Discretization of Continuous Linear Anisochronic Models

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Abstract: Much work has been done on analysis and control of continuous or discrete time-delay systems. However, the discretization of continuous time-delay systems has not been extensively studied. This paper introduces an original approach to the discrete approximation of continuous-time linear anisochronic models by discrete models for the purposes of numerical modelling, simulation, identification and control. Depending on the required numerical accuracy of the approximation, this method employing the Taylor series enables us to determine the order and the resulting structure of a discrete linear model appropriate for numerical processing. The technique is simultaneously applied, more in greater detail, to a continuous anisochronic model of the 2nd order which is often used in applications including various forms of delays that are very closely related to the existence of continuously distributed parameters. In addition, this work includes derived, original formulae which enable us to convert directly an original continuous linear anisochronic model of the 2nd order into the corresponding discrete approximation. Finally, an example is given of the application of the method, and a measure of agreement is shown between the original continuous anisochronic model representing the system and a discrete approximation of it.

Keywords: discretization, delay, anisochronic model, internal point delay

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1. Introduction

The rapid development of digital technology has affected the fields of modelling, simulation, identification and control of so-called hereditary systems, i.e. systems where time delays and latencies are an essential feature of process dynamics.

Time delays and latencies are closely connected with mass, energy and information transport. Considerable attention has therefore been paid to these systems in recent years. To describe such models, so-called anisochronic models (as opposed to isochronic models) are nowadays often applied. Anisochronic models enable multi-parameter systems (multi-dimensional systems) to be described including not only transport delays of these input variables but also internal delays of their state variables. For more detailed information about anisochronic models, see e.g. [7] and [9].

The linear models represented by the matrix differential equation (1) and the matrix equation (2) are anisochronic models which have been often used.

$$\dot{x}(t) = \sum_{i=0}^{m} A_i \cdot x(t-\vartheta_i) + \sum_{j=0}^{n} B_j \cdot u(t-\tau_j) \tag{1}$$

$$y(t) = C \cdot x(t) \tag{2}$$

where, $\dot{x}(t)$ is the first derivative of state vector x at time t, $x(t-\vartheta_i)$ is the state vector at time $t-\vartheta_i$, $u(t-\tau_j)$ is its input vector at time $t-\tau_j$,

$A_i, i=0,1,\cdots,m$ and $B_j, j=0,1,\cdots,n$ and C are particular constant matrices,

$\vartheta_0 = 0$, $\vartheta_i > 0, i=1,2,\cdots,m$, and $\tau_0 = 0$, $\tau_j > 0, j=1,2,\cdots,n$ are the delays of the state variables and of the input variables respectively.

If anisochronic model (1) is to be used for digital modelling, simulation, identification or control, it is necessary to find the corresponding discrete approximation of this linear model.

In the case of discrete approximation of continuous linear models where the influences of transport delays are not considered, the field of theory of alternative discrete model design has been well covered, see, e.g. [3]. Although much work has been done on analysis and control of continuous time-delay systems, there are only a few published methods of discrete approximation of linear anisochronic models for cases which respect internal and external point delays. The basic reason is that the characteristic equations of anisochronic models are not algebraic but transcendental. Such equations admit an arbitrarily large number of roots in the complex domain, and each of the roots tends to influence the resulting response of the system even in the case of an unlimited set of roots. For this reason not all the roots of the characteristic equation can be determined, as required in common methods of classic discrete approximation of continuous models. This problem has been solved by searching for dominant roots [4], [6], [8], using FIR filters [5] or with the help of recursive solution of the state vector [2]. In this paper the proposed discretization method does not require determination of the roots of the characteristic equation, and allows us to design the discrete model with respect to the required accuracy.

2. Discrete Approximation of a Linear Anisochronic Model

In order to avoid looking for the roots of the characteristic equation of an anisochronic model, the Taylor series expansion (3) evaluated in the neighbourhood of the operating point $x(t)$ is used to obtain its discrete approximation.

$$x(t+\Delta) = x(t) + \dot{x}(t)\cdot\Delta + \ddot{x}(t)\cdot\tfrac{\Delta^2}{2!} + \dddot{x}(t)\cdot\tfrac{\Delta^3}{3!} + \cdots \qquad (3)$$

where Δ is a step of discrete approximation

The derivatives of the state vector can be obtained using equation (1); e.g. it holds for $\ddot{x}(t)$

$$\ddot{x}(t) = \sum_{i=0}^{m} A_i \cdot \dot{x}(t-\vartheta_i) + \sum_{j=0}^{n} B_j \cdot \dot{u}(t-\tau_j)\,. \qquad (4)$$

The goal is to obtain a model suitable for digital control where the input vector changes only at discrete moments. Assumption (5) can be accepted for time instants $t = k\cdot\Delta, k=0,1,2,\cdots$

$$\dot{u}(t-\tau_j) = [0] \text{ for } j=0,1,\cdots,\text{n} \qquad (5)$$

where $[0]$ is the vector of zeros.

Applying formula (1) once again, formula (4) yields

$$\ddot{x}(t) = \sum_{i1=0}^{m} A_{i1} \cdot \left(\sum_{i2=0}^{m} A_{i2} \cdot x(t-\vartheta_{i1}-\vartheta_{i2}) + \sum_{j=0}^{n} B_j \cdot u(t-\vartheta_{i1}-\tau_j) \right) =$$

$$=\sum_{i1=0}^{m}\sum_{i2=0}^{m} A_{i1}\cdot A_{i2}\cdot x(t-\vartheta_{i1}-\vartheta_{i2})+\sum_{i1=0}^{m}\sum_{j=0}^{n} A_{i1}\cdot B_j\cdot u(t-\vartheta_{i1}-\tau_j). \qquad (6)$$

Each state vector derivative occurring in the Taylor series expansion (3) can be evaluated by application of the method introduced above, and the resulting formula (7) is obtained after substitution into those derivatives.

$$x(t+\Delta)=x(t)+\Delta\cdot\left(\sum_{i1=0}^{m} A_{i1}\cdot x(t-\vartheta_{i1})+\sum_{j=0}^{n} B_j\cdot u(t-\tau_j)\right)+$$
$$+\tfrac{\Delta^2}{2!}\cdot\left(\sum_{i1=0}^{m}\sum_{i2=0}^{m} A_{i1}\cdot A_{i2}\cdot x(t-\vartheta_{i1}-\vartheta_{i2})+\sum_{i1=0}^{m}\sum_{j=0}^{n} A_{i1}\cdot B_j\cdot u(t-\vartheta_{i1}-\tau_j)\right)+$$
$$+\tfrac{\Delta^3}{3!}\cdot\left(\sum_{i1=0}^{m}\sum_{i2=0}^{m}\sum_{i3=0}^{m} A_{i1}\cdot A_{i2}\cdot A_{i3}\cdot x(t-\vartheta_{i1}-\vartheta_{i2}-\vartheta_{i3})+\right.$$
$$\left.+\sum_{i1=0}^{m}\sum_{i2=0}^{m}\sum_{j=0}^{n} A_{i1}\cdot A_{i2}\cdot B_j\cdot u(t-\vartheta_{i1}-\vartheta_{i2}-\tau_j)\right)+\cdots \qquad (7)$$

The number of elements of the Taylor series expansion is optional and should be chosen with respect to the required accuracy of the approximation. This results from the fact that the approximation error of Taylor's polynomial of the l^{th} order at time $t+\Delta$ is vector $e(t+\Delta)$,

$$e(t+\Delta)=\frac{\Delta^{l+1}}{(l+1)!}\cdot x^{(l+1)}(\xi)$$

where $\xi\in(t,t+\Delta)$. The particular value of the $(l+1)^{th}$ state vector derivative at time ξ, i.e. $x^{(l+1)}(\xi)$, can again be calculated according to formula (1).

If the time arguments of the state vectors or the input vectors are not commonly integer multiples of the discrete approximation time step Δ, it is necessary to apply a method that uses known vectors obtained at time instants $k\cdot\Delta$, where $k=0,1,2,\cdots$ and which thus enables the state vectors to be estimated. Various interpolation methods [4] can be utilized for this purpose. So-called linear interpolation is one of the simplest methods and is satisfactory in most cases. Interpolation using spline functions is a more sophisticated method that takes advantage of the application of formula (1).

3. Discrete Approximation of a Anisochronic and Order Model

In this section, the described common method of discrete approximation will be applied to a linear anisochronic model of the 2nd order with the s-operator transcendental transfer function (8)

$$G(s)=\frac{K\cdot e^{-s\cdot\tau}}{(T_1\cdot s+1)\cdot(T\cdot s+e^{-s\cdot\vartheta})} \qquad (8)$$

where K is the static gain, τ is the transport delay of the input, T,T_1 are time constants, and ϑ is the feedback delay (which is determined by the location of the inflex point of the transient response, see Fig.1)

Transfer function (8) corresponds to differential equation (9)

$$T_1\cdot T\cdot\ddot{y}(t)+T\cdot\dot{y}(t)+T_1\cdot\dot{y}(t-\vartheta)+y(t-\vartheta)=K\cdot u(t-\tau) \qquad (9)$$

Because of its considerable universality (its structure remains the same regardless of the plant which is to be approximated), this model has been used increasingly in recent years for describing real processes in practice, e.g. [1],[6],[7],[9].

Using the standard method of successive integration, model (9) will be transformed into the state-space representation.

$$\dot{x}_1(t) = \frac{-x_2(t-\vartheta)}{T \cdot T_1} - K \cdot u(t-\vartheta) \tag{10}$$

$$\dot{x}_2(t) = x_1(t) - \frac{x_2(t)}{T_1} - \frac{x_2(t-\vartheta)}{T}$$

$$y(t) = -\frac{x_2(t)}{T \cdot T_1}. \tag{11}$$

Then, the set of equations (10) can be modified as given by (12), which corresponds to the matrix notation (1)

$$\dot{x}(t) = A_0 \cdot x(t) + A_1 \cdot x(t-\vartheta) + B_1 \cdot u(t-\tau). \tag{12}$$

where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ and

$$A_0 = \begin{bmatrix} 0 & 0 \\ 1 & \frac{-1}{T_1} \end{bmatrix} \tag{13}$$

$$A_1 = \begin{bmatrix} 0 & \frac{-1}{T \cdot T_1} \\ 0 & \frac{-1}{T} \end{bmatrix} \tag{14}$$

$$B_1 = \begin{bmatrix} -K \\ 0 \end{bmatrix}. \tag{15}$$

By applying the technique introduced in section 1, the elements of the Taylor expansion series (3) can be determined with the use of formula (12). As it holds

$$A_0 \cdot A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{16}$$

the state-space model representation below can be derived

$$\begin{aligned} x(t+\Delta) &= \Phi_0(\Delta) \cdot x(t) + A_1 \cdot \Phi_1(\Delta) \cdot x(t-\vartheta) + A_1^2 \cdot \Phi_2(\Delta) \cdot x(t-2\cdot\vartheta) + \\ &+ A_1^3 \cdot \Phi_3(\Delta) \cdot x(t-3\cdot\vartheta) + \cdots + \Phi_1(\Delta) \cdot B_1 \cdot u(t-\tau) + \\ &+ A_1 \cdot \Phi_2(\Delta) \cdot B_1 \cdot u(t-\tau-\vartheta) + A_1^2 \cdot \Phi_3(\Delta) \cdot B_1 \cdot u(t-\tau-2\cdot\vartheta) + \cdots \end{aligned} \tag{17}$$

where $\Phi_0(\Delta)$ is the fundamental matrix which is defined by the infinity series

$$\Phi_0(\Delta) = e^{A_0 \cdot \Delta} = I + A_0 \cdot \Delta + A_0^2 \cdot \frac{\Delta^2}{2} + A_0^3 \cdot \frac{\Delta^3}{3!} + \cdots \tag{18}$$

where I is the unit matrix, and it holds for $\Phi_i(\Delta),\ i = 1, 2, 3, \cdots$ that

$$\Phi_i(\Delta) = \int_0^{\Delta} \Phi_{i-1}(x)\, dx. \tag{19}$$

The fundamental matrix $\Phi_0(\Delta)$ may easily be derived by the inverse Laplace transform

$$\Phi_0(\Delta) = L^{-1}\left\{ \left[s \cdot I - A_0 \right]^{-1} \right\} \Big|_{t=\Delta} \tag{20}$$

where L^{-1} denotes the inverse Laplace transform, and s is a complex variable which is related to the Laplace transform.

It follows from formulae (13) and (20) for $\Phi_0(\Delta)$

$$\Phi_0(\Delta) = \begin{bmatrix} 1 & 0 \\ -T_1 \cdot (e^{\frac{-\Delta}{T_1}} - 1) & e^{\frac{-\Delta}{T_1}} \end{bmatrix}. \tag{21}$$

By recurrent computation using (19) and (21), $\Phi_i(\Delta), i = 1, 2, 3, \cdots$ can be evaluated. Thus, for example, it holds for $\Phi_1(\Delta), \Phi_2(\Delta), \Phi_3(\Delta)$ that

$$\Phi_1(\Delta) = \begin{bmatrix} \Delta & 0 \\ T_1 \cdot \left(\Delta - T_1 + T_1 \cdot e^{\frac{-\Delta}{T_1}} \right) & T_1 \cdot \left(1 - e^{\frac{-\Delta}{T_1}} \right) \end{bmatrix} \tag{22}$$

$$\Phi_2(\Delta) = \begin{bmatrix} \frac{\Delta^2}{2} & 0 \\ T_1 \cdot \left(\frac{\Delta^2}{2} - T_1 \cdot \Delta - T_1^2 \cdot e^{\frac{-\Delta}{T_1}} + T_1^2 \right) & T_1 \cdot \left(\Delta - T_1 + T_1 \cdot e^{\frac{-\Delta}{T_1}} \right) \end{bmatrix} \tag{23}$$

$$\Phi_3(\Delta) = \begin{bmatrix} \frac{\Delta^3}{3!} & 0 \\ T_1 \left(\frac{\Delta^3}{3!} - \frac{\Delta^2}{2} T_1 + T_1^3 e^{\frac{-\Delta}{T_1}} + T_1^2 \Delta - T_1^3 \right) & T_1 \left(\frac{\Delta^2}{2} - T_1 \Delta - T_1^2 e^{\frac{-\Delta}{T_1}} + T_1^2 \right) \end{bmatrix}. \tag{24}$$

If the formulae introduced above are utilized to obtain the discrete approximation of the anisochronic model represented by the operator transfer function (9), i.e. by differential equation (10) with parameters $T = 2\,\text{s}, T1 = 1\,\text{s}, K = 1,\ \vartheta = 0.32\,\text{s},\ \tau = 0.45\,\text{s}$, then the resulting anisochronic model discrete approximation with the discrete time step $\Delta = 0.1\,\text{s}$ is

$$\begin{aligned} & x(k+1) = P_0 \cdot x(k) + P_1 \cdot x(k-3.2) + \mathrm{P}_2 \cdot x(k-6.4) + P_3 \cdot x(k-9.6) + \cdots \\ & \cdots + U_0 \cdot u(k-4.5) + U_1 \cdot u(k-7.7) + U_2 \cdot u(k-10.9) + \cdots \\ & y(k) = C \cdot x(k) \end{aligned} \tag{25}$$

where $P_0 = \Phi_0(\Delta) = \begin{bmatrix} 1 & 0 \\ 0.095 & 0.905 \end{bmatrix}$, $P_1 = A_1 \cdot \Phi_1(\Delta) = \begin{bmatrix} -2.419 \cdot 10^{-3} & -0.048 \\ -2.419 \cdot 10^{-3} & -0.048 \end{bmatrix}$,

$P_2 = A_1^2 \cdot \Phi_2(\Delta) = \begin{bmatrix} 4.063 \cdot 10^{-5} & -1.209 \cdot 10^{-3} \\ 4.063 \cdot 10^{-5} & -1.209 \cdot 10^{-3} \end{bmatrix}$, $C = \begin{bmatrix} 0 & -0.5 \end{bmatrix}$,

$P_3 = A_1^3 \cdot \Phi_3(\Delta) = \begin{bmatrix} -5.106 \cdot 10^{-7} & -2.032 \cdot 10^{-5} \\ -5.106 \cdot 10^{-7} & -2.032 \cdot 10^{-5} \end{bmatrix}$, $U_0 = \Phi_1(\Delta) \cdot B_1 = \begin{bmatrix} -0.1 \\ -4{,}837 \cdot 10^{-3} \end{bmatrix}$,

$U_1 = A_1 \cdot \Phi_2(\Delta) \cdot B_1 = \begin{bmatrix} 8.129 \cdot 10^{-5} \\ 8.129 \cdot 10^{-5} \end{bmatrix}$, $U_2 = A_1^2 \cdot \Phi_3(\Delta) \cdot B_1 = \begin{bmatrix} -1.021 \cdot 10^{-6} \\ -1.021 \cdot 10^{-6} \end{bmatrix}$, and it holds for the discrete time k that $t = k \cdot \Delta$. The decreasing influence of the former state–space variables and also of the inputs can be observed on the previous formulae, and consequently the order of the discrete model may easily be set with respect to the required accuracy. Simultaneously, it is easily seen that the columns of particular matrices $P_i, i = 1,2,\cdots$ and of each vector $U_j, j = 1,2,\cdots$ contain the same values.

For the case of considering only matrices P_0, P_1, P_2 and vector U_0 for a discrete model, the transient responses (Fig.2) and frequency responses (Fig.3) of the two compared models are shown to demonstrate agreement between the anisochronic model (9) and its discrete approximation (25).

The influence of former values of the state variables and of the inputs are neglected (elements of matrices $P_i, i = 3,4,\cdots$ and vectors $U_j, j = 1,2,\cdots$ are less than 10^{-4}).

4. Conclusion

Discrete approximation of continuous systems is a necessary step when their models are to be numerically implemented. The original method of discrete approximation of a linear anisochronic model introduced here can be applied to the field of numerical modelling, simulation, identification and control of systems with external and internal delays. The method does not require knowledge of the roots of a characteristic transcendental equation, and allows the structure and the order of the discrete model to be estimated in relation to the required accuracy of the approximation. The method also enables derivation of expressions (17), (18) and (19), which are applicable to the wide class of hereditary systems described by the anisochronic model. Further research will be aimed toward further generalization and seeking for explicit formulae applicable in the field of technical cybernetics.

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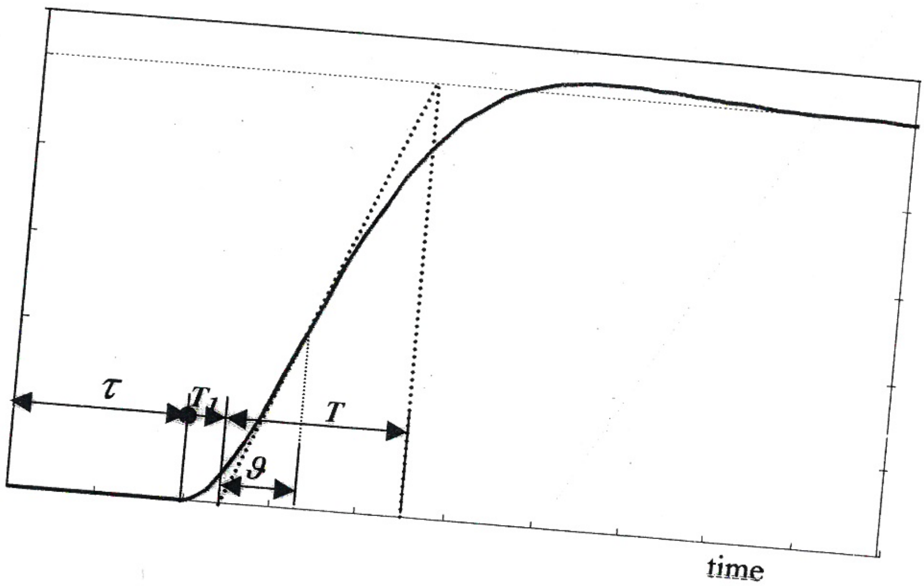

Figure 1.

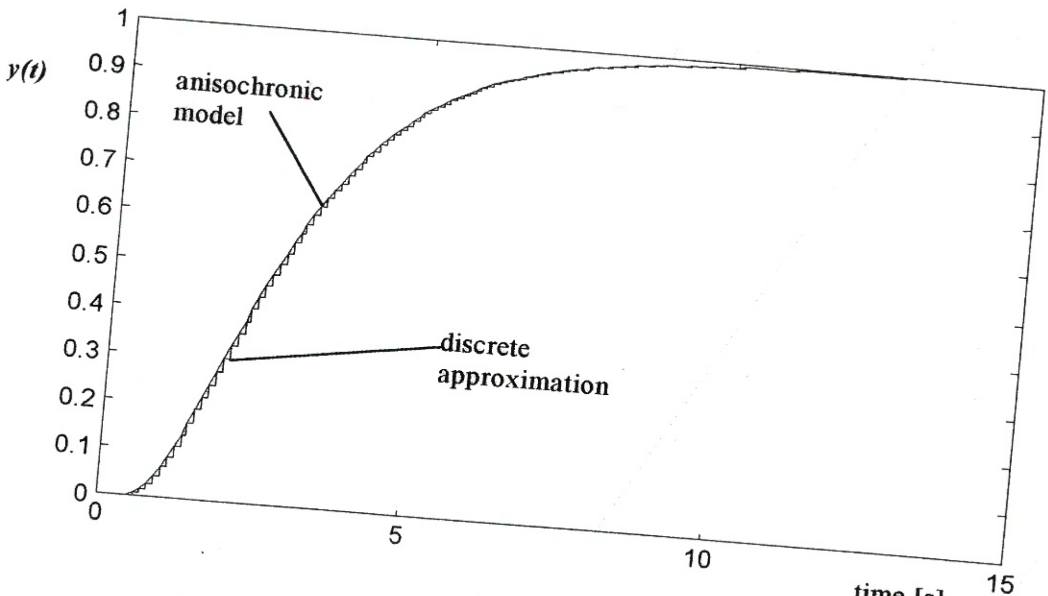

Figure 2.

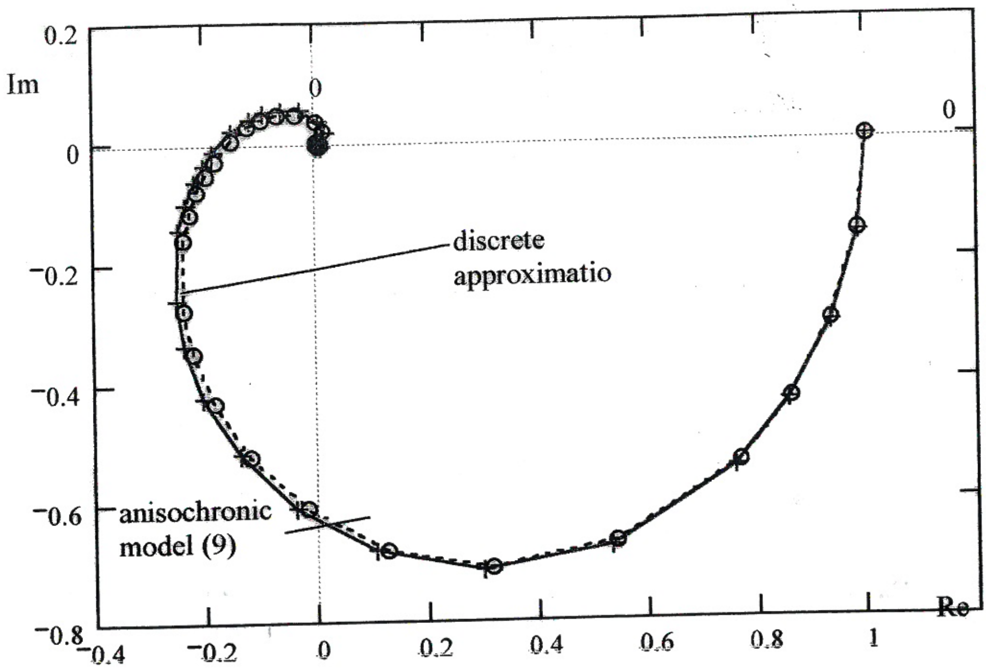

Figure 3.

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