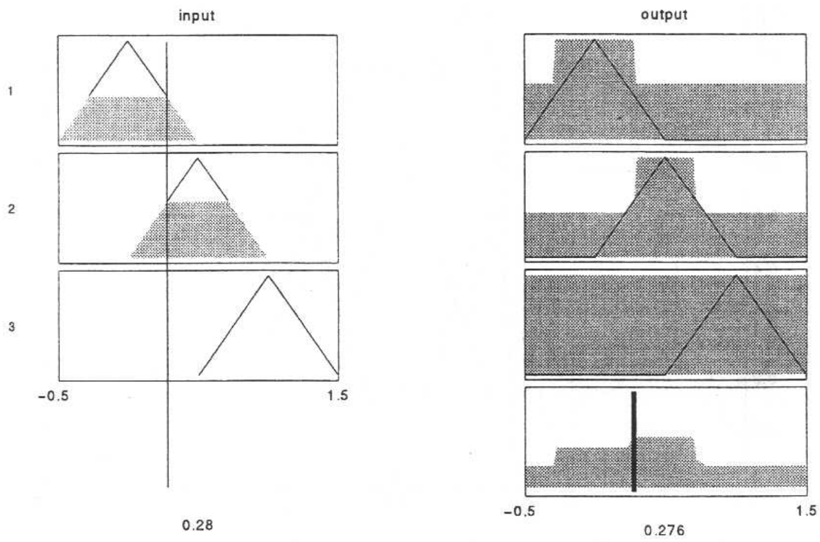

Figure 5: Illustration of the inference system

Closing this section, we present Figure 6. This illustrates a numerical example based on the previous inference system. A static transfer function for the implication I_0 with the application of a modified Center of Gravity defuzzification method is shown there. Two aggregation methods have been applied for the rules: product (continuous line) and bounded sum (dotted line) with the FITA method.

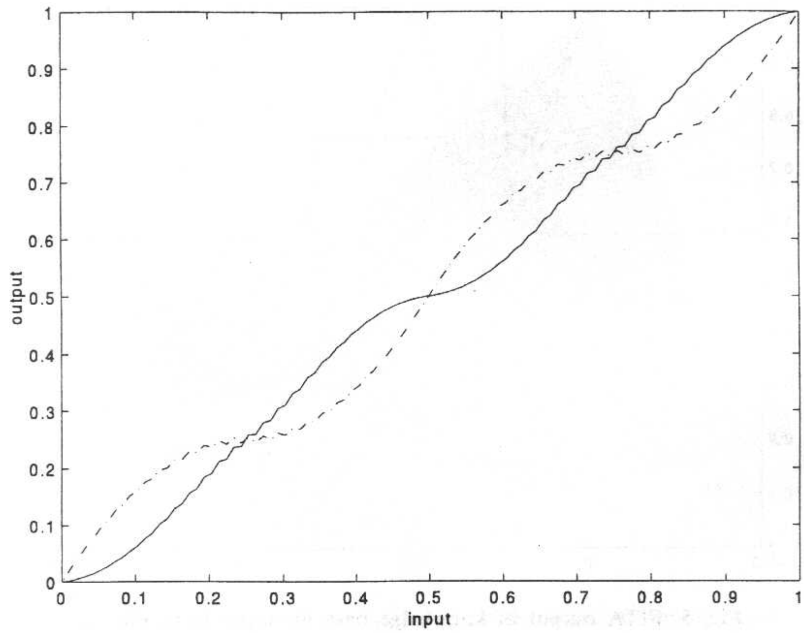

Figure 6: A static transfer function for I_0

4. Binary Operations in Generalized Modus Ponens

We already mentioned that the compositional rule of inference is also linked to the Generalized Modus Ponens (GMP) inference pattern with fuzzy predicates. In this framework, by using the notations introduced above, the general expression for B' is given as follows:

(7) $B'(y) = \sup_x M(A'(x), I_{A \to B}(x, y))$

where M is a fuzzy conjunction (that is, an extension of the classical Boolean conjunction, i.e., $M(x, y) \in [0,1]$ for every $x, y \in [0,1]$, and $M(0,0) = M(0,1) = M(1,0) = 0$, $M(1,1) = 1$) and $I_{A \to B}$ is a fuzzy binary relation (usually an implication) on $X \times Y$.

Usually, GMP is expected to meet a number of intuitively desirable requirements. Most papers on GMP investigate this problem by choosing first particular classes of conjuntions (e.g. t-norms) and implications (e.g. S- or R-implications based on t-norms) then testing whether the different requirements are fulfilled.

Opposed to these approaches, we choose a constructive way to investigate properties of GMP. First we fix only a few basic requirements to be fulfilled, in our opinion, by GMP. They lead to a system of functional equations for M and $I_{A \to B}$. In order to find a solution we assume some reasonable properties of conjunction and implication operators. Then a particular solution for M and $I_{A \to B}$ is given and some further properties of GMP are verified as consequences, however they usually appear as requirements in the rich literature on GMP (see e.g. references in [9]).

In the literature it is generally required that

if $A' = A$ then $B' = B$ ($A, B \not\equiv 0$);

if $\mathrm{Supp}A' \cap \mathrm{Supp}A = ?$ then $B' \equiv 1$ ($A, B \not\equiv 0$);

monotonicity: $B'(y)$ is non-decreasing with respect to $A'(x)$ and $B(y)$ and non-increasing with respect to $A(x)$;

if $A' \equiv 0$ then $B' \equiv 0$.

In our paper [4] we wanted to find at least one pair (M, I) such that R1--R4 are satisfied by using (7). Now we summarize those axiomatic results briefly.

Axioms for Operations in GMP

First we assume that $I_{A \to B}$ is defined pointwise, that is,

$I_{A \to B}(x, y)$ depends only on $A(x)$ and $B(y)$ i.e. $I_{A \to B}(x, y) = J(A(x), B(y))$

and so (7) turns into

(8) $B'(y) = \sup_x M\big(A'(x), J(A(x), B(y))\big)$

J is non-increasing with respect to its first argument and non-decreasing with respect to its second argument (shortly $J(?,?)$);

$J(0, v) = 1 \quad \forall v \in [0,1]$;

$J(1, v) \le v \quad \forall v \in [0,1]$.

M is non-decreasing with respect to both arguments (shortly $M(?,?)$);

$M(0, v) = 0 \quad \forall v \in [0,1[$;

A.3. $M(u,v) \le v \quad \forall u,v \in [0,1]$;

Obviously, these axioms are fulfilled when $M = T$ is a t-norm and J is either an R-implication or an S-implication based on T .

Naturally, the GMP should satisfy properties R1 - R4 also when A, A', B, B' are crisp sets. After simple calculations we finally get from the above equations and from R3 nd R4 the following system of equations for any $u, v \in [0,1]$

$$
\begin{aligned}
M(0, J(u,v)) &= 0 \\
M(1, J(0,v)) &= 1 \\
M(u, J(1,0)) &= 0 \\
M(1, J(u,1)) &= 1
\end{aligned} \tag{9}
$$

Replacing A , A' and B by fuzzy singletons (fuzzy points) of height u and v respectively, we have from R1 for any $u, v \in]0,1]$ the following equation:

$$M(u, J(u,v)) = v \tag{10}$$

Note that this last equation cannot be satisfied by using a t-norm T and R- or S-implication based on T . Therefore, we have to find solutions of (10) outside the class of t-norms and corresponding R- or S-implications.

By using our axioms A1 - A7, it is easy to see that we have

$$
\begin{aligned}
M(1,v) &= v \\
J(1,v) &= v \\
J(u,1) &= 1
\end{aligned} \tag{11}
$$

Under some continuity conditions, any solution (M,J) of (9)-(10)-(11) possesses further nice properties, as we state in the following theorem. For proof see [4].

Suppose that (M,J) is any solution of (9)-(10)-(11) satisfying axioms A1 - A8 and M is left-continuous in the first place while J is right-continuous in its first argument. Then the following properties are also satisfied by using (8):

if $A' \subset A$ then $B' \subseteq B$;

if $A' \equiv 1$ and $\inf_x A(x) = 0$ then $B' \equiv 1$;

if $A \equiv 0$ and $A' \not\equiv 0$ then $B' \equiv 1$.

B. Idempotent Solutions

In this section we are looking for solutions (M,J) of the system (9)-(10)-(11) such that both M and J are idempotent, i.e.,

$$
\begin{aligned}
M(x,x) = x &\quad \text{for all } x \in [0,1], \\
J(x,x) = x &\quad \text{for all } x \in]0,1].
\end{aligned}
$$

Note that idempotency of conjunctions is useful in dealing with redundancies in knowledge bases, see [2]. Moreover, idempotency of implications is not a very common property. The equality $J(x,x)=x$ can hold only on $]0,1]$ since $J(0,0)=1$.

Let us introduce, for typographical reasons, the following function:

$$m_\varphi(u,v)=\varphi^{-1}\left(1-(1-\varphi(u))^\alpha(1-\varphi(v))^{1-\alpha}\right),$$

where φ is an automorphism of the unit interval and $\alpha\in[0,1[$. In [4] we proved the following result.

Theorem 2. For any automorphism φ of the unit interval, functions M_φ and J_φ defined by

$$M_\varphi(u,v)=\begin{cases} m_\varphi(u,v) & \text{if } 0<u\le v\\ v & \text{if } u>v\\ 0 & \text{if } u=0\end{cases},$$

$$J_\varphi(u,v)=\begin{cases} \varphi^{-1}\left(1-\left(\frac{1-\varphi(v)}{(1-\varphi(u))^\alpha}\right)^{\frac{1}{1-\alpha}}\right) & \text{if } 0<u\le v\\ v & \text{if } u>v\\ 1 & \text{if } u=0\end{cases}$$

with $0\le\alpha<1$ *are solutions of the system (9)-(10)-(11). Moreover, both* M_φ *and* J_φ *are idempotent.*

C. Nonidempotent Solutions

In this section we are looking for appropriate new operations (both for conjunctions and implications) in the following form:

$$\frac{T(x,y)}{x},$$

where $x\in]0,1]$ and $y\in[0,1]$ and T is a t-norm. For motivation and more details see [4].

By important characterization results in [fodker95ham], the Hamacher family $\{T_\gamma\}_{\gamma\ge1}$ (see [6] for details on this family) of t-norms defined by

$$T_\gamma(x,y)=\frac{xy}{\gamma+(1-\gamma)(x+y-xy)}$$

is such that the following functions

$$M_\gamma(u,v)=\begin{cases} \frac{v}{\gamma+(1-\gamma)(u+v-uv)} & \text{if } u>0\\ 0 & \text{if } u=0,\end{cases}$$

are fuzzy conjunctions ($\gamma\ge1$). In addition, R- and S-implications based on each M_γ ($\gamma\ge1$) are the same and their common expression is given as follows:

$$J_\gamma(u,v) = \begin{cases} \frac{\gamma v+(1-\gamma)uv}{\gamma+(1-\gamma)(1-v+uv)} & \text{if } u>0 \\ 1 & \text{if } u=0 \end{cases}.$$

Each pair $(M_\gamma, J_\gamma)_{\gamma\geq 1}$ of fuzzy conjunctions and implications are solutions of our system (sys1)-(sys2)-(sys3), satisfying also axioms A1 - A8.

5. Conclusion

In this paper we summarized two approaches to approximate reasoning. In the first one the applicability of the implication based on nilpotent minimum in fuzzy inference was demonstrated. The obtained results are promising. The static transfer functions does not differ too much from the expected linear characteristic.

In the second part of the paper we investigated fuzzy conjunctions and implications from different points of view. By the results it became clear that one must be rather flexible in choosing connectives for particular reasons. Especially, non-commutative and non-associative conjunctions and the corresponding implications given above can fulfil the expected properties better than t-norms and related implications. Therefore, we would like to encourage readers to use more advanced operators not only in theoretical problems but also in practice.

REFERENCES

1. E. CZOGAŁA, J. FODOR, and J. ŁESKI. **The Fodor fuzzy implication in approximate reasoning**, Systems Science, 23 (1997), pp. 17-28.

2. D. DUBOIS and H. PRADE. **Fuzzy sets in approximate reasoning**, Part 1: Inference with possibility distributions, Fuzzy Sets and Systems 40 (1991), pp. 143-202.

3. J. FODOR, **Contrapositive symmetry of fuzzy implications**, Fuzzy Sets and Systems 69 (1995), pp. 141-156.

4. J. FODOR and T. KERESZTFALVI. **Non-standard conjunctions and implications in fuzzy logic**, Internat. J. Approx. Reason, 12 (1995), pp. 69-84.

5. J.C. FODOR and T. KERESZTFALVI. **A characterization of the Hamacher family of t-norms,** Fuzzy Sets and Systems 65 (1994), pp. 51-58.

6. H. HAMACHER. **ber logische Verknpfungen unscharfer Aussagen und deren zugehrige Bewertungsfunktionen**, Working Paper 75/14, RWTH Aachen, Aachen, Germany, 1975.

7. E.P. KLEMENT, R. MESIAR, and E. PAP. **Triangular Norms**, (Kluwer Academic Publishers, Dordrecht, 2000).

8. L. MADARASZ. **Intelligent Technologies and their Applications in the Complex Systems**. (University Press, ELFA, TU Košice, Slovakia, 2004), 384 p.

9. P. MAGREZ and P. SMETS. **Fuzzy modus ponens: a new model suitable for applications in knowledge-based systems**, Internat. J. Intelligent Systems 4 (1989), pp. 181-200.

10. E. H. MAMDANI and S. ASSILIAN. **An experiment in linguistic synthesis with a fuzzy logic controller**, Intern. J. Man-Machine Stud. 7 (1975), pp. 1-13.

11. J. TICK and J. FODOR. **Fuzzy implications and inference processes**, in: Proc. of the IEEE 3rd International Conference on Computational Cybernetics (ICCC 2005), (April 13-16, 2005, Hotel Le Victoria, Mauritius).

12. J. VAŠČAK and L. MADARASZ. **Similary relations in diagnosis fuzzy systems**. Journal of Advanced Computional Intelligence 4 (2000), pp. 246-250.

13. L.A. ZADEH. **Outline of a new approach to the analysis of complex systems and decision processes**, IEEE Trans. Syst., Man Cybern. 3 (1973), pp. 28-44.