Modern Control Theory
- A historical perspective -

Neculai Andrei

Research Institute for Informatics,

Center for Advanced Modeling and Optimization,

8-10, Averescu Avenue, Bucharest 1,

ROMANIA

E-mail: nandrei@ici.ro

Abstract. The purpose of this paper is to present a brief sketch of the evolution of modern control theory. Systems theory witnessed different stages and approaches, which will be very shortly presented. The main idea is that, at present, Control Theory is an interdisciplinary area of research where many mathematical concepts and methods work together to produce an impressive body of important applied mathematics. A general conclusion is that the main advances in Control of Systems would come both from mathematical progress and from technological development. We start with frequency-domain approach and end our historical perspective with structural-digraph approach, passing through time-domain, polynomial-matrix-domain frequential and geometric approaches.

Dr. Neculai Andrei graduated in Mathematics, "A.I. Cuza" University – Iassy, and in Electrical Engineering from "Politehnica" University – Bucharest. He took his Ph.D degree for his contributions to Digraph Approach of Large-Scale Sparse Linear Dynamic Systems in 1984. He has been a senior scientific researcher at National Institute for R&D in Informatics, ICI since 1973. He is author of 13 books and text books, as well as over 50 published papers and 300 Technical Reports in area of mathematical modeling and optimization. He is the key architect of *ALLO language and compiler* for modeling in mathematical programming, as well as of some professional packages for linear programming and large-scale unconstrained optimization. His main current scientific interests centre on modeling for mathematical programming and on large-scale non-linear optimization, conjugate gradient, interior point methods, penalty and barrier approaches. Dr. Andrei is a founding member of the *Operations Research Society* of Romania; founding member of the *Center for Advanced Modeling and Optimization*; member of the Editorial Board of *Computational Optimization and Applications* – Springer Verlag (since 1992); member of *Computational Optimization and Applications - Optimization Forum*; member of the Editorial Board of *Journal of Systems Science and Complexity* – Science Press – China and Allerton Press – USA (since 2001). He is Editor-in-Chief of the *Advanced Modeling and Optimization – An Electronic International Journal* – ICI Press (since 1999). Dr. Andrei is an Alexander-von-Humblodt fellow, member ERCIM and of Academy of Romanian Scientists (since 2001).

1. Introduction

This contribution is dedicated to present the main approaches and their mathematical support of development the control systems theory. The underlying idea is that, at present, Control Theory is an interdisciplinary area of research where many mathematical concepts and methods work together to produce an impressive body of important applied mathematics. Control systems theory witnessed different stages and approaches, which we will shortly describe here. The idea is to produce a general overview concerning the basics of these approaches, their origins, history and the way applications and interactions with mathematics and technologies has generated the development of the discipline. It is worth saying that all these approaches of Control Theory have their own value and still continue to represent important contributions both in theory and practice.

The word *control* has two main meanings. First, it is understood as the activity of testing or checking that a physical or mathematical device has a satisfactory behaviour. Secondly, *to control* is to act, to implement decisions that guarantee that device behaves as desired. To control is to get "*ordo ab chao*".

Control idea trace back in times of Aristotle (384-322 BC), [Bennet, 1979]. In his book "*Politics*", one of the most influencing books ever written, in Chapter 3, Book 1, he has written:

"*… if every instrument could accomplish its own work, obeying or anticipating the will of others … if the shuttle weaved and the pick touched the lyre without a hand to guide them, chief workmen would not need servants, nor masters slaves.*"

We see that Aristotle described in a very transparent manner the purposes of Control Theory: to automatize processes in such a way to achieve their purposes they have constructed of, and to let the human being in liberty and freedom.

For the very beginning, the human being was under the ***primacy of existence***. The description of physical or artificial systems was more linguistic, often not they are, but more they like to be. Even the mathematics in old times, as synthesised by Euclid (325-265 îChr.) and Diophantus of Alexandria (200-284 BC), was expressed in terms of syllogisms. The world was seen as a Noun or as a Verb. The ancient philosophers sentenced that the world can be discussed in terms of processes governed by laws, all living under the commandment "*nothing too much*". The first attempt to introduce computation was given by Fibonacci (1170-1250) in his *Liber abaci.* However, after three centuries the importance of computation came into being.

René Descartes (1596-1650) came with the concept of *method.* Galileo Galilei (1564-1642) brought into use the *physical experiments* and later Sir Isaac Newton (1642-1727) and Baron Gottfried Leibniz (1646-1716), introducing *calculus* achieved the first metamorphosis of science by transforming the operational foundations of science, that were mainly the heritage of Aristotle, into its modern form we know today. They established the ***primacy of mathematics***, under which we still live today. This new attitude consisted of the use of *physical experiments* and the use of *mathematical models* involving differential equations. This primacy of mathematics is so much active that the scientists still continue to refer to the mathematical representation of the creation, to the mathematical models, avoiding to get down to existence. They are interested to solve these mathematical models, to get their properties and to study their solution.

However, the character and structure of science, mainly over the past century, has been going through a second metamorphosis, which is the result of two classes of discoveries [Jackson, 1994]. The first group of discoveries refers to the limitations we can obtain about the dynamic behaviours of the creation from the mathematical models. It was discovered that all mathematical models have limitations to make: *analytical mathematical deductions*, *deterministic physical predictions*, and *structurally stable models of closed systems*. Additionally, the discovery of Kurt Gödel (1906-1978), that neither the consistency nor the completeness of any sufficiently general mathematical system can be proved within that system by generally accepted logical principles, struck at the foundations of mathematics that the mathematical systems can establish any result which is true. The second group of discoveries is in close connection with the raising of the *computer science and informatics*. These enlarged the operational basis for scientific investigations by introducing the *computer experiments*. The second metamorphosis of science enlarged the operational basis of physical experiments and mathematical models by including the third operational basis to get knowledge as *computational or numerical experiments*. These numerical experiments gave the scientists the access to a strange world. Intensive and very sophisticated computational experiments with mathematical models determined the re-addressing to the real existence. Rephrased, *numerical experiments restored the primacy of existence, and the discipline that achieved this restoration is informatics.* Basically we can define informatics as "*coming down in computational of the mathematical concepts, turning these mathematical concepts in algorithms, the study of the associated algorithms subject to convergence and complexity*". This is the substance of informatics – *transforming the advanced mathematical concepts in algorithms, implementation of algorithms in computing programs*. In a way, *informatics is computational linear algebra*. The control systems theory determined the development of science in all its aspects and this in turn influenced the development of control theory.

The structure of this work is as follows. In section 2 we present the key concepts used in control theory. We emphasize the concept of feedback, the need for fluctuations and the optimization. Section 3 is for Frequency-Domain approach of systems control theory, where fundamental is the concept of transfer function. In section 4 we describe the Time-Domain Algebraic approach which is based on the theory of differential equations. The Polynomial-Matrix-Domain Frequential approach is presented in section 5. This is a very natural extension of classical transfer function description to multi-input, multi-output systems. The next section is dedicated to Geometric approach which is an extension to algebraic approach. Finally, in the next section the Structural-Digraph approach is considered where the structure of the system is crucial in designing its control.

2. Fundamental Concepts in Control Theory

Although the mathematical formulation of control problems, based on the mathematical models of physical systems, is intrinsically complex, the fundamental ideas in control theory are enough simple very intuitive. These key ideas can be found in Nature, in the evolution and behaviour of living beings. There are three fundamental concepts in control theory.

The first one is that of *feedback.* One of the most important contribution of Charles Darwin (1805-1882) was the theory that feedback over long time periods is responsible for the evolution of species. Later, Vito Volterra (1860-1940) used this concept to explain the balance between two populations of fish in a closed pond. But, the most influencing was Norbert Wiener (1885-1964) who introduced the fruitful concepts of positive and negative feedback in biology. In engineering, this term has been early introduced by the engineers of the Bell Telephone Laboratory [Mayr, 1970]. Now it is an active concept in practically all area of activity. A feedback process is one in which the state of the system, or its output, determines the way in which the control has to be computed at any time instant.

The second key concept in control theory is that of the *need for fluctuations*. This is a basic principle that we apply and use many times in our every day life. Basically, the idea behind this concept is that we do not have necessarily to stress the system and drive it so brutally to the desired state immediately or directly. Very often, it is more efficient and physically realisable to control the system letting it to fluctuate, and trying to find that dynamics that will drive the system to the desired state without forcing it too much. This concept has been revealed early by Hall [1907] when he compared the action of political economists, who admitted that a proper action of the law of supply and demand must admit fluctuations, and the engineers who generally not recongnized the need of fluctuations in steam engine governors. The need for having fluctuations is a very general principle we find it also in penalty function or interior point methods from mathematical programming [Andrei, 1999a, 1999b, 2004a, 2004b].

The third very important concept in control theory is that of *optimization.* This is a very well established branch of mathematics, whose goal is to find the values for variables in order to maximize the profit or to minimize the costs subject to some constraints. It is in close connection to Control Theory mainly because a large variety of problems arising in system and control theory can be reduced to a few standard convex or quasi-convex optimization problems involving linear matrix inequalities. The important aspect is that the resulting optimization problems can be solved numerically very efficiently using the interior point methods. Therefore, the reduction of control problems to optimization constitutes a solution to the original problems, clearly in a very practical sense [Andrei, 2001], [Boyd, El Ghaoui, Feron and Balakrishnan, 1994]. Additionally, the contributions of Richard Bellman (1920-1984) by introducing the *dynamic programming* and of Lev Pontryagin (1908-1988) with its *maximum principle* for nonlinear optimal control, established the foundations of Modern Control Theory.

3. Frequency-Domain Approach

One of the first mathematical analysis of control systems was the frequency-domain approach. This is based on the developments of Pierre-Simon de Laplace (1749-1827), Joseph Fourier (1768-1830), Augustin Louis Cauchy (1789-1857), and others. The central concept of frequency-domain approach is that of *transfer function*. The transfer function of a linear time-invariant system is defined as $Y(s)/U(s)$, where $Y(s)$ is the Laplace transform of the output, and $U(s)$ is the Laplace transform of the input of the system. It turns out that the transfer function is the Laplace transform of the *system impulse response* $h(t)$. Therefore, $H(s)=Y(s)/U(s)$, i.e. $H(s)$ embodies the *transfer characteristics* of the system. This approach is appropriate for linear time-invariant systems, especially for single-input/single-output systems where the graphical techniques are very efficient.

Harry Nyquist (1889-1976)

Frequency-domain approach originated in the process of solving of a major problem referring to the mass communication systems over long distances. To reduce distorsions in amplifiers, after six years of persistence, Harold S. Black (1898-1983) revolutionized telecommunications by introducing the *negative feedback* in 1927 [Black, 1934]. As a method of system control, this has had a great impact in a large number of applications.
The theory of design the stable amplifiers was developed by Harry Nyquist (1889-1976) at Bell Laboratories. He derived his stability criterion, generally called the *Nyquist stability theorem*, based on the polar plot of the transfer function [Nyquist, 1932].

Later on, also at Bell Laboratories, Hendrik Bode (1905-1982) used the magnitude and phase frequency response plots of the transfer function to investigate the closed-loop stability, and introducing the notions of *gain* and *phase margin* [Bode, 1940]. In 1947, at MIT Radiation Laboratory, Nathaniel B. Nichols (1914-1997) developed his *Nichols chart* for the design of feedback systems, establishing the theory of servomechanisms [James, Nichols and Phillips, 1947]. A major step in design of control systems is the *root locus method* introduced by Walter R. Evans (1920-1999) at North American Aviation. The idea behind this method is to use the poles and zeroes of the open loop system to determine the properties of the closed loop system when one parameter is changing.

Hendrik Bode (1905-1982)

The classical control theory was expressed in the frequency domain and the s-plane using the methods of Nyquist, Bode, Nichols and Evans. All that is needed is the magnitude and phase of the frequency response, or the poles and zeroes of the open loop transfer function. This is very implementable for single-input/single-output systems since all these elements, the frequency response, and poles and zeroes of a transfer function, can accurately be determined. More than this, robust design is implemented using notions of gain and phase margin. To determine the transfer function of complex systems the block diagram algebra is very intensive used. It is not necessary for an internal description of the system dynamics; that is, only the input/output behavior of the system is needed.

The graphical techniques are difficult to apply for multi-input/multi-output, or multi-loop systems. Due to the interactions among the control loops in a multivariable system, even that each single-input/single-output transfer function has acceptable properties concerning the step response and robustness, the whole system can fail to be acceptable. The *quantitative feedback theory* developed by Horowitz, overcome many of these limitations, providing an effective approach for the design the multivariable systems [Horowitz, 1963], [Horowitz and Sidi, 1972]. Quantitative feedback theory is a frequency-domain technique utilising the Nichols chart in order to achieve a robust design over a specified region of plant uncertainty. Basically, the desired time-domain responses are translated into frequency domain tolerances, which lead to bounds on the transfer function. Concerning the nonlinear systems, the classical techniques can be considered on a linearized version of a nonlinear system, at an equilibrium point where the system behaviour is approximately linear.

The above description of systems is useful in some circumstances, but is still very limited. The important omitted factors are the dynamical changes and the internal mechanisms by which the system transforms the inputs in outputs. Consequently, some new representation of a system has been considered, as we will see in the following.

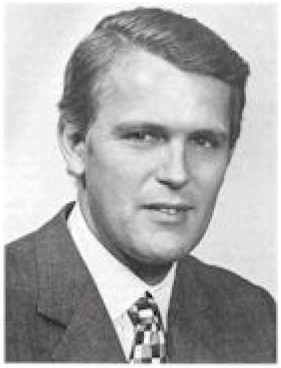
4. Time-Domain Algebraic Approach

This approach is based on the theory of differential equations. This theory is developed due to the infinitesimal calculus created by Newton and Leibniz, and the work of brothers Bernoulli, Jacopo Riccati (1676-1754), Leonhard Euler (1707-1783) and others. The analysis of motion of the dynamical systems by means of differential equations has been considered by Joseph-Louis Lagrange (1736-1813) and William Rowan Hamilton (1805-1865).

One of the most important problem considered in this representation was that of *stability*. George Airy (1801-1892) was the first to discuss the instability of a closed-loop system using the differential equations [Airy, 1840]. James Clerck Maxwell (1831-1879) analyzed the stability of Watt's governor [Maxwell, 1868]. His idea was to linearize the differential equation of motion in order to find the characteristic equation of the system. He proved that the system is stable if the roots of the characteristic equation have negative real parts. Later, Edward Routh (1831-1907) provided a numerical technique for determining when a polynomial has negative roots, giving a treatise on the stability of a given state of motion [Routh, 1877]. Using differential equations, independently of Maxwell, Vishnegradsky [1877] analyzed the stability of regulators. But, the most elegant and general theory of stability was created by Alexander Lyapunov (1857-1918). He studied the stability of nonlinear differential equations using a generalized notion of energy [Lyapunov, 1893]. He introduced some concepts and techniques which are still in use. Following the idea of Lyapunov, Yakov Tsypkin (1919-1997) considered the *phase plane* for stable nonlinear control design and Vasile Mihai Popov [1961] provided the *circle criterion* for nonlinear stability analysis.

Another important problem considered in time-domain representation was that of *optimal control* and *estimation*. Johann Bernoulli (1667-1748) was the first who articulated the principle of optimality. The Brachistochrone problem was independently solved by Bernoulli and Newton, thus firmly establishing the power of calculus. Later on, various optimality principles were formulated by Pierre de Fermat (1601-1665) (in optics), Carl Friedrich Gauss (1777-1855), Jean d'Alembert (1717-1783), Pierre de Maupertuis (1698-1759), Euler, Lagrange and Hamilton, and Albert Einstein (1879-1955) (in mechanics). In 1957 Richard Bellman formulated the *dynamic programming* principle to the optimal control of discrete-time systems [Bellman, 1957], and in 1958 Lev Pontryagin developed the *maximum principle* for solving nonlinear optimal control problems [Pontryagin *et al*., 1962]. Both these optimality principles characterize the optimal control by means of a feedback law. The main idea of Bellman was to introduce the *value function* (the Bellman function) which satisfies the Hamilton-Jacobi equation. On the other hand, the Pontryagin maximum principle is based on maximization of the Hamiltonian associated to the system by means of the adjoint state equation. It is worth saying that in both approaches the conclusions are the same. However, while the Pontryagin's principle extends the concept of Lagrange multiplier from mechanics, the Bellman's principle provides a novel viewpoint in which the value function and its time evolution has a vital role [Ionescu and Popeea, 1981].

The modern era in control theory started with the work of Rudolf Kalman who published a number of books in which the main problems of nonlinear systems theory was presented. In [Kalman and Bertram, 1960] the Lyapunov stability in time-domain of nonlinear systems is considered. The optimal control of systems as well as the design of linear quadratic regulator is discussed in [Kalman, 1960a]. The optimal filtering and estimation theory, and the design equation for the discrete Kalman filter was presented in [Kalman, 1960b]. The continuous version of Kalman filter was developed in [Kalman and Bucy, 1961]. To overcome the limitation of frequency-domain approach, which is very much an art and provide a non-unique feedback, Kalman introduced the concept of "*state*", a mathematical entity that mediates between inputs and outputs. The importance of this concept is based on the fact that the state of a dynamical system emphasizes the notions of causality and internal structure.


Rudolf Kalman

For finite-dimensional systems, i.e. systems for which the state belongs to a finite-dimensional vector space, the representation is given by a first-order vector differential equation of the following form:

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t),$$

where $x(t)$ is the vector of internal variables, or system states, $u(t)$ is the vector of control inputs and $y(t)$ is the vector of measured outputs. The matrices A, B and C describes the system dynamical interconnections.

Using this representation Kalman formalized the notion of feedback control and optimality in control and estimation theory. In control theory he introduced the fundamental notions of *controllability, observabillity, detectability*, etc. and used them to determine a feedback control of the form:

$$u(t) = -Kx(t),$$

in order to achieve suitable closed-loop properties. In the standard linear quadratic regulator problem the feedback matrix K is determined to minimize a quadratic index:

$$J = \int_0^\infty (x^T Qx + u^T Ru)\mathrm{d}t,$$

where Q and R are weighting matrices, that is design parameters. The importance of linear quadratic regulator design is the fact that if the Q and R are correctly chosen, than the feedback gain matrix K can be computed to make J finite, i.e. the integral involving the weighted norms of $u(t)$ and $x(t)$ is bounded, and therefore $x(t)$ and $u(t)$ go to zero in time. This property guarantees the closed-loop stability of the system [Kalman, Falb and Arbib, 1969]. Feedback laws like $u(t) = -Kx(t)$ are called static. An alternative to static feedback is the *dynamic compensator* of the form:

$$\dot{z}(t) = Ez(t) + Fu(t) + Gy(t),$$

$u(t) = Hz(t),$

where the inputs of the compensator are the system inputs and outputs. Now, the design problem is to select matrices E, F, G and H in such a manner to get good closed-loop performance of the system.

An efficient solution to the fundamental problems of linear systems theory like: pole placement, localization and exact disturbance rejection, restriced decoupling, extended decoupling, left invertibility, simultaneous decoupling and pole assignment, simultaneous disturbance localization and decoupling etc. can be obtained by means of dynamic compensators. However, a disadvantage and a limitation with this design using dynamic compensator is that the dimension of the compensator is the same as of the plant. The controllability or observability subspaces does not have a minimal dimension. There is a lack of transparency, relationship to frequency response methods is not apparent. The geometric approach and the structural-digraph approach consider these subspaces of minimal dimension giving thus very elegant and efficient design algorithms for solving the fundamental problems of linear systems theory. Besides, the linear quadratic regulator using static or dynamic feedback design procedures has no guaranteed robustness properties.

The problem of designing controllers that satisfy both the robust stability and some performance criteria is called robust control. H_∞ control theory is one of the cornerstones of modern control theory. It was developed to solve such problems with very strong practical implications. The widely accepted modern technique for solving robust control problems now is to reduce them to linear matrix inequalities problems (LMI). Historically, the first LMIs appeared around 1890 when Lyapunov showed that *the linear dynamic system* $\dot{x}(t) = Ax(t)$ *is stable, i.e. all its trajectories converge to zero, if and only if there exists a solution to the matrix inequalities:* $A^T P + PA < 0,\ P = P^T > 0,$ *which are linear in unknown matrix* P. In 1940 Lu're, Postnikov and others, applied Lyapunov approach to control problems with nonlinearity in the actuator thus obtaining stability criteria in the form of LMIs. These inequalities were polynomial (frequency dependent) inequalities.

Later on in 1960 Vladimir Yakubovich, Popov, Kalman, Anderson, and others obtained the positive real lemma which reduces the solution of the LMIs to simple graphical citerion: the circle criterion by Popov and Tsypkin's criteria. It is fair to say that Yakubovich is the father of the LMI field. His results on the solution of certain special matrix inequalities, published early in 1962, are widely known. Ionescu and Stoica, [1999] considered the robust stabilization and H_∞ problems based on the generalized Popov-Yakubovich theory. The reduction of a robust control problem to an LMI problem provides a solution. The key concept is the Kalman-Yakubovich-Popov Lemma:

Vladimir A. Yakubovich

Given a number $\gamma \geq 0,$ *two* n *vectors* b, c *and an* $n \times n$ *Hurwitz matrix* A*, of the single-input single-output minimal system* (A, b, c)*, if the pair* (A, b) *is completely controllable, then* q *satisfying:*

$$A^T P + PA = -qq^T,$$

$$Pb - c = \sqrt{\gamma} q,$$

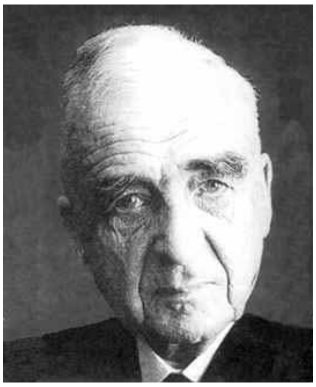
exists if and only if

$$\gamma + 2\,\mathrm{Re}\left[c^T (j\omega I - A)^{-1} b\right] \geq 0$$

for all real ω.

The Kalman-Yakubovich-Popov lemma connects two areas of control theory: frequency methods and time-domain algebraic methods. It leads to the Positive Real Lemma, the Bounded Real Lemma, Circle Criterion, network theory, adaptive control etc. Popov [1962] gave the famous Popov frequency-domain stability criterion for the absolute stability of nonlinear systems. Popov's criterion could be checked using graphical means, by verification that the Nyquist plot of the "linear part" of the nonlinear system was confined to a specific region in the complex plane. Yakubovich [62,64] established the connection between the Popov criterion and the existence of a positive definite matrix satisfying certain matrix inequalities, establishing the area of linear matrix inequalities in control theory. The importance of LMI is that it can be efficiently solved using interior point methods. The interior point methods started a revolution in mathematical programming with the work of Karmarkar, published in 1984. In 1988 Nesterov and Nemirovskii developed interior point methods that apply directly to linear matrix inequalities showing that LMIs can be efficiently solved with convex optimization techniques. [Boyd, Ghaoui, Feron and Balakrishnan, 1994]. Generally, in control problems we do not encounter the LMI in canonical or semidefinite form, but rather with matrix variables. The most software packages for solving LMI work with the canonical or semidefinite forms. Therefore, a pre-processing phase is required. To convert a nonlinear convex matrix inequality into an LMI we can use the Schur complement:

$$\begin{bmatrix} A(x) & B(x) \\ B^*(x) & C(x) \end{bmatrix} > 0 \quad \Leftrightarrow \quad C(x) > 0, \;\; A(x) - B(x)C^{-1}(x)B^*(x) > 0.$$



Paul Finsler (1894-1970)

A very useful trick in robustness is the Lemma given by Paul Finsler (1894-1970), which directly follows from the theorem of alternatives in LMIs theory:

The following statements are equivalent

$x^* Ax > 0$ *for all* $x \neq 0$ *subject to* $Bx = 0$,

$\tilde{B}^* A\tilde{B} > 0$ *where* $B\tilde{B} = 0$,

$A + \rho B^* B > 0$ *for some scalar* ρ,

$A + XB + B^* X^* > 0$ *for some matrix* X.

Recently, Nesterov showed that the positivity of a polynomial can be expressed as an LMI. This gives a new unified insight into several problems encountered in control: spectral factorization of polynomials (H_2 and H_∞ optimal control), global optimization over polynomials (robust stability analysis), positive real and bounded real lemma (nonlinear systems and H_∞ control), sufficient stability conditions for polynomials (robust analysis and design).

5. Polynomial-Matrix-Domain Frequential Approach

In this approach a *matrix-fraction description* and *polynomial equation design* is considered, where a multi-input/multi-output system is described not in state-space form, but in input/output one, being a direct extension of the classical transfer function description with very powerfull design capabilities.

William Volovich

Rosenbrock [1974] and Wolovich [1974] considered this approach based on polynomials in a complex variable s obtained as Laplace transforms of the differential equations under consideration in zero initial conditions. The dynamical behavior of a $m-$ input, $p-$ output, linear, time invariant, system may be represented by a proper $p \times m-$ transfer matrix $T(s)$, where $y(s) = T(s)u(s)$, and the proper transfer matrix can always befactored as $T(s) = R(s)P(s)^{-1}$, where $R(s)$ and $P(s)$ are relatively right prime polynomial matrices of dimensions $p \times m$ and $m \times m$ respectively.

Using this representation Wolovich presented a very general compensator able to achieve any desired closed loop transfer matrix. In this context the compensator is specialized in order to solve the fundamental problems of arbitrary pole placement, static and dynamic decoupling and exact model matching. The advantage of this approach is that the desing objectives can best be described in the frequency domain in terms of a desired transfer matrix, and the polynomial-matrix compensation scheme can be employed to achieve any desired transfer matrix.

The latest development of this approach is the *polynomial systems theory* [Blomberg and Ylinen, 1983], [Ylinen, 2003]. The fundamental idea in the case of differential systems is that the differential operator $p = d / dt$ is interpreted as a linear mapping from the space X of differentiable time functions into itself. Thus, the differential equations are represented by equations in $p-$polynomials. Problems referring to stabilization, non-interacting systems, estimation and diagnostics can be solved. The advantage of this theory is that many features are in common with the classical transfer function methods, but works in an efficient way in the multivariable case. The main drawback of polynomial systems theory is that the essential operations are carried out in the ring of $p-$polynomials which is a weak algebraic structure. The equations in $p-$polynomials generally can not be solved with respect to an unknown signal. Thus, the equation solving is replaced by structural considerations.

Later, Willems [1991, 1997] has introduced the concept of *behavioural systems theory*, which in principle is the same as the polynomial systems theory.

LMI gives a technical support to the polynomial-matrix-domain frequential approach of systems theory. Indeed, the set of polynomials that are positive on the real axis is a convex set that can be described by an LMI. This idea originating from the work by Shor is related with David Hilbert's (1862-1943) 17th problem about algebraic sum of squares decompositions. A sufficient stability condition of a polynomial matrix can be characterized by an LMI: polynomial matrix $R(s)$ is stable if and only if there exists a polynomial matrix $P(s)$ and a matrix $P = P^* > 0$ satisfying the LMI

$$P^* R + R^* P - S(P) > 0,$$

where $S(P) = V^*(S \otimes P)V$ and V is a special permutation matrix.

6. Geometric Approach

The geometric approach of linear systems theory is an extension of the algebraic approach and originated by the papers of Basile, Laschi and Marro [1969] and Basile and Marro, [1969a,b], where the *controlled* and *conditioned invariants* were introduced. Wonham and Morse [1970] renamed these objects as *(A,B)-invariants* and *(C,A)-invariants* which played a crucial role in multivariable control, establishing the geometric design approach of closed-loop multivariable systems.

Using the abstract geometric concepts of linear spaces, Wonham [1979] consolidated the geometrical apprach and articulated a compact and coordinate-free formulation and solution for many problems in linear control theory, including: model matching, disturbance rejection, reference tracking, decoupling and pole placement, etc. Basile and Marro [1982], and Schumacher [1983] introduced new geometric objects, the so-called *self-bounded controlled invariants* and *self-hidden conditioned invariants*, which proved to be very effective in minimizing the complexity of the dynamic compensators, solution of the above mentioned problems with stability. The minimality of compensators is the key aspect of geometric approach of linear system theory.

W.M. Wonham

Now the efforts are directed to link the Kalman control and filtering (the H_2 control and filtering) to the geometric decoupling, thus giving the possibility for solving the singular problems both in discrete and continuous-time case [Stoorvogel, 1992], [Saberi, Sannuti and Chen, 1995], [Marro, Prattichizzo and Zattoni, 2002].

The main drawback of this approach is that the comparatively simple language of matrix algebra is translated into the more abstract language of high-dimensional vector space where the intuition is almost lost.

7. Structural-Digraph Approach

All the above described approaches of control theory have serious limitations:

- The description of the systems, both in differential equation form and in polynomial-matrix frequential form, has entries regarded as numerical values which are exactly known.
- The algorithms associated for system analysis and controller synthesis are based on matrix manipulation without considering any particular structure of the system or of the feedback.
- The procedures for feedback synthesis are suitable for systems with a reduced number of inputs, outputs and states. A typical feature of real large-scale systems, *sparsity*, is not considered in these procedures.

In order to overcome these limitations a digraph-theoretic approach has been considered. The idea behind this approach is to elaborate structural methods for obtaining the classes of the feedback matrices with minimal structure, solutions of the fundamental problems of linear control theory, which change in a minimal sense the original structure of the system. The first comprehensive analysis of the digraph-theoretic approach of large-scale linear systems has been considered by Andrei [1984, 1985]. The method considered grasp the structure of the system and its properties, which are invariant when the numerical parameters are modified, and design the feedback matrices with minimal structure. Thus, *the solving process of the fundamental problems is embedded in some subspaces of smaller dimensions, where the numerical computations are non-trivial.*

For any large-scale, time invariant, linear dynamic system a digraph is associated in a very canonical manner. The digraph consists of a number of nodes and edges associated to the nonzero entries of the matrices of the system. In this representation a number of digraph objects are introduced as: *the supremal (A,B)-invariant subdigraph*, *the supremal (A,B)-controllable subdigraph* and *the infimal (C,A)-invariant subdigraph*. Using these digraphs the conditions to determine the minimal structure of the feedback matrices are given [Andrei, 1983, 1984, 1985]. The main problems considered are the digraph pole placement, the digraph exact disturbance rejection, the combined pole placement with digraph exact disturbance rejection, digraph decoupling, and the combined problem of digraph decoupling with exact disturbance rejection. *The key aspect of the digraph-theoretic approach is that the structure of the feedback matrices is determined directly from the structure of the system.* The numerical computations are dramatically reduced in some subspaces of small dimensions. Using this technique a better insight into the structural nature of the design procedures is obtained. Thus, *the structure of the feedback matrix and some of its elements are determined in such a way that a desired property of the closed-loop system holds generically (independently of numerical value of parameters), the rest of parameters of the feedback matrix may be considered for some other design requirements or optimization*. The digraph-theoretic representation suggests *the minimal modification of the structure of the system* in order to fulfil a desired property [Andrei, 1984, 1985]. Later, Reinschke [1988] and Wend (1993) consolidated this approach.

8. Conclusion

In this presentation we travelled through Control Systems Theory developments starting with the Frequency-Domain approach and continuing with Time-Domain Algebraic, Polynomial-Matrix-Domain Frequential, Geometric and Structural-Digraph approaches. A general conclusion is that, at present, Control Theory is an interdisciplinary area of research where many advanced mathematical concepts, techniques and methods work together to produce an impressive body of important applied mathematics. The advances in Control of Systems are coming both from mathematical progress and from technological development. The main source of methods and techniques for solving the fundamental problems in Control Theory is given by Mathematical Programming Theory and Optimal Control. The development of semidefinite programming and second order cone programming together with the interior point methods lead to a corpus of methods and algorithms able to solve in a unified manner real design and management problems in a very large diversity of domains.

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