H_∞ Guaranteed Cost Fuzzy Control for Non-linear Systems: An LMIs Approach

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Abstract: In this paper a state feedback controller is designed to stabilize a discrete-time nonlinear and uncertain system. The Takagi Sugeno (TS) fuzzy model with norm-bounded uncertainties is considered for modelling the nonlinear system. Beyond stability, a mixed (LQ/H_∞) index is also considered, which simultaneously guarantees an upper bound of LQ cost and is robust for unknown external disturbance in the sense of induced H_∞ norm. Sufficient conditions for existence of fuzzy state feedback controller are derived in terms of linear matrix inequalities. Finally the effectiveness of the proposed controller design methodology is illustrated through numerical simulations.

Keywords: Fuzzy control, uncertain non linear system, guaranteed cost, robust stability, linear matrix inequalities

1. Introduction

Most industrial plants have sever nonlinearities and parametric uncertainties for which the design of suitable controls presents many difficulties. To overcome these difficulties, the fuzzy logic control has been found an effective approach.

Recently, a great amount of efforts has been developed in an attempt to describe nonlinear systems using the Takagi-Sugeno (TS) fuzzy model [18].

The TS fuzzy model represents a nonlinear system by a set of local linear models, which are smoothly blended together trough membership functions. This representation allows the designers to take advantage of conventional linear system to analyse and design the fuzzy model control. [6,9,15,21,23]

The control design is carried out based on the fuzzy models via the so-called parallel-distributed compensation PDC scheme [5,7,15,16,19].

Since uncertainties often degrade system performance and may even lead to instability, a number of results have appeared on stability analysis and control synthesis for uncertain fuzzy systems [3,4,5,7,9,15,16,19,25,26].

In the last two decades, many researchers have worked on robust linear quadratic problem in attempt to guarantee robust stability and robust performance in the presence of plant uncertainties.

The H_∞ control, against unknown disturbance, has also been studied by a number of researchers [8,11,13,17,27].

Recently many works on TS fuzzy model-based control for nonlinear system were developed with adequate performance [4,12,19,21,22,23,24,25]. Y Cao et al [3] presented robust H_∞ controller design for a class of uncertain discrete time fuzzy dynamic systems with norm bounded uncertainties. H.N. Wu et al in [25] develop an LMI-based control method for uncertain nonlinear systems with H_2 performance. B.S. Chen et al. [9] presented mixed H_2/H_∞ controllers for a nonlinear system via observer-based output feedback.

The aim of this paper is to design a state feedback controller for a class of fuzzy uncertain system that guarantees the mixed (LQ/H_∞) index. This study implies that LQ cost has an upper bound and that the induced H_∞ norm, from external disturbance to LQ cost, is less than a prescribed level. Based on Lyapunov stability and an LMI approach, sufficient conditions for stabilization of the uncertain TS fuzzy model with mixed (LQ/H_∞) performance are derived in terms of a family of linear matrix inequalities [1,9,15,21].

The paper is organized as follows. Problem formulation is presented in Section II. The robust stabilization with mixed (LQ/H_∞) performance is given in Section III. The synthesis of the controller, which is based on the feasibility of certain LMIs, is formulated in Section IV. Section V provides some controller design examples and simulation results. Finally the conclusion is given in Section VI.

2. Problem Formulation

A fuzzy dynamic model has been proposed by Takagi and Sugeno (TS) to represent a non linear and an uncertain system. The TS model is a piecewise interpolation of several linear models through membership functions. The fuzzy model is described by fuzzy If/Then rules and will be employed here to deal with control design problem for a nonlinear system [18].

The i^{th} rule of the fuzzy linear model for the nonlinear discrete-time system has the following form [3,15]

Plant rule i:

$$\begin{aligned}&\text{if} \quad x_1(k) \text{ is } M_{i1} \text{ and... and } x_p(k) \text{ is } M_{ip}\\ &\text{then} \quad x(k+1) = \left(A_i + \Delta A_i(k)\right)x(k) + \left(B_{ui} + \Delta B_{ui}(k)\right)u(k) + \left(B_{wi} + \Delta B_{wi}(k)\right)w(k)\\ &\qquad z(k) = Cx(k) + Du(k)\\ &\qquad x_0 = x(0), \qquad \text{for } i = 1,2,...,r\end{aligned} \tag{1}$$

where M_{ij} is the fuzzy set with $j = 1,2,...,p$, r is the number of rules, $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control input, $w(k) \in L_2[0,\infty)$ is the external disturbance and $\Delta A_i(k), \Delta B_{ui}(k)$ and $\Delta B_{wi}(k)$ are time-varying but norm bounded uncertainties with the following form

$$\begin{aligned}&\left[\Delta A_i(k), \Delta B_{ui}(k), \Delta B_{wi}(k)\right] = H_i F(k)\left[E_i, E_{ui}, E_{wi}\right]\\ &\text{for} \quad i = 1,2,...,r\end{aligned} \tag{2}$$

where H_i, E_i, E_{ui}, E_{wi} are known constant matrices of appropriate dimensions and $F(k)$ is an unknown real time varying matrix with Lebesgue-measurable elements satisfying $F^T(k)F(k) \le I$, in which I is the identity matrix of appropriate dimension.

The dynamic fuzzy model can be represented by the following overall model, which combines all the local models through membership functions.

$$x(k+1) = \sum_{i=1}^{r} h_i(x)\left\{\left(\left(A_i + \Delta A_i(k)\right)x(k) + \left(B_{ui} + \Delta B_{ui}(k)\right)u(k) + \left(B_{wi} + \Delta B_{wi}(k)\right)w(k)\right)\right\} \tag{3}$$

where

$$\omega_i(x(k)) = \prod_{j=1}^{p} M_{ij}(x(k)) \tag{4}$$

$$h_i(x(k)) = \frac{\omega_i(x(k))}{\sum_{l=1}^{r}\omega_l(x(k))} \tag{5}$$

and $M_{ij}(x(k))$ is the grade of member of $x(k)$ in M_{ij} and $h_i(x(k))$ is the normalised membership function of the inferred fuzzy set $\omega_i(x(k))$.

Assume for all time k,

$$\omega_i(x(k)) \ge 0 \text{ and } \sum_{i=1}^{r} h_i(x(k)) = 1 \; ; \text{for} \quad i = 1,2,...,r \tag{6}$$

Conformably to this description, it is quite natural to seek a state feedback parallel-distributed compensation (PDC) for (1) in the form of

Control Rule j:

$$\begin{aligned} &\text{if} \quad x_1(k) \text{ is } M_{j1} \text{ and... and } x_p(k) \text{ is } M_{jp} \\ &\text{then} \quad u(k+1) = K_j x(k) \\ &\text{for} \quad j = 1,2,\dots,r \end{aligned} \tag{7}$$

Hence, the final overall fuzzy controller is given by

$$u(k+1) = \sum_{j=1}^{r} h_j(x) K_j x(k) \tag{8}$$

Then, the resulting closed-loop system is described by

$$x(k+1) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(x) h_j(x)\left[\overline{G}_{ij} x(k) + \overline{B}_{wi} w(k)\right] \tag{9}$$

where $\overline{G}_{ij} = \left(A_i + \Delta A_i(k) + \left(B_{ui} + \Delta B_{ui}(k)\right)K_j\right)$, $\overline{B}_{wi} = B_{wi} + \Delta B_{wi}(k)$

Associated with the closed-loop system (9), the linear quadratic *LQ* is defined as [10,12,15,17]

$$J = \sum_{k=0}^{\infty}\left[x^T(k) Q x(k) + u^T(k) R u(k)\right] \tag{10}$$

where $Q^T = Q \geq 0$, $R^T = R \geq 0$ are given weighting matrices of state and control input respectively.

If we consider that $C = \begin{bmatrix} Q^{\frac{1}{2}} & 0 \end{bmatrix}^T$ and $D = \begin{bmatrix} 0 & R^{\frac{1}{2}} \end{bmatrix}^T$, the *LQ* cost function (10) can be rewritten as

$$J = \sum_{k=1}^{\infty} z^T(k) z(k) = \|z\|_2^2 \tag{11}$$

Since H_∞ control is popular with its efficiency to eliminate the effect of the disturbance on the control system, it will be employed to deal with the robust performance control in (9). Let us consider the following H_∞ control performance [3,15]

$$J = \sum_{k=1}^{\infty} z^T(k) z(k) \leq \gamma^2 \sum_{k=1}^{\infty} w^T(k) w(k) \tag{12}$$

where γ characterizes the impact of external disturbance w on *LQ* cost J.

Since there exist uncertainties in the parameter of the fuzzy model, it is very difficult to get an optimal *LQ* cost. In this situation, a sub-optimal *LQ* control design is proposed to solve this problem by minimizing the upper bound of the *LQ* cost, i.e. $J \leq \alpha$

Definition [15]**:** Associated with the closed-loop system (9), the mixed (LQ/H_∞) performance is defined as

$$J \leq \alpha + \gamma^2 \|w\|_2^2 \tag{13}$$

where $\alpha > 0$ is upper bound on *LQ* and γ is an attenuation level of disturbance.

Problem formulation: Find a suitable set of controller $\{K_j\}$ such as the closed-loop system (9) turns to be stable and satisfies the mixed (LQ/H_∞) index (13).

3. Robust Stabilization and Guaranteed Cost of the TS Fuzzy Model

In this section we develop the PDC controller for uncertain fuzzy model (2) with taking account of the problem formulated in the previous section.

The following theorem investigates the existence condition of (LQ/H_∞) controller and indicates that the closed-loop system satisfying the (LQ/H_∞) performance is always robustly stable.

Theorem 1*:* For the fuzzy system (3), there exists a state feedback fuzzy control law (8) such that the closed-loop system (9) is stable and the (LQ/H_∞) performance in (13) is guaranteed, if there exists a positive-definite matrix P which is the common solution of the following matrix inequalities

$$\widetilde{A}_{ii}\widetilde{P}\widetilde{A}_{ii}-\overline{P}<0 \tag{14}$$

$$\left(\frac{\widetilde{A}_{ij}+\widetilde{A}_{ji}}{2}\right)^T\widetilde{P}\left(\frac{\widetilde{A}_{ij}+\widetilde{A}_{ji}}{2}\right)-\overline{P}<0 \tag{15}$$

where, $\widetilde{A}_{ij}=\begin{bmatrix}\overline{G}_{ij} & \overline{B}_{wi}\\ \overline{C}_i & 0\end{bmatrix}$, $\widetilde{P}=\begin{bmatrix}P & 0\\ 0 & I\end{bmatrix}$ and $\overline{P}=\begin{bmatrix}P & 0\\ 0 & \gamma^2 I\end{bmatrix}$, for $i<j=1,2,\dots,r$

Proof:

From (11), we obtain

$$J=\sum_{k=0}^{\infty}\left\{z^T(k)z(k)+x^T(k+1)Px(k+1)-x^T(k)Px(k)\right\}-x^T(k_f+1)Px(k_f+1)+x^T(0)Px(0)$$

$$\le\sum_{k=0}^{\infty}\left\{z^T(k)z(k)+\Delta V(x_k)-\gamma^2 w^T(k)w(k)\right\}+x^T(0)Px(0)+\gamma^2\|w\|_2^2$$

Substituting (8) in the auxiliary output yields

$$z(k)=\sum_{i=1}^{r}h_i(C+DK_i)x(k)=\sum_{i=1}^{r}h_i\overline{C}_i x(k)=\overline{C}x(k)$$

$$\bullet z^T(k)z(k)-\gamma^2 w^T(k)w(k)=x(k)^T\overline{C}^T\overline{C}x(k)-\gamma^2 w^T(k)w(k)$$

$$=\xi(k)^T\left\{\begin{bmatrix}\overline{C}^T\\ 0\end{bmatrix}\begin{bmatrix}\overline{C} & 0\end{bmatrix}-\begin{bmatrix}0 & 0\\ 0 & \gamma^2 I\end{bmatrix}\right\}\xi(k)$$

$$\bullet\ \Delta V(x_k)=\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{p=1}^{r}\sum_{q=1}^{r}h_ih_jh_ph_q\left\{\left(\overline{G}_{ij}x(k)+\overline{B}_{wi}w(k)\right)^T P\left(\overline{G}_{pq}x(k)+\overline{B}_{wp}w(k)\right)\right\}-x^T(k)Px(k)$$

$$=\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{p=1}^{r}\sum_{q=1}^{r}h_ih_jh_ph_q\xi(k)^T\left\{\begin{bmatrix}\overline{G}_{ij}^{\,T}\\ \overline{B}_{wi}^{\,T}\end{bmatrix}P\begin{bmatrix}\overline{G}_{pq} & \overline{B}_{wp}\end{bmatrix}-\begin{bmatrix}P & 0\\ 0 & 0\end{bmatrix}\right\}\xi(k)$$

where $\xi(k)=\begin{bmatrix}x(k)^T & w(k)^T\end{bmatrix}^T$

If we substitute (18) and (19) in (16), we obtain

$$J\le\sum_{k=0}^{\infty}\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{p=1}^{r}\sum_{q=1}^{r}h_ih_jh_ph_q\xi(k)^T\left\{\begin{bmatrix}\overline{G}_{ij}^{\,T} & \overline{C}_i^T\\ \overline{B}_{wi}^{\,T} & 0\end{bmatrix}\begin{bmatrix}P & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\overline{G}_{pq} & \overline{B}_{wp}\\ \overline{C}_p & 0\end{bmatrix}-\begin{bmatrix}P & 0\\ 0 & \gamma^2 I\end{bmatrix}\right\}\xi(k)\right\}$$

$$+x^T(0)Px(0)+\gamma^2\|w\|_2^2$$

$$=\sum_{k=0}^{\infty}\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{p=1}^{r}\sum_{q=1}^{r}h_ih_jh_ph_q\xi(k)^T\left(\widetilde{A}_{ij}^T\widetilde{P}\widetilde{A}_{pq}-\overline{P}\right)\xi(k)\right\}+x^T(0)Px(0)+\gamma^2\|w\|_2^2$$

$$=\sum_{k=0}^{\infty}\frac{1}{4}\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{p=1}^{r}\sum_{q=1}^{r}h_ih_jh_ph_q\xi(k)^T\left(\left(\widetilde{A}_{ij}+\widetilde{A}_{ji}\right)^T\widetilde{P}\left(\widetilde{A}_{pq}+\widetilde{A}_{qp}\right)-4\overline{P}\right)\xi(k)\right\}+x^T(0)Px(0)+\gamma^2\|w\|_2^2$$

$$\leq \sum_{k=0}^{\infty}\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i h_j \xi(k)^T\left(\frac{\left(\widetilde{A}_{ij}+\widetilde{A}_{ji}\right)^T}{2}\widetilde{P}\frac{\left(\widetilde{A}_{ij}+\widetilde{A}_{ji}\right)}{2}-\overline{P}\right)\xi(k)\right\}+x^T(0)Px(0)+\gamma^2\|w\|_2^2$$

$$=\sum_{k=0}^{\infty}\left\{\xi(k)^T\left\{\sum_{i=1}^{r}h_i^2\left(\widetilde{A}_{ii}\widetilde{P}\widetilde{A}_{ii}-\overline{P}\right)+2\sum_{i=1}^{r}\sum_{j>i}^{r}h_i h_j\left(\frac{\left(\widetilde{A}_{ij}+\widetilde{A}_{ji}\right)^T}{2}\widetilde{P}\frac{\left(\widetilde{A}_{ij}+\widetilde{A}_{ji}\right)}{2}-\overline{P}\right)\right\}\xi(k)\right\}$$
$$+x^T(0)Px(0)+\gamma^2\|w\|_2^2$$

By the property of $h_i\left(x(k)\right)$ in (6) and conditions (14)-(15), we obtain

$$J \leq x^T(0)Px(0)+\gamma^2\|w\|_2^2$$

It is easy to get $x^T(0)Px(0)\leq\alpha$, consequently we have $J\leq\alpha+\gamma^2\|w\|_2^2$. This completes the proof.

To prove the closed-loop system (9) is locally quadratically stable at the equilibrium $x=0$, let us define a Lyapunov function as

$$V\left(x(k)\right)=x^T(k)Px(k) \tag{16}$$

where the weighting matrix *P* is definite positive.

From (16) we obtain

$$\Delta V = V\left(x(k+1)\right)-V\left(x(k)\right)$$
$$=x^T(k+1)Px(k+1)-x^T(k)Px(k)$$
$$=\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{p=1}^{r}\sum_{q=1}^{r}h_i h_j h_p h_q \xi(k)^T\begin{bmatrix}\overline{G}_{ij}^T P\overline{G}_{pq} & \overline{G}_{ij}^T P\overline{B}_{wp}\\ \overline{B}_{wi}^T P\overline{G}_{pq} & 0\end{bmatrix}\xi(k)$$
$$\leq\frac{1}{4}\sum_{i=1}^{r}\sum_{j=1}^{r}h_i h_j\xi(k)^T\begin{bmatrix}\left(\overline{G}_{ij}+\overline{G}_{ji}\right)^T P\left(\overline{G}_{pq}+\overline{G}_{qp}\right) & \left(\overline{G}_{ij}+\overline{G}_{ji}\right)^T P\left(\overline{B}_{wp}+\overline{B}_{wq}\right)\\ \left(\overline{B}_{wi}+\overline{B}_{wj}\right)^T P\left(\overline{G}_{pq}+\overline{G}_{qp}\right) & \left(\overline{B}_{wi}+\overline{B}_{wj}\right)^T P\left(\overline{B}_{wi}+\overline{B}_{wj}\right)\end{bmatrix}\xi(k)$$

$$=\sum_{i=1}^{r}h_i^2\xi(k)^T\begin{bmatrix}\overline{G}_{ii}^T P\overline{G}_{ii} & \overline{G}_{ii}^T P\overline{B}_{wi}\\ \overline{B}_{wi}^T P\overline{G}_{ii} & \overline{B}_{wi}^T P\overline{B}_{wi}\end{bmatrix}\xi(k)$$
$$+2\sum_{i=1}^{r}\sum_{j>i}^{r}h_i h_j\xi(k)^T\begin{bmatrix}\left(\frac{\overline{G}_{ij}+\overline{G}_{ji}}{2}\right)^T P\left(\frac{\overline{G}_{ij}+\overline{G}_{ji}}{2}\right) & \left(\frac{\overline{G}_{ij}+\overline{G}_{ji}}{2}\right)^T P\left(\frac{\overline{B}_{wi}+\overline{B}_{wj}}{2}\right)\\ \left(\frac{\overline{B}_{wi}+\overline{B}_{wj}}{2}\right)^T P\left(\frac{\overline{G}_{ij}+\overline{G}_{ji}}{2}\right) & \left(\frac{\overline{B}_{wi}+\overline{B}_{wj}}{2}\right)^T P\left(\frac{\overline{B}_{wi}+\overline{B}_{wj}}{2}\right)\end{bmatrix}\xi(k)$$

Considering conditions (14) and (15), the previous inequalities implies that

$$\Delta V < -x^T(k)\overline{C}^T\overline{C}x(k)+\gamma^2 w^T(k)w(k)$$
$$\Delta V < -z^T(k)z(k)+\gamma^2 w^T(k)w(k) \tag{17}$$

Summing (17) from *k=0* to $k=\infty$ yields

$$\sum_{k=0}^{\infty}\Delta V\left(x\left(k\right)\right)=V\left(x\left(\infty\right)\right)-V\left(x\left(0\right)\right)\leq -J+\gamma^2\left\|w\right\|_2^2$$

If we assume that $\lim_{k\to\infty} x\left(k\right)=0$, we have

$$J\leq x^T\left(0\right)Px\left(0\right)+\gamma^2\left\|w\right\|_2^2$$

By keeping a cost upper bound α on $x^T\left(0\right)Px\left(0\right)$, obviously, the resulting closed-loop system (9) satisfies the (LQ/H_∞) index.

4. Fuzzy Controller Synthesis

This section is devoted to the design of the robust fuzzy (LQ/H_∞) controller defined in the previous section. Using inequality manipulations and parallel distributed compensation (PDC) techniques, conditions (14) and (15) in theorem 1 are reduced to a set of coupled LMIs. The local state feedback controllers are then derived by the numerical solutions of the coupled LMIs. The overall robust fuzzy (LQ/H_∞) controller is made up of fuzzy controller blending of the local linear controllers.

The following theorem presents a solution to (LQ/H_∞) fuzzy state feedback control problem for TS model with parametric uncertainties in terms of LMIs

Theorem 2: Consider system (3), if there exists a common positive-definite matrix X, some matrices Y_i, and some positives scalars ε_{ij} $(i<j=1,\dots,r)$ satisfying the following LMIs:

$$\text{(a)}\quad \begin{pmatrix} -X & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_iX+B_{ui}Y_i & B_{wi} & -X & * & * & * \\ CX+DY_i & 0 & 0 & -I & * & * \\ E_iX+E_{ui}Y_i & E_{wi} & 0 & 0 & -\varepsilon_{ii}I & * \\ 0 & 0 & H_i^T & 0 & 0 & -\varepsilon_{ii}^{-1}I \end{pmatrix}<0 \qquad (18)$$

$$\text{(b)}\quad \begin{pmatrix} -4X & * & * & * & * & * & * & * \\ 0 & -4\gamma^2 I & * & * & * & * & * & * \\ \begin{pmatrix} A_iX+B_{ui}Y_j \\ A_jX+B_{ui}Y_i \end{pmatrix} & B_{wi}+B_{wj} & -X & * & * & * & * & * \\ 2CX+D\left(Y_i+Y_j\right) & 0 & 0 & -I & * & * & * & * \\ E_iX+E_{ui}Y_j & E_{wi} & 0 & 0 & -\varepsilon_{ij}I & * & * & * \\ E_jX+E_{uj}Y_i & E_{wj} & 0 & 0 & 0 & -\varepsilon_{ij}I & * & * \\ 0 & 0 & H_i^T & 0 & 0 & 0 & -\varepsilon_{ij}^{-1}I & * \\ 0 & 0 & H_j^T & 0 & 0 & 0 & 0 & -\varepsilon_{ij}^{-1}I \end{pmatrix}<0 \qquad (19)$$

then there exists a (LQ/H_∞) fuzzy state feedback controller (8) such that the resulting closed-loop overall fuzzy system (9) is asymptotically stable and the cost (13) is satisfied.

Where $X=P^{-1}$, $Y_i=K_iP^{-1}$ and where * denotes the transposed elements in the symmetric positions for $i\leq j=1,2,\dots,r$.

Proof:

Before proving the theorem we recall the following lemma.

Lemma 2 [26]**:** Given constant matrices *D* and *E* and a symmetric constant matrix *P* of appropriate dimensions and a scalar $\varepsilon > 0$, the following inequality holds

$$P + DFE + E^T F^T D^T < 0$$

where *F* satisfies $F^T F \le R$, if and only if

$$P + \begin{bmatrix} \varepsilon^{-1} E^T & \varepsilon D \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \varepsilon^{-1} E \\ \varepsilon D^T \end{bmatrix} < 0$$

We prove the second condition and the first can be established in the same manner.

From (15) and by applying the Schur complement we get

$$\begin{bmatrix} -4P & * & * & * \\ 0 & -4\gamma^2 I & * & * \\ \overline{G}_{ij} + \overline{G}_{ij} & \overline{B}_{wi} + \overline{B}_{wj} & -P^{-1} & * \\ \overline{C}_i + \overline{C}_j & 0 & 0 & I \end{bmatrix} = \Omega_{ij} + \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ \begin{matrix} \Delta A_i + \Delta B_{ui} K_i \\ +\Delta A_j + \Delta B_{uj} K_i \end{matrix} & \Delta B_{wi} + \Delta B_{wj} & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} < 0 \qquad (20)$$

where $\Omega_{ij} = \begin{bmatrix} -4P & * & * & * \\ 0 & -4\gamma^2 I & * & * \\ \begin{pmatrix} A_i + B_{ui} K_j \\ +A_j + B_{uj} K_i \end{pmatrix} & B_{wi} + B_{wj} & -P^{-1} & * \\ \overline{C}_i + \overline{C}_j & 0 & 0 & I \end{bmatrix}$

Substituting (2) into (20), we obtain the following inequality

$$\Omega_{ij} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ H_i & H_j \\ 0 & 0 \end{bmatrix} F \begin{bmatrix} E_i + E_{ui} K_j & E_{wi} & 0 & 0 \\ E_j + E_{uj} K_i & Ej & 0 & 0 \end{bmatrix} + \begin{bmatrix} E_i + E_{ui} K_j & E_{wi} & 0 & 0 \\ E_j + E_{uj} K_i & Ej & 0 & 0 \end{bmatrix}^T F^T \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ H_i & H_j \\ 0 & 0 \end{bmatrix}^T < 0$$

. (21)

According to lemma 2, matrix inequality (21) holds for all $F^T F < I$, if and only if there exists a constants $\varepsilon_{ii}^{1/2}$ such that

$$\Omega_{ij} + \begin{bmatrix} (E_i + E_{ui} K_j) & E_{wi} & 0 & 0 \\ (E_j + E_{uj} K_i) & E_{wj} & 0 & 0 \\ 0 & 0 & H_i^T & 0 \\ 0 & 0 & H_j^T & 0 \end{bmatrix}^T \begin{bmatrix} -\varepsilon_{ij}^{-1} I & 0 & 0 & 0 \\ 0 & -\varepsilon_{ij}^{-1} I & 0 & 0 \\ 0 & 0 & -\varepsilon_{ij} I & 0 \\ 0 & 0 & 0 & -\varepsilon_{ij} I \end{bmatrix} \begin{bmatrix} (E_i + E_{ui} K_j) & E_{wi} & 0 & 0 \\ (E_j + E_{uj} K_i) & E_{wj} & 0 & 0 \\ 0 & 0 & H_i^T & 0 \\ 0 & 0 & H_j^T & 0 \end{bmatrix} < 0$$

(22)

Applying the Schur complement to (22), we get

$$
\begin{bmatrix}
-4P & * & * & * & * & * & * & * \\
0 & -4\gamma^2 I & * & * & * & * & * & * \\
\begin{pmatrix} A_i + B_{ui}K_j \\ A_j + B_{ui}K_i \end{pmatrix} & B_{wi} + B_{wj} & -P^{-1} & * & * & * & * & * \\
2C + D\left(K_i + K_j\right) & 0 & 0 & -I & * & * & * & * \\
E_i + E_{ui}K_j & E_{wi} & 0 & 0 & -\varepsilon_{ij} I & * & * & * \\
E_j + E_{uj}K_i & E_{wj} & 0 & 0 & 0 & -\varepsilon_{ij} I & * & * \\
0 & 0 & H_i^T & 0 & 0 & 0 & -\varepsilon_{ij}^{-1} I & * \\
0 & 0 & H_j^T & 0 & 0 & 0 & 0 & -\varepsilon_{ij}^{-1} I
\end{bmatrix} < 0 \qquad (23)
$$

Pre- and post-multiplying both sides of last matrix by $diag\left(P^{-1}, I, I, I, I, I, I, I\right)$ and denoting $X = P^{-1}$ and $Y_j = K_j P^{-1}$, matrix inequality (19) is satisfied.

Then we complete the proof.

Therefore, we can minimize the upper bound on *LQ* cost to obtain a sub-optimal mixed (*LQ/H*$_\infty$) control design based on the minimization problem given by the following corollary:

Corollary: To obtain the better (*LQ/H*$_\infty$) performance, the control problem can be formulated taking into account the following minimization problem

$\min \alpha$ and γ

subject LMIs (18)-(19) and

$$
\begin{bmatrix} -\alpha & * \\ x(0) & -X \end{bmatrix} < 0 \qquad (24)
$$

It is easy to get LMI (24) if we consider that $x^T(0) P x(0) < \alpha$ and applying the Schur complement.

4. Numerical Examples

Example 1:

To illustrate the effectiveness of the proposed controller design strategy, we consider the backing-up of a computer simulated truck trailer. We use the following truck-trailer model formulated as [3] [20]

$$
\begin{cases}
x_1(k+1) = \left(1 - \dfrac{vt}{L}\right) x_1(k) + \dfrac{vt}{l} u(k) \\
x_2(k+1) = x_2(k) + \dfrac{vt}{L} x_1(k) \\
x_3(k+1) = x_3(k) + vt\sin\left(x_2(k) + \dfrac{vt}{2L} x_1(k)\right)
\end{cases}
$$

The model parameters are given by $l = 2.8,\ L = 5.5,\ v = -1.0$ and $t = 2.0$.

We assume that the fuzzy model with disturbance is given by:

Plant rule 1:

if $\left(x_2(k) + \dfrac{vt}{2L} x_1(k)\right)$ is about 0

then $x(k+1) = \left(A_1 + \Delta A_1(k)\right) x(k) + \left(B_{u1} + \Delta B_{u1}(k)\right) u(k) + \left(B_{w1} + \Delta B_{w1}(k)\right) w(k)$

Plant rule 2:

if $\left(x_2(k) + \frac{vt}{2L} x_1(k) \right)$ is about π or $-\pi$

then $x(k+1) = \left(A_2 + \Delta A_2(k)\right)x(k) + \left(B_{u2} + \Delta B_{u2}(k)\right)u(k) + \left(B_{w2} + \Delta B_{w2}(k)\right)w(k)$

where

$$A_1 = \begin{bmatrix} 1-\frac{vt}{L} & 0 & 0 \\ \frac{vt}{L} & 1 & 0 \\ \frac{v^2t^2}{2L} & vt & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1-\frac{vt}{L} & 0 & 0 \\ \frac{vt}{L} & 1 & 0 \\ \frac{dv^2t^2}{2L} & dvt & 1 \end{bmatrix}, \quad B_{u1} = B_{u2} = \begin{bmatrix} \frac{vt}{L} \\ 0 \\ 0 \end{bmatrix}, \quad B_{w1} = B_{w2} = \begin{bmatrix} 0 \\ 0.2 \\ 0.1 \end{bmatrix}$$

Based on assumption (2), we define

$$H_1 = H_2 = \begin{bmatrix} 0 & 0 & 0.0023 \end{bmatrix}^T$$

$$E_1 = E_2 = \begin{bmatrix} \frac{vt}{2L} & 1 & 0 \end{bmatrix}, \quad E_{u1} = E_{u2} = 0, \quad E_{w1} = E_{w2} = 0$$

We set $d = \frac{0.01}{\pi}$ and the membership functions as follows

$$h_1(\theta_k) = \frac{\sin(\theta_k)}{\theta_k}, \; h_2(\theta_k) = 1 - h_1(\theta_k) \text{ where } \theta_k = x_2(k) + \frac{vt}{2L} x_1(k)$$

Considering theorem 2, with $Q = 10^{-3} I_3, R = \frac{1}{2} 10^{-3}$, starting point $x_0 = [\frac{\pi}{2} \;\; \frac{3\pi}{4} \;\; -10]^T$ and *w(k)* is stochastic disturbance which is generated by MATLAB function (*rand*(.)*-0.5)/(1+0.01k*), we get the local state feedback gains given by $K_1 = [\; 2.881 \quad -3.5 \quad 0.405]$ and $K_2 = [\; 2.736 \quad -2.70 \quad 0.0095]$, the *LQ* upper bound $\alpha = 1.077$ and the attenuation level $\gamma = 1.6$.

The simulation result is shown in figure 1.

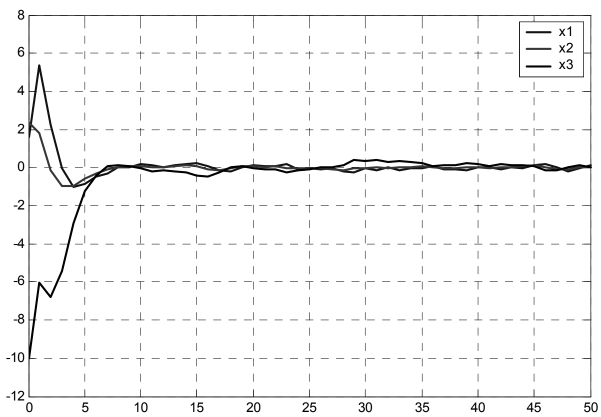

Figure 1: State trajectories of closed-loop track-trailer system

Example 2

In this example, we considered the following discrete-time Lorenz system, which is based on the example given in [4] [5] with the sampling time $T_s=0.002$ *s.*

Plant rule 1:

if $x_1(k)$ is about M_1

then $x(k+1)=(A_1+\Delta A_1(k))x(k)+(B_{u1}+\Delta B_{u1}(k))u(k)+(B_{w1}+\Delta B_{w1}(k))w(k)$

Plant rule 2:

if $x_1(k)$ is about M_2

then $x(k+1)=(A_2+\Delta A_2(k))x(k)+(B_{u2}+\Delta B_{u2}(k))u(k)+(B_{w2}+\Delta B_{w2}(k))w(k)$

where

$$A_1=\begin{bmatrix}1-\sigma T_s & \sigma T_s & 0\\ rT_s & 1-T_s & -M_1T_s\\ 0 & M_1T_s & 1-bT_s\end{bmatrix},\quad A_2=\begin{bmatrix}1-\sigma T_s & \sigma T_s & 0\\ rT_s & 1-T_s & -M_2T_s\\ 0 & M_2T_s & 1-bT_s\end{bmatrix}$$

$$B_{u1}=B_{u2}=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad B_{w1}=\begin{bmatrix}0.1\\-0.1\\0.1\end{bmatrix} B_{w2}=\begin{bmatrix}0.1\\0.1\\0.1\end{bmatrix}$$

and the membership functions are

$$h_1(x_1(k))=\frac{M_2-x_1(k)}{M_2-M_1},\ h_2(x_1(k))=\frac{-M_1+x_1(k)}{M_2-M_1}$$

The nominal values of (σ,r,b) are (10,28,8/3) for chaos to emerge. We assume that all system parameters are uncertain but bounded within 30% of their nominal values.

Based on assumption as (2), we define

$$H_1=H_2=\begin{bmatrix}0.0006 & 0 & 0\\ 0 & 0.0006 & 0\\ 0 & 0 & 0.0006\end{bmatrix},\quad E_1=E_2=\begin{bmatrix}-\sigma & \sigma & 0\\ r & 0 & 0\\ 0 & 0 & -b\end{bmatrix}$$

$$E_1=E_2=E_{u1}=E_{u2}=\begin{bmatrix}0 & 0 & 0\end{bmatrix},\qquad E_{w1}=E_{w2}=\begin{bmatrix}0 & 0 & 0\end{bmatrix}$$

Considering theorem 2, with $Q=\frac{1}{2}10^{-3}I_3, R=\frac{1}{2}10^{-1}$, starting point $x_0=[10\ \text{-}10\ \ -10]^T$ and $w(k)=0.005\sin(0.0125k)$, we get the local state feedback gains given by $K_1=[\ \text{-}0.303\quad \text{-}0.135\ \ \text{-}0.0024]$ and $K_2=[\ \text{-}0.315\quad \text{-}0.129\quad 0.0075]$, the *LQ* upper bound $\alpha=14.337$ and the attenuation level $\gamma=\ 1.0901$.

The simulation result is shown in figure 2 with $M_1=-20$ and $M_2=30$.

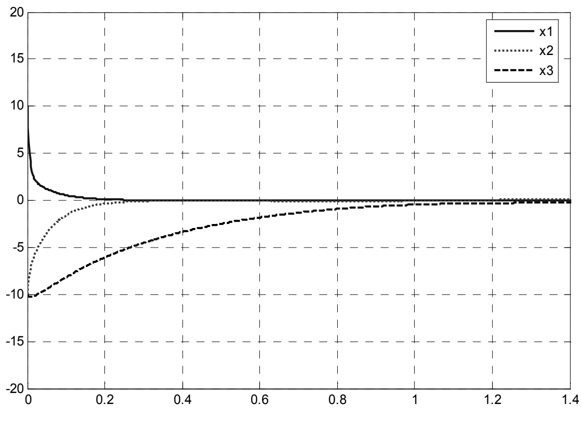

Figure 2: State trajectories of closed-loop Lorenz chaotic system

5. Conclusion

In this paper, we have studied the H_∞ guaranteed cost control problem of uncertain nonlinear system. The TS fuzzy model is used to represent a nonlinear system with uncertainties. The controller is designed by solving the minimization problem that minimizes the upper bound of a (LQ/H_∞) performance index.

The resulting fuzzy controllers guarantee that the close-loop overall fuzzy system is asymptotically stable on one hand, and provides an optimised bound on the value of the cost for all admissible parametric uncertainties on the other hand.

This technique has been applied to a nonlinear system. It has been shown that this approach is both simple and effective.

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