Inverse Fuzzy Sum-product Composition and its Application to Fuzzy Linguistic Modelling

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Abstract: This paper provides, under some assumptions, a simple algorithm for the inversion of a fuzzy relational equation with sum-product composition. This result is applied to the inversion of a discrete fuzzy linguistic model. It is shown that only a matrix inversion is necessary to obtain the solution which is unique, whenever it exists.

Keywords *:* Fuzzy relations, Fuzzy system models, Inverse Problem, Sum-product composition

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1. Introduction

The inverse problem of fuzzy relational equations has been widely tackled since the work of Sanchez [7]. In general, the operators studied are *max-t* operators, where t is a *t-norm* such as the *min* or *product* operators (e.g. [1] [5] [6] [8] [9]), and it was shown that, the solutions, whenever existing, form an upper semi-lattice with upper and lower bounds.

The *addition* (*bounded sum*) operator is a possible alternative to the *max* operator for implementing the union of fuzzy sets. Fuzzy systems which use the *product* as the intersection operators and the *addition* as the union operator (referred to as the *Sum-Prod* inference) are more robust with respect to input changes than that which use truncation operators, and provide a smoother output [2]. Moreover, the *Sum–Prod* composition, when used together with a Center of Gravity Method for defuzzication assure a linear interpolation of the output between the rules which is of interest for control purposes [4].

This note presents a special case of the inverse problem which consists of finding the fuzzy set A for which $A \circ R = B$, where R is a fuzzy relation, B a fuzzy set and $\circ$ the Sum-Prod composition. The incidence of the result on the inversion of fuzzy linguistic models is shown in a second part.

2. Inversion of Fuzzy Relational Equations with Sum-prod Composition

The product operator of two scalars a and b is noted $a \wedge b = a.b$. The bounded sum operator is noted $a \vee b = \min\left(a+b,1\right)$.

Let the row vector $a = \left(a_1,..., a_m \right)$ correspond to the fuzzy subset A, and the matrix $R = \left[r_{ij} \right]_{\substack{1\le i\le m\\ 1\le j\le n}}$ denotes a fuzzy relation on $U \times V$,where U,V are fuzzy universes of discourse.

The product-sum composition of a with the matrix R , noted $a \circ R = b$, is equal to the row vector $b = \left(b_1,...,b_n \right)$ corresponding to the fuzzy subset B such that :

$$b_j = \bigvee_i a_i.r_{ij} . \tag{1}$$

Lemma 1. Suppose that $a \circ R = b$, where $\circ, a, R, b$ are defined above, and $\forall j \in \{1 \cdots n\}$, $\sum_{i=1}^{m} a_i r_{ij} \le 1$ then

$$b = aR . \tag{2}$$

Proof Since $\sum_{i=1}^{m} a_i r_{ij} \le 1$, $b_j = \min\left(\sum_{i=1}^{m} a_i r_{ij}, 1 \right) = \sum_{i=1}^{m} a_i r_{ij}$. Thus, $b = aR$.

Corollary 1. Suppose that $a \circ R = b$, where $\circ, a, R, b$ are defined above, and $\sum_{i=1}^{m} a_i \le 1$, then $b = aR$.

Let $b' = \left(b'_1, ... ,b'_n \right)$ a fuzzy sequence of V , and $R = \left[r_{ij} \right]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$, a fuzzy relation of $U \times V$.

Let us note $p = n - m$, $\Omega = \left\{ k / b'_k < 1 \right\}$, and $q = \operatorname{card}(\Omega)$.

Let $\tilde{b}' = \left\{ b'_i / i \in \Omega \right\}$, which denotes vector b' for which all elements equal to 1 have been removed, and note $\tilde{R} = \left\{ r_{ij} \;/ j \in \Omega \right\}$. Suppose that now $m \le q$, we note $\overline{b}' = \left(\tilde{b}' \right)_{(m \text{ values})}$, which consists of m elements of $\tilde{b}'$, and $\overline{R} = \left(\tilde{r}_{ij} \right)_{j=1 \cdots m}$ the matrix which consists of the corresponding m columns of $\tilde{R}$. Note that $\overline{b}'$ may consist of any combination of m elements of $\tilde{b}'$ and that the choice of these elements can be chosen anyhow. We can now state the following theorem:

Theorem 1.

If $m \le q$, if $\overline{R}$ is invertible, and if there exists a fuzzy sequence a' of V such that $a' \circ R = b'$, then the solution is unique and given by:

$$a' = \overline{b}' \, \overline{R}^{-1} . \tag{3}$$

Proof

From the construction of $\overline{b}'$, $\overline{b}'_j = \min\left(\sum_{i=1}^{m} a'_i \overline{r}_{ij}, 1 \right) < 1$. By the Lemma, $\overline{b}' = a' \overline{R}$, and $a' = \overline{b}' \overline{R}^{-1}$ since, by the hypotheses of the Theorem , $\overline{R}$ is a square and invertible matrix.

Corollary 2. Let us note $\widehat{b}' = b' \setminus \overline{b}'$ the complement of the sequence $\overline{b}'$ with respect to b' , and $\widehat{R}$ the corresponding matrix. Suppose that $\overline{R}$ is invertible and that the dimension of vector $\tilde{b}'$ is at least m , the solution $a' = \overline{b}' \overline{R}^{-1}$ to the problem $a' \circ R = b'$ exists and is unique iff

$$a' \circ \widehat{R} = \widehat{b}' . \tag{4}$$

Remark The corollary expresses that the remaining p equations are compatible with the solution $a' = \overline{b}' \overline{R}^{-1}$, so that indeed $a' \circ R = b'$. As an example, in the case where all elements are smaller than 1, one can choose arbitrarily m elements of b' to build $\tilde{b}'$, and verify that the remaining equations provide the same result.

If this assumption is not verified, there is no solution to the inversion problem (item "iv" in the inversion procedure). It is also clear that, when $\dim \tilde{b}' < m$ (for example, all elements of b' are 1's), the solution will not be unique. This, along with the case where $\overline{R}$ is not invertible –for which the solution is not unique- will be tackled in future work.

The result in the Corollary 2 is quite important since fuzzy relational inversion with other operators provide only bounds for solutions which may prevent the use of adaptive or inverse-model-based control.

The inversion procedure to find a' from R and b' is as now follows:

i) Remove all the 1 's from vector b', and check that the dimension of vector $\tilde{b}'$ is at least m

ii) Take m columns of $\tilde{b}'$ to build vector $\overline{b}'$, and build the appropriate square matrix $\overline{R}$

iii) Check that $\overline{R}$ is invertible and compute $a' = \overline{b}'\overline{R}^{-1}$

iv) Check that $a' \circ \widehat{R} = \widehat{b}'$, i.e.

- if $j \in \Omega$ and $j > m$, $\sum_{i=1}^{m} a_i' \tilde{r}_{ij} = \tilde{b}_j'$
- if $j \notin \Omega$ $\sum_{i=1}^{m} a_i' r_{ij} \geq 1$, so that $b_j' = 1$.

3. Application to the Inversion of Fuzzy Linguistic Models

3.1. A Fuzzy Linguistic Model

The linguistic fuzzy model [3] has been introduced as a way to integrate qualitative knowledge under the form of N_R if-then rules R_i .

$$R_i: \text{IF } \tilde{x} \text{ IS } A_i \text{ THEN } \tilde{y} \text{ IS } B_i, \tag{5}$$

where $\tilde{x}$ and $\tilde{y}$ are respectively the input and output linguistic variables, and their base variables are $x \in X \subset \mathbb{R},\ y \in Y \subset \mathbb{R}$. The antecedent linguistic terms A_i and consequent terms B_i belong respectively to the fuzzy sets A and B. The membership functions of the antecedent (resp. consequent) fuzzy sets are the mappings $\mu_{A_i}(x): X \to [0,1]$ and $\mu_{B_i}(y): Y \to [0,1]$. Each of the fuzzy rules can be regarded as a fuzzy relation of $X \times Y$ onto $[0,1]$. For values x, y in $X \times Y$, the membership function $\mu_{R_i}(x,y)$ is defined in the product space $A \times B$ as

$$\mu_{R_i}(x,y) = \mu_{A_i}(x) \wedge \mu_{B_i}(y). \tag{6}$$

The fuzzy relation R representing the entire model is given by the fusion of the individual rules R_i :

$$\mu_R(x,y) = \vee_i \mu_{R_i}(x,y), \tag{7}$$

where the operators $\wedge, \vee$ are defined above.

Once a rule base has been formed, the membership function $\mu_{B'}(y)$ to the fuzzy set B' , induced by the fuzzy input set A' is given by:

$$\mu_{B'}(y) = \mu_{A'}(x) \circ R, \tag{8}$$

where $\circ$ is the *Sum-Product* composition.

It is possible to discretize the sets X and Y with respectively m and n sample points, where $x^d = \left(x_1^d, \cdots, x_m^d \right)$ and $y^d = \left(y_1^d, \cdots, y_n^d \right)$ are the sampling points with respect to X and Y. In this case, fuzzy sets and relations of equation (7) are represented by the fuzzy membership values at a set of discrete points such that :

$$\mu_B(y^d) = \mu_A(x^d) \circ R^d, \tag{9}$$

where $\mu_A(x^d) = \left(\mu_A(x_1^d), \cdots, \mu_A(x_m^d) \right)$, $\mu_B(y^d) = \mu_A(x^d) \circ R^d$, where $R^d = \left[r_{ij}^d \right]$ is a $m \times n$ matrix such that $R^d = \left[r_{ij}^d \right] = \min\left(\sum_{k=1}^{N_R} \mu_{A_k}(x_i^d)\, \mu_{B_k}(y_j^d), 1 \right)$,

where A_k and B_k refer to the fuzzy subsets attached to the k[th] rule R_k.

3.2. Fuzzy Model Inversion

Theorem 2

Let matrix R^d a fuzzy relation $x^d \times y^d \to [0,1]$, where $x^d = \left(x_1^d, \cdots, x_m^d \right)$, $y^d = \left(y_1^d, \cdots, y_n^d \right)$ and B' a fuzzy set of V. We note $b' = \left(b'_1, ..., b'_n \right) = \left(\mu_{B'}\left(y_1^d\right), ..., \mu_{B'}\left(y_n^d\right) \right)$, $p = n - m$, $\Omega = \left\{ k / b'_k < 1 \right\}$, and $q = \text{card}(\Omega)$.

We can now build $\tilde{b}' = \left\{ b'_i / i \in \Omega \right\}$, $\tilde{R}^d = \left\{ r_{ij}^d \,/\, j \in \Omega \right\}$, and, supposing that $m \le q$ we note $\bar{b}' = \left(\tilde{b}' \right)_{i=1\cdots m}$, $\bar{R}^d = \left(\tilde{r}_{ij}^d \right)_{i,j=1\cdots m}$, $\hat{b}' = b' \setminus \bar{b}'$, $\hat{R}^d$ the matrix corresponding to $\hat{b}'$.

Suppose that $m \le q$ and $\bar{R}^d$ is invertible. The unique fuzzy subset A' of U, with $a' = \left(a'_1, ..., a'_m \right) = \left(\mu_{A'}\left(x_1^d\right), ..., \mu_{A'}\left(x_m^d\right) \right)$, such that $a' \circ R^d = b'$, given by:

$$a' = \bar{b}' \left(\bar{R}^d \right)^{-1}, \tag{10}$$

exists iff $a' \circ \hat{R}^d = \hat{b}'$.

Proof by direct application of Theorem 1 and Corollary 2.

3.3. Example

Let us consider a simple fuzzy model which describes how the heating power of a burner depends on the oxygen supply [adapted from [10]); fuzzy sets $A = \left\{ LOW, OK, HIGH \right\}$ and $B = \left\{ LOW, MEDIUM, HIGH \right\}$ are given in Figure 1.

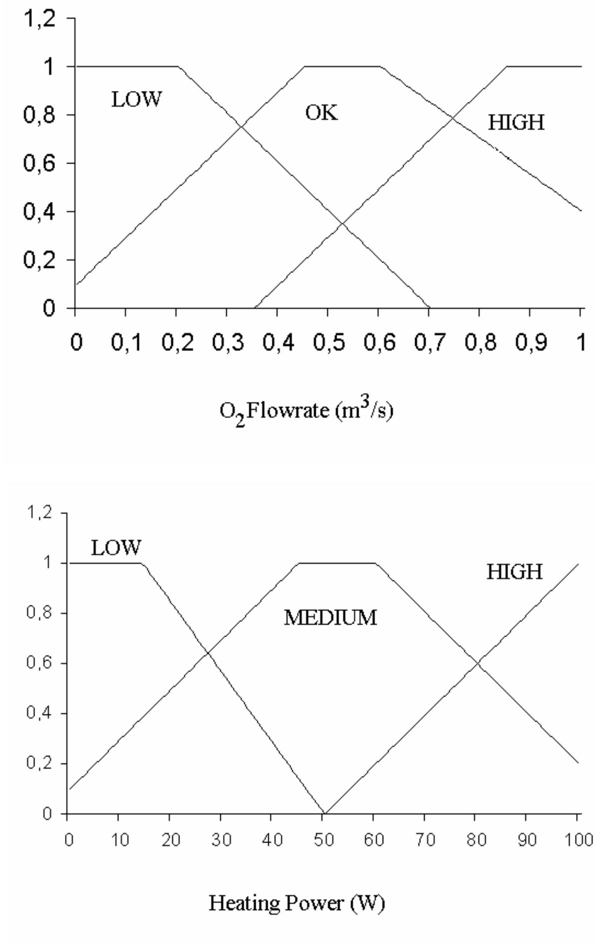

Figure 1. Membership Functions.

The 3 rules are:

If O_2 flowrate IS LOW THEN heating power is LOW

If O_2 flowrate IS OK THEN heating power is MEDIUM

If O_2 flowrate IS HIGH THEN heating power is HIGH

The sample points are $x^d = \{ 0, 0.5, 1 \}$ and $y^d = \{ 0, 25, 50, 75, 100 \}$.

From the figure 1, we have $\mu_{LOW}\left(x^d\right) = \{1 \quad 0.4 \quad 0\}$, $\mu_{OK}\left(x^d\right) = \{0.1 \quad 1 \quad 0.4\}$, $\mu_{HIGH}\left(x^d\right) = \{0 \quad 0.3 \quad 1\}$, and one can derive the same for y^d.

We thus have from relation (9) , taking the first rule:

$$R_1^d = \mu_{LOW}\left(x^d\right) \vee \mu_{LOW}\left(y^d\right) = \begin{pmatrix} 1 & 0.76 & 0 & 0 & 0 \\ 0.4 & 0.28 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \text{ one obtains identically}$$

$$R_2^d = \begin{pmatrix} 0.01 & 0.06 & 0.1 & 0.07 & 0.02 \\ 0.1 & 0.6 & 1 & 0.7 & 0.2 \\ 0.04 & 0.24 & 0.4 & 0.28 & 0.08 \end{pmatrix}, \; R_3^d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.15 & 0.3 \\ 0 & 0 & 0 & 0.5 & 1 \end{pmatrix} \text{ and}$$

$$R^d = \min\left(\sum R_i, 1 \right) = \begin{pmatrix} 1 & 0.82 & 0.1 & 0.7 & 0.02 \\ 0.5 & 0.88 & 1 & 0.85 & 0.5 \\ 0.04 & 0.24 & 0.4 & 0.78 & 1 \end{pmatrix}.$$

Let $b' = \{ 0.808 \quad 0.986 \quad 0.73 \quad 1 \quad 0.51 \}$ corresponding to a “rather LOW” heating;

taking the 3 first columns gives $\overline{R}^d = \begin{pmatrix} 1 & 0.82 & 0.1 \\ 0.5 & 0.88 & 1 \\ 0.04 & 0.24 & 0.4 \end{pmatrix}$, $\overline{b}' = \{ 0.808 \quad 0.986 \quad 0.73 \}$ and

$a' = \overline{b}'\left(\overline{R}^d\right)^{-1} = \{ 0.5 \quad 0.6 \quad 0.2 \}$ corresponds to a “rather LOW” flowrate such that $a' \circ \overline{R}^d = b'$.

One can check easily that $a' \circ \widehat{R}^d = \widehat{b}'$ and that, indeed, the two first values are the right ones, i.e. $b_4' = 1, b_5' = 0.51$.

Conclusion

A sufficient condition has been given to invert a sum-product relation which has been applied to a fuzzy linguistic model. It is shown that a simple matrix inversion is necessary to obtain a solution. This algorithm could find applications in adaptive control. This work will be extended to the case of non-invertible relational matrices for which a unique solution does not exist.

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