Cyclic Hilbert Spaces

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Dedicated to Professor Andrei Neculai to his 60th birthday

Abstract: We analyse in this paper a concept related to the Connes Embedding Problem [Co]. A type II_1 algebra is an algebra with a trace, and CEP requires for the multiplication to be approximated by matrices. Here we start the analysis of four products, which is the study of cyclic Hilbert spaces.

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In this paper we introduce the notion of a cyclic Hilbert space, which is by definition a Hilbert space, that carries a special cyclic scalar product on $H \otimes H$. We prove that such spaces can be embedded into finite unbounded (separable) von Neumann algebras.

Given are arbitrary II_1 factor M, and V a subspace of selfadjoint elements, the Connes embedding Problem is reducible ([Ra]) to the problem to approximation of four products: that is if V is a finite dimensional real vector space of M, find an approximate embedding (that preserves approximately $\tau(abcd)$, $a,b,c,d \in V$) into $M_n(C)$ with the normalized trace.

This consists into proving that every cyclic Hilbert Space, as defined bellow is embeddable into a II_1 factor.

Definition. (Cyclic prehilbertian space) Let V be a real prehilbertian space, with pointed vector 1, of norm 1 and assume that there is an additional bilinear complex valued, positive form $<<,>>$ ($<<\alpha,\beta>> = \overline{<<\beta,\alpha>>}$) on $(V \otimes_R V) \otimes_R C$ with the following properties:

1) V embeds isometrically into $V \otimes V$, via the map $v \to v \otimes 1 = 1 \otimes v$;

2) $<<,>>$ is cyclic in the following sense $<< a \otimes b, c \otimes d >> = << c \otimes a, d \otimes b >>$ for all $a,b,c,d \in V$;

3) $<<,>>$ is autoadjunct in the sense that for all a,b,c,d in V

$$<< a \otimes b, c \otimes d >> = \overline{<< b \otimes a, d \otimes c >>}.$$

Such a space will be called a cyclic space.

Note. Such a cyclic space will also have the following additional property:

2') $<< a \otimes b, c \otimes d >> = << b \otimes d, a \otimes c >>$.

Moreover 2') and 3) are equivalent to 2) and 3).

4) The map from $V \otimes V \to V \otimes V$ (extended then by antilinearity to $(V \otimes V) \otimes_R C$ by $J(\otimes b) = b \otimes a$ is an involution.

Proof. Assume 2), 3) are true. Then because of 2) we have

$$<< b \otimes d, a \otimes c >> = << a \otimes b, c \otimes d >>.$$

Clearly,

$$<< J(a\otimes b), J(c\otimes d) >>= \langle b\otimes a, d\otimes c\rangle = \overline{<< a\otimes b, c\otimes d >>}.$$

Note also that $<< a\otimes b, c\otimes d >>= \overline{<< a\otimes c, b\otimes d >>}$ which follows by aplly iteratively properties 2) and 3). □

In the next proposition we prove that the cyclic structure on *V*, can be extended to a larger space *W*, such that if *y* is an element in $V\otimes V$, the product identification in the scalar product given by << ,>>, we have $y\in W(=W\otimes 1)$.

Proposition. *Let V be a finite dimensional cyclic vector space. Let y be a selfadjoint element in* $(V\otimes V)\otimes_R C$ *(that is J y= y) of lenght 1 and that is not identified (via <<,>>) with an element in* $V\otimes 1$ *(or equivalently* $1\otimes V$*).*

Fix an orthonormal basis $x_1, x_2, \cdots, x_n$ *of V, and asuume that* x_1*=1, the pointed vector of V.*

Then for every ε > 0, there exists an ε-perturbation $<<,>>_\varepsilon$ *of the original structure << , >> that is*

$$\left|<<x_j\otimes x_j, x_k\otimes x_l>> - <<x_i\otimes x_j, x_k\otimes x_l>>\right| < \varepsilon$$

for all i,j,k,l=1,2,…,n, with the folowing properties:

Let Y be an undeterminate and let $W = V\otimes RY$*. Consider a scalar product on W such that* $W, X_1, X_2, \cdots, X_n, Y$ *as an orthonormal basis.*

Then, W has cyclic vector space structure $<<,>>_W$*, such that*

1) $<<,>>_W$ *extends the structure* $<<,>>_\varepsilon$ *on* $V\otimes V$*;*

2) $y = Y\otimes 1$ *(modulo* $<<,>>_W$*).*

Proof. We will use in the proof the notation $x_i x_j, x_i Y, Yx_i, Y^2$ for $x_i\otimes x_j$, $x_i\otimes Y$, $Y\otimes x_i$, $Y\otimes Y$

We need to define $<< Y\otimes x_i, x_j\otimes x_l >>$, for *i,j,l*. Note that $<< Y\otimes x_i, 1\otimes x_l >>$ is already defined and required to be equal to $<< y, x_l x_i >>$, as is $<< Y, x_l x_i >>$ required to be $\langle y, x_l x_i\rangle$.

We consequently will start by constructing the vector ξ_a in $(V\otimes V)\otimes_R C$, which will correspond to the projections of the vectors Yx_a in $(W\otimes_R W)\otimes_R C$ onto $(V\otimes V)\otimes_R C$ (projection with respect to the scalar product induced by << ,>>).

The vector, ξ_a *a*=2,…,*n*, are subject to

$$<< \xi_a, x_b x_c >>=< J\xi_b, x_c x_a > =<< J\xi_b, Jx_a x_c >>= \overline{<< \xi_b, x_a x_c >>} \quad (1)$$

for all *a*,*b*,*c*=2,…,*n*, conform with properties 2), 3). We will use the notation << ,>> for $<<,>>_\varepsilon$ until we define the requirements on the deformation.

We also require that the projection of $x_a Y$ to be $J\xi_a$, and hence properties 2), 3) will be satisfied at least for the projections of the vector ξ_a.

Let *P* be the projection of $(V\otimes_R V)\otimes_R C$ onto $(W\otimes_R C)^\perp$. Note that quantities $\lambda^c_{ab} = \langle \xi_a, (1-P)x_b x_c\rangle$ are predetermined (as the numbers $<< \xi_a, \xi_l >>=<< y, x_l x_a >>$ are all determined).

Hence to verify relations (1) we have to determine vectors η_a in $(V\otimes V)\otimes_R C - V\otimes_R C$ (that will be equal to $P\xi_a$) that verify the equations

$$< \eta_a, x_b x_c > - < J\eta_b, x_c x_a >= \theta^c_{ab} = -\lambda^c_{ab} + \overline{\lambda}^c_{ba},$$
$$a,b,c = 2,...,n,\ b > a. \quad (2)$$

(Note that in relations (1) the *a*,*b* are interchangable and that for *b*=*a* the relations are redundant.)

We denote by $R(\eta_a)$, $I(\eta_a)$ the real and imaginary part of η_a with respect to *J*.

Then let ν_0 be the real part with respect to *J* of the complex Hilbert space $[(V\otimes_R V)\otimes_R C - V\otimes_R RC]$. The problem to solve the equations (2) reduces to finding 2(n-1) vectors $R(\eta_a)$, $I(\eta_a)$, a=2,…,n, in $\mathcal{V}$ that verify the following conditions:

$$<< R(\eta_a), R(x_b x_c) >> + << I(\eta_a), I(x_b x_c) >> + << R(\eta_b), R(x_b x_c) >> + << I(\eta_b), I(x_b x_c) >>= \mathrm{Re}(\theta^c_{ab}), \quad (3)$$

$$<< I(\eta_a), R(x_b x_c) >> + << R(\eta_a), I(x_b x_c) >> + << I(\eta_b), R(x_c x_a) >> + << R(\eta_b), I(x_c x_a) >>= \mathrm{Re}(\theta^c_{ab}), \quad (4)$$

$a<b$, $a,b,c \in \{2,\ldots,n\}$.

We will do so by showing that the equations (3), (4) are non-contradictory. We consider the vectors $(R(\eta_a), I(\eta_a))_{a\in\{2,\cdots,n\}}$ as vectors in $\mathcal{V}_0^{n-1}$, and hence, we have to verify that the relations

$$\langle(..R(\eta_a), I(\eta_a),...,R(h_b), I(\eta_b),...), v^c_{ab}\rangle = \mathrm{Re}\theta^c_{ab}$$

and similar set for the imaginary part are non-contradictory. The corresponding vectors are

$$v^c_{ab} = (0...R(x_b x_c), I(x_b x_c)...0 - R(x_c x_x), I(x_c x_a)...)$$
$$w^c_{ab} = (0...-I(x_b x_c), R(x_b x_c)..0\ I(x_c x_x), R(x_c x_a)...),$$

where the non-components are exactly on the components corresponding to a and b.

We will choose the deformation so that the vectors v^c_{ab}, w^c_{ab} are linearly independent so that one can solve equations (3), (4).

Having a linear combination that gives 0 would correspond to

$$\sum_{\substack{2\le a<b\le n\\ 2\le c\le n}} \alpha^c_{ab} v^c_{ab} + \sum_{\substack{2\le a<b\le n\\ 2\le c\le n}} \beta^c_{ab} w^c_{ab} = 0.$$

We fix an a and look what this relation correspond on the real imaginary components in the a-components. There are two possible situations. We have a contribution to this column for some $b>a$, or it may be obtained from a $b'<a$ (roles beeing switched in this last case).

Thus it may happen that (for the real component)

$$\sum_{\substack{2\le a<b\le n\\ c}} (\alpha^c_{ab} R(x_b x_c) + \beta^c_{ab}(-I(x_b x_c))) + \sum_{\substack{2\le b'<a\le n\\ 2\le c'\le n}} (\alpha^{c'}_{b'a} - R(x_{c'} x_{b'}) + \beta^{c'}_{b'a} I(x_{c'b'})) = 0 \quad (5)$$

and for the imaginary component:

$$\sum_{\substack{2\le a<b\le n\\ c}} (\alpha^c_{ab} I(x_b x_c) + \beta^c_{ab} R(x_b x_c)) + \sum_{\substack{2\le b'<a\le n\\ c'}} (\alpha^{c'}_{b'a} I(x_{c'} x_{b'}) + \beta^{c'}_{b'a} R(x_{c'b'})) = 0. \quad (6)$$

At this moment we construct a deformation of the original scalar product that should enable us to conclude linear independence.

We construct the following deformation of V. Add in direct sum $\sqrt{(1-\varepsilon')}x_i \oplus \sqrt{\varepsilon'} s_i = \widetilde{x}_i$, where s_i are semicircular [Vo], so that $\widetilde{x}_i$ remains an orthonormal basis. Moreover, with respect the new $\widetilde{x}_i$ the scalar product <<,>> is deformed from the old one by less than ε, ε' is small enough.

In this deformation $(x_a x_b)^n_{a,b\ge 2}$ are linearly independent and independent from $(x_a)^1_{a=1}$. In particular, over the reals, $I(x_{ab})$, $R(x_{ab})$ are linearly independent. Because $P\eta_a = \eta_a$, we have to take into consideration that the projection P of the elements in (5), (6) is 0, i.e., that these linear combinations will produce a conflict if they arrive in V.

From (5), (6) it follows that

$$\sum_{\substack{2\le a<b\le n\\ c}} \alpha^c_{ab} R(x_b x_c) + \sum_{\substack{2\le b'<a\le n\\ c'}} \alpha^{c'}_{b'a}(-R(x_{c'} x_{b'})) = 0 \quad (7)$$

$$\sum_{\substack{2\le a<b\le n\\ c}} \alpha^c_{ab} I(x_b x_c) + \sum_{\substack{2\le b'<a\le n\\ c'}} \alpha^{c'}_{b'a} I(x_{c'} x_{b'}) = 0 \quad (8)$$

Similarly, we have the relations

$$\sum_{\substack{2\le a<b\le n\\ 2\le c\le n}} \beta^c_{ab}(-I(x_b x_c)) + \sum_{\substack{2\le b'<a\le n\\ 2\le c'\le n}} \beta^{c'}_{b'a} I(x_{c'b'}) = 0 \quad (9)$$

and

$$\sum_{\substack{2\le a<b\le n\\ c}} \beta^{c}_{ab} R(x_b x_c) + \sum_{\substack{2\le b'<a\le n\\ c'}} \beta^{c'}_{b'a} R(x_{c'b'}) = 0 \quad (10)$$

Now, in the relations (6), (7), clearly if in the first sum *c*>*a* or in the second sum *c*'<*a*, these terms cannot cancel each other, so their coefficients must be zero.

The only possibility that a term in the first part of the sum is equal to one in the second half is when *c*<*a*, *c*'>*a* and *c* corresponds to *b*' in the second sum while *c*' corresponds to *b* in the first.

Note that in this case one has to have $\alpha^{c}_{ab} - \alpha^{c'}_{b'a} = 0$ from the first sum and the opposite from the second. Hence also these coefficients may be zero likewise for the *p*-coefficients.

Step 2 of proof. We define the matrix $<< Yx_a, Yx_b >>$ and $<< Yx_a, x_b Y >>$. We are obliged to take $<< x_b Y, x_a Y >> = \overline{<< Yx_a, Yx_b >>}$.
Moreover, one should have that the matrix $<< Yx_a, Yx_b >>$ has any property that $<< Y^2, x_a x_b >>$ that is:

(*) if $\sum \theta_{ab} x_a x_b = 0$ with respect to <<,>> then $\sum \theta_{ab} <<, Yx_a, Yx_b >>$ should be 0.

In order that <<,>> is a positive scalar product it is necessary and sufficient that the matrix

$$<< \alpha(Yx_a), \beta(Yx_b) >>^{n}_{a,b=1,\alpha,\beta=1,j} \ge \sum_{\varepsilon} << \alpha(Yx_a), \varepsilon >> \overline{<< \beta(Yx_b), \varepsilon >>},$$

Where ε runs over a basis of $V \otimes V$. Moreover, the only condition (*) comes to the requirement that $\left(\langle Yx_a, Yx_a\rangle + \langle Yx_{a^c}, Yx_{a^c}\rangle\right)^{n}_{a=2}$ be proportional to the numbers c_a.

We impose $<< Yx_a, x_b Y >> = 0$ and clearly we can find a large enough positive matrix with this property (since * doesn't imply that the matrix $\langle Yx_a, Yx_b\rangle$ should be singular).

Step 3. We define $<< Yx_a, Y^2 >>$ to be 0 and equal to $<< Y, Y^2 >>$ taking $<< Y^2, Y^2 >>$ to be large enough we get a scalar product.

Theorem. *Let V be a cyclic finite dimensional vector space with basis* $x_1, \cdots, x_n$. *Then for every ε>0, there exists an ε deformation of the original structure on V (that preserves the square conditions) and there is an infinite dimensional cyclic prehilbertian space W, with orthonormal bases* $x_1, .. x_n, x_{n+1}, ...$, *such that* $d(x_i x_j, W \otimes 1) = 0$, *∀i,j, the distance is relative to the norm induceded by positive bilinear form on W.*

Proof. We apply successivelly the previous proposition. Moreover, at every step *n* we use $\varepsilon / 2^n$. At every step we are extending the basis from the previous step. Let $<<,>>_n$ be the scalar product on W_n at step *n*. Then the sequence $<< x_i x_j, x_k x_l >>_n$ is convergent and defines a cyclic structure on the reunion *W* of W_n. Moreover, the distance $d(x_i x_j, W_n)$ tends to zero.

Note. If *W* can be obtained as an algebra of bounded operators, with trace then it would follow that [Ra] every positive polynomial *p* of degree four, that is positive under the trace in any II_1 factor, then *p* is a sum of squares (modulo universal terms of zero trace).

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