Study of Financial Systems Volatility Using Suboptimal Estimation Algorithms

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Abstract: This paper presents and implements a novel stochastic volatility (SV) model, based on the structure of the GARCH model, to describe the relationship between an observed financial return series and its standard deviation, namely volatility. The proposed approach has been compared to the standard GARCH as the underlying modeling structure within a particle-filtering-based scheme for state estimation. The proposed structure has been implemented to estimate the volatility of both a simulated return series and the NASDAQ Composite index during the period July 21, 2008 - July 17, 2009. The results of this procedure show that the parameters of the GARCH model can be used in the uGARCH structure, allowing the latter representing the hidden volatility as accurate as the standard GARCH, but also providing an estimate of the whole probability density.

Keywords: Parameter identification, state estimation, time-series analysis, financial systems, particle filters, GARCH model.

1. Introduction

In the field of Mathematical Finance –i.e., the branch of Applied Mathematics that aims to model the behavior of variables in a financial system– it is of great interest to quantify the stability of an asset given its realized returns. A measure of this stability is the so-called *volatility*, which refers to the standard deviation of the continuously compounded returns of a financial instrument.

From a Bayesian standpoint, it is possible to relate the volatility to the observed returns by a state-space model, and in order to account for the returns stylized facts, such as higher order moments and volatility clustering, this model should include non-linearities, non-Gaussian innovations, and unobservable states. A structure that is commonly used to model volatility of financial instruments is the Generalized AutoRegressive Conditional Heteroskedasticity (GARCH) model proposed by [1], which assumes that the volatility evolves in a deterministic fashion given the observations.

On the other hand, the stochastic volatility model considers that the volatility is a hidden state driven by an innovation process. Although the models that consider stochastic variables allow representing the uncertainty of some observed phenomena, the identification of its parameters is not straightforward due to a likelihood function defined as an intractable integral. Moreover, even if a suitable set of parameters is available, when the model is non-linear/non-Gaussian the estimation of the hidden states turns out into a difficult issue, since for this kind of structures there is no algorithm that guarantees optimal estimation as the Kalman filter does for the linear/Gaussian case. Consequently, sub-optimal filtering and identification approaches are needed to overcome the estimation of the volatility when a hidden-state, non-linear/non-Gaussian model is used.

The implementation of numerical techniques to estimate unobserved components has received the attention of several scientific disciplines due to the high computational power, and increasing storage capacity, developed throughout the last decades. Among the suboptimal techniques for state estimation, particle filters (PF) have recently caught the attention of the scientific community [2]-[5]. These methods are capable to approximate expectations w.r.t. a sequence of time-varying, growing dimension, probability density functions (pdf) through a finite set of weighted samples. In the case of the filtering problem, this pdf is the posterior density of the state. PF have been widely used –particularly in Financial Mathematics– due to their flexibility, and capability to be implemented along dynamical models characterized, for instance, by non-linearities, jump-diffusions, and non-Gaussian or multiplicative noise [6]-[9].

This paper is organized as follows. Section 2 introduces the concept of volatility, and presents two important classes of volatility models. Section 3 proposes a novel stochastic volatility model based on the structure of a widely used deterministic model, while Section 4 presents the results of both the previous and the introduced approach. Finally, Section 5 states the concluding remarks and suggests areas of further study according to the results obtained.

2. Volatility Models

In the field of Mathematical Finance, the term *volatility* is known as the standard deviation of the –continuously compounded– returns of a financial instrument (e.g., a share price, an equity index, or an exchange rate) and it is representative of the instrument's risk. The volatility σ_t is given by the expression:

$$\sigma_t^2 = \mathrm{var}(r_t) = E\left\{(r_t - \mu_t)^2\right\} \tag{1}$$

where r_t and μ_t are, respectively, the compounded return and its expected value at time t. The return series is known to be *heteroskedastic* if the volatility is time-varying.

Regarding the behavior of financial time series, the realized returns have shown the presence of *volatility clustering*; i.e., the fact that the returns enter in periods of high –or low– volatility. Mandelbrot [10] was one of the first authors in realizing this phenomenon stating that: *Large changes [in asset prices] tend to be followed by large changes –of either sign– and small changes tend to be followed by small changes*. Furthermore, the observations suggest that the returns are best described by non-Normal distributions, but by peaked and asymmetric ones; i.e., with high 3rd and 4th moments (a.k.a. skewness and kurtosis, respectively) [11], [12].

In order to represent these stylized facts, for each $t \in \mathbb{N}$ the return r_t –or its bias w.r.t. a known expected value– can be modeled as a product of two processes:

$$r_t = \sigma_t \varepsilon_t \tag{2}$$

where $\varepsilon_t, t \in \mathbb{N}$, is a zero-mean, unit-variance, i.i.d sequence, independent of σ_t and r_t [13].

As far as the volatility dynamics concerns, the evolution of σ_t, $t \in \mathbb{N}$, can be modeled by a deterministic, or stochastic, difference equation. The first approach leads to – easy to identify – structures that are able to model the hidden volatility only as a deterministic sequence, while the stochastic approach treats the state as a random process, hence, allowing the inference of its statistical properties, but requiring advanced techniques for parameter identification.

Subsection 2.1 presents the model known as Generalized AutoRegressive Conditional Heteroskedasticity (GARCH), a common example of the previously mentioned deterministic approach, while Subsection 2.2 shows the general form of the Stochastic Volatility (SV) model.

2.1 Generalized autoregressive conditional heteroskedasticity (GARCH)

This structure models the conditional variance at time t with respect to the observations until time $t-1$; i.e.,

$$\sigma_{t|t-1}^2 = E\left\{(r_t - \mu_{t|t-1})^2 \mid \Sigma_{t-1}\right\} \tag{3}$$

where Σ_τ is the σ-algebra constructed upon the information contained in the returns $r_{1:\tau}$, and $\mu_{t|t-1} = E\left\{r_t \mid \Sigma_{t-1}\right\}$ is the conditional expectation of r_t.

The model proposed in [1] is given by:

$$\begin{aligned} \sigma_{t|t-1}^2 &= \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1|t-2}^2 \\ r_t &= \mu_{t|t-1} + u_t \end{aligned} \tag{4}$$

where the return r_t is the observed process, the conditional volatility $\sigma_{t|t-1}$ is the state, and the innovation process u_t is defined by:

$$u_t = r_t - \mu_{t|t-1} \tag{5}$$

Also, according to (2), it is possible to assume that $\forall t$ this innovation process is distributed by:

$$u_t = \sigma_t \varepsilon_t \sim N(0, \sigma_t^2) \tag{6}$$

where $\left\{\varepsilon_t\right\}_{t\in\mathbb{N}} \sim \mathrm{NID}(0,1)$.

Additionally, $\mu \in \mathbb{R}$ and $\omega, \alpha, \beta \in \mathbb{R}^+$, $\alpha + \beta < 1$, so that $\sigma_{t|t-1}^2 > 0$ for any value of ε_t. It is important to note that, given r_{t-1}, $\sigma_{t|t-1}$ is known without uncertainty, since the GARCH model is deterministic.

Regarding identification, an optimal set of parameters –given a set of observations– can be found by means of maximum likelihood (ML) provided that the likelihood function (7) is known (assuming that $\mu_{t|t-1} = \mu$ during the period of study). However, the right hand side of (7) is usually difficult to maximize, and requires the consideration of numerical techniques.

$$L(\omega,\alpha,\beta,\mu \mid r_{1:\tau}) = p(r_{1:\tau} \mid \omega,\alpha,\beta,\mu) \qquad (7)$$

Although the GARCH model is theoretically capable of representing the stylized facts mentioned above, it models the conditional volatility as a deterministic signal, hence, it does not allow the estimation of its higher order moments or confidence intervals. This drawback suggests the consideration of structures that model the volatility as a stochastic process. This class of models is presented in the following subsection.

2.2 Stochastic volatility models (SVM)

The stochastic volatility models assume that σ_t is a hidden variable, driven by an innovation sequence which is independent of the observed process. This assumption is consistent with long-term observations showing clear, and apparently random, changes in volatility. Further reading about SVM can be found in [14]-[15]. The general form of the SVM is given by:

$$\begin{aligned} \sigma_t &= f(\sigma_{t-1},\eta_t) \\ r_t &= \sigma_t \varepsilon_t \end{aligned} \qquad (8)$$

where the return r_t is the observation, the volatility σ_t is the state of the system, and both $\{\varepsilon_t\}_{t\in\mathbb{N}}$ and $\{\eta_t\}_{t\in\mathbb{N}}$ are i.i.d. sequences.

Regarding the identification of the SVM, denoting by θ the set of parameters of the model, the likelihood function

$$\begin{aligned} L(\theta \mid r_{1:T}) &= p(r_{1:T} \mid \theta) = \\ &= \int p(r_{1:T} \mid \sigma_{1:T})\, p(\sigma_{1:T} \mid \theta)\, d\sigma_{1:T} \end{aligned} \qquad (9)$$

is defined as an intractable integral due to the presence of a hidden, stochastic, state. Therefore, an optimal set of parameters cannot be obtained from a maximum likelihood point of view, suggesting alternative schemes such as Expectation-Maximization, or Quasi-Maximum Likelihood [16].

The SVM are theoretically appropriate to explain the observed features in financial time series, however, the use of these structures implies complex distributions and considerable computational resources, hence, requiring advanced schemes to overcome the problem of volatility inference, and parameter identification.

In order to make use of the advantages of both approaches, the deterministic and the stochastic one, in the following Section an SVM with a structure similar to the one of the GARCH model is proposed, so that the GARCH parameters found by means of ML, are assumed to be suitable approximations of the SVM's optimal set of parameters.

3. Development of a Novel, GARCH-based, Stochastic Volatility Model

In the previous section it was stated that the GARCH model in (4), although allows straightforward parameter identification using ML, it does not account for the statistical properties of the variables of interest. On the other hand, the SVM (8) treats the volatility as an unobserved process allowing the inference of its pdf, at the price of not having a closed-form of the parameter likelihood function. In this regard, the necessity of a SVM with a straightforward identification procedure arises naturally. To overcome this issue, an SVM based on the structure of the GARCH model will be proposed as follows.

Regarding previous attempts to find stochastic counterparts for the GARCH model, [17] proposes an SVM called SGARCH (Stochastic GARCH) with a stochastic innovation in the evolution step. Although the authors in this work state that the SGARCH likelihood function is, unlikely most of the SV models, "relatively easy to derive", and its maximization is more time-consuming than for a standard GARCH.

3.1 Model definition

The GARCH-based SVM proposed in this work, assumes that the volatility in (4) is not driven by the realized shocks $u_t = r_t - \mu$, but by an unobserved process u'_t that is normally distributed as assumed in Subsection 2.1. The supporting idea behind this concept relies on the belief that the process σ_t will be best described if the whole residual density –instead of the realized shock u_t– is considered.

Based on Equation (6), the idea explained above can be implemented replacing u_t in (4) by u'_t defined by:

$$\begin{aligned} u'_t &= \sigma_t \eta_t \\ \eta_t &\sim N(0,1)\ i.i.d.\ \forall t \end{aligned} \tag{10}$$

Based on the relationships above, the proposed model can be expressed as in (11), and it is called uGARCH (*unobserved* GARCH).

$$\begin{aligned} \sigma_t^2 &= \omega + \alpha \sigma_{t-1}^2 \eta_{t-1}^2 + \beta \sigma_{t-1}^2 \\ r_t &= \mu + \sigma_t \varepsilon_t \end{aligned} \tag{11}$$

where r_t is a return process, σ_t is the stochastic volatility, $\mu \in \mathbb{R}$ and $\omega, \alpha, \beta \in \mathbb{R}^+$ are parameters, $\alpha + \beta < 1$; $\varepsilon_t \sim N(0,1)$ and $\eta_t \sim N(0, \sigma^2)$ are i.i.d. sequences.

An important issue regarding this model is that the moments of its posterior distribution:

$$p(\sigma_t^2 \mid r_{1:t}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(r_t - \mu)^2}{2\sigma_t^2} \right) \frac{|r_t - \mu|}{\sigma_t^3} \tag{12}$$

do not exist; this means that the expectations related to the filtering problem do not exist either. However, as a PF-based state estimation framework is to be used [2], the consideration of finite number of samples guarantees that the expectations are taken w.r.t. a pdf which is representative of $p(\sigma_t^2 \mid r_{1:t})$ –i.e. similar in the areas of high concentration of probability mass– , but with finite support. Also, although the moments of the posterior do not exist, the confidence intervals are properly defined; i.e., for any $\lambda \in [0,1)$, exists $c < 1$ such that:

$$\int_0^c p(\sigma_t^2 \mid r_{1:t}) d\sigma_t^2 = \lambda \tag{13}$$

3.2 Model identification based on standard GARCH structure

In order to perform a successful implementation of the proposed model, a set of parameters accurately representing the process under study is required. Unfortunately, the likelihood function of the uGARCH, given the observations, is defined as an intractable integral. Considering $\theta = [\mu, \alpha, \beta]$, the likelihood function, derived from (9), is:

$$\begin{aligned} L(\theta \mid r_{1:t}) = \int \prod_{j=1}^{t} & \frac{1}{2\pi \sigma_j \sigma_{j-1} \sqrt{\alpha(\sigma_j^2 - \omega - \beta \sigma_{j-1}^2)}} \\ & \cdot \exp\left(\frac{\omega + \beta \sigma_{j-1}^2 - \sigma_j^2}{2\alpha \sigma_{j-1}^2} + \frac{(r_j - \mu)^2}{2\sigma_j^2} \right) d\sigma_{1:t}^2 \end{aligned} \tag{14}$$

Hence, the maximization of $L(\theta|r_{1:t})$ is not straightforward. Based on the similarities of the models GARCH and uGARCH, it will be assumed that the optimal set of parameters for the GARCH model, derived using ML, represents a suitable estimation of the parameters for the uGARCH model.

This means that, given a sequence of T observed returns $r_{1\ldots T}$, it is possible to fit a GARCH model by means of ML; i.e., to derive an optimal set of values for the parameters ω, α, β, and μ in (4), and use them in the uGARCH model (11) to represent the relationship between a return series and its underlying stochastic volatility.

4. PF-based Analysis of Financial Data

In order to validate the proposed structure, a particle-filtering-based estimation scheme has been implemented, using the uGARCH model, to filter the volatility of both simulated and real financial return series.

The filtering structure to be used is the so-called Bootstrap filter proposed by [18]. This filter approximates the posterior distribution $p(\sigma_t|r_{1:t})$ by a set of N weighted samples, computed recursively every time a new observation is available; i.e., it is assumed that exists $\{\sigma_t^{(i)}, w_t^{(i)}\}_{i=1\ldots N}$, such that:

$$p(\sigma_t \mid r_{1:t}) \simeq \sum_{i=1}^{N} w_t^{(i)} \delta(\sigma_t - \sigma_t^{(i)}) \tag{15}$$

where $\delta(\bullet)$ is the Dirac function. Moreover, to ensure that the set of samples represents accurately the posterior distribution, the Bootstrap filter includes a resampling step to avoid degeneracy of the empirical distribution in the right hand side of (15).

4.1 Simulated time series

The motivation behind the utilization of the proposed model structure, to solve the problem of estimating the volatility of a simulated return series, relies on the fact that the volatility process is known in such case; hence, the estimation error can be computed w.r.t. the actual process. In this sense, 250 samples have been generated for the volatility and its corresponding return process $\{\sigma_t, r_t\}_{t=1...250}$ using the GARCH model in (16).

$$\begin{aligned} &\sigma^2_{t|t-1} = 10^{-6} + 0.2u^2_{t-1} + 0.6\sigma^2_{t-1|t-2} \\ &r_t = 9 \cdot 10^{-4} + u_t \end{aligned} \tag{16}$$

With this parameters, a uGARCH model has been implemented as shown in (17) as the underlying modeling structure of a Bootstrap filter to estimate the process $\{\sigma_t\}_t$=1...250 considering the observations $\{r_t\}_t$=1...250.

$$\begin{aligned} &\sigma^2_t = 10^{-6} + 0.2\sigma^2_{t-1}\eta^2_{t-1} + 0.6\sigma^2_{t-1} \\ &r_t = 9 \cdot 10^{-4} + \sigma_t \varepsilon_t \end{aligned} \tag{17}$$

Results of this estimation procedure are shown in Figure 1. It is possible to note that the set of parameters of the GARCH model that generated the volatility sequence $\{\sigma_t\}_{t=1...250}$, and its corresponding returns $\{r_t\}_{t=1...250}$, can be also used for the uGARCH SVM, since the generated process and the expectation of the volatility estimated by the Bootstrap filter are qualitatively close. Additionally, the mentioned filter gives a notion of the whole volatility pdf; characterized by its 95% confidence interval (shown in Figure 1).

Although simulation results validate the proposed structure, it is also important to test it in scenarios with real data, where the hidden state and the model parameters are unknown. This case is revised as follows.

4.2 Estimation of NASDAQ composite index volatility

Since February 5th, 1971, the NASDAQ Composite Index (NCI) represents all of the components listed in the NASDAQ stock market, meaning that it has over 3000 financial instruments, and it is widely known as a technology industry indicator. The study of the volatility of the NCI –motivated by speculators and shareholders both aiming to trade derivatives over the NCI, or to take part of the technology market– is an interesting challenge given the strong fluctuations, and the high uncertainty, that such index has shown since its very early beginning. These features make the NCI a suitable candidate to validate the proposed structures.

4.2.1 Data preprocessing

Given that the models presented in the Sections 2 and 3 assume that the observed process is a return time series, the financial data –usually

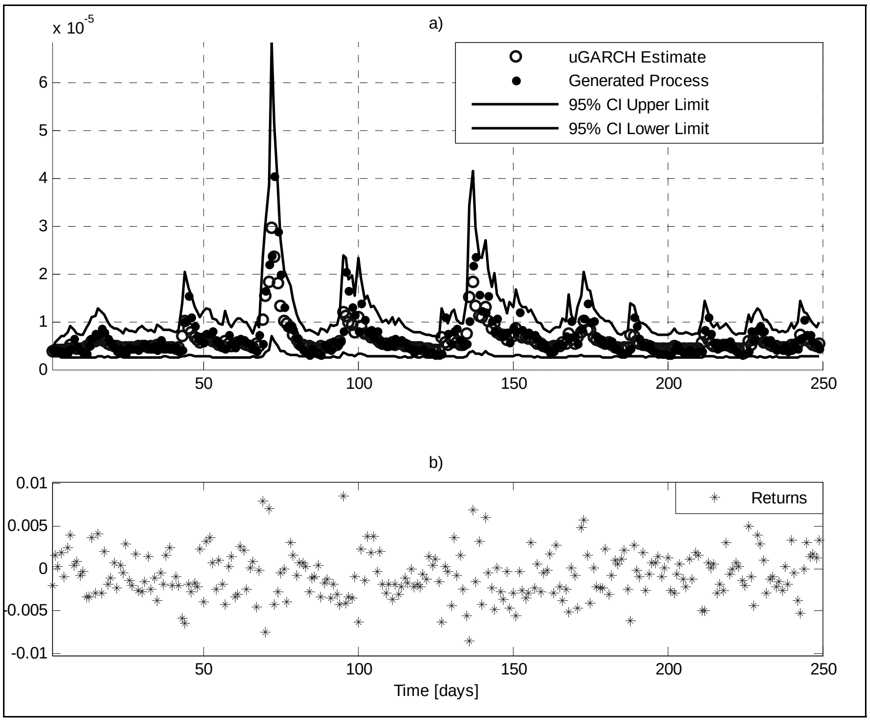

Figure 1. Volatility estimates of a simulated process $r_{1:250}$ using the uGARCH model and the PF. a) black dots: simulated volatility process using a GARCH; circles: uGARCH-PF estimate; fine solid line: uGARCH estimate of the 95% CI. b) Simulated return process

available in the form of prices, or index, series– have to be converted into returns. This process is performed using continuous compounding; i.e., the index values p_t, and the returns r_t are related by:

$$r_t = \log\left(p_{t+1} / p_t \right) \quad (18)$$

Figure 2 presents both the NCI values and their returns computed according to (18) for the last 4 years. The first signs of the late-2008 recession can be also observed in this figure, where the 600th sample represents September 4, 2008. Also, based on the statistics in Table 1, the presence of higher-order moments suggests the consideration of volatility models as the ones presented in Sections 2 and 3. In the following both the standard and the unobserved GARCH are implemented to estimate the volatility of the NASDAQ Composite Index.

2008 and July 17, 2009 (250 samples corresponding only to weekdays, continuously compounded, and denoted by $r_{1:250}$). The motivation to consider such period was to evaluate the performance of the estimation structures under considerable index changes, since September 2008 is the start of the late-2000s recession (according to the National Bureau of Economic Research).

Out of the index values $r_{1:250}$, it was considered the subset $r_{1:80}$, to identify a standard GARCH model using maximum likelihood. These parameters, as in Subsection 4.1, have been also used to implement an uGARCH model (19).

$$\begin{aligned} \sigma_t^2 &= 1.53 \cdot 10^{-5} + 0.121 \sigma_{t-1}^2 \eta_{t-1}^2 + 0.863 \sigma_{t-1}^2 \\ r_t &= 2.7 \cdot 10^{-4} + \sigma_t \varepsilon_t \end{aligned} \quad (19)$$

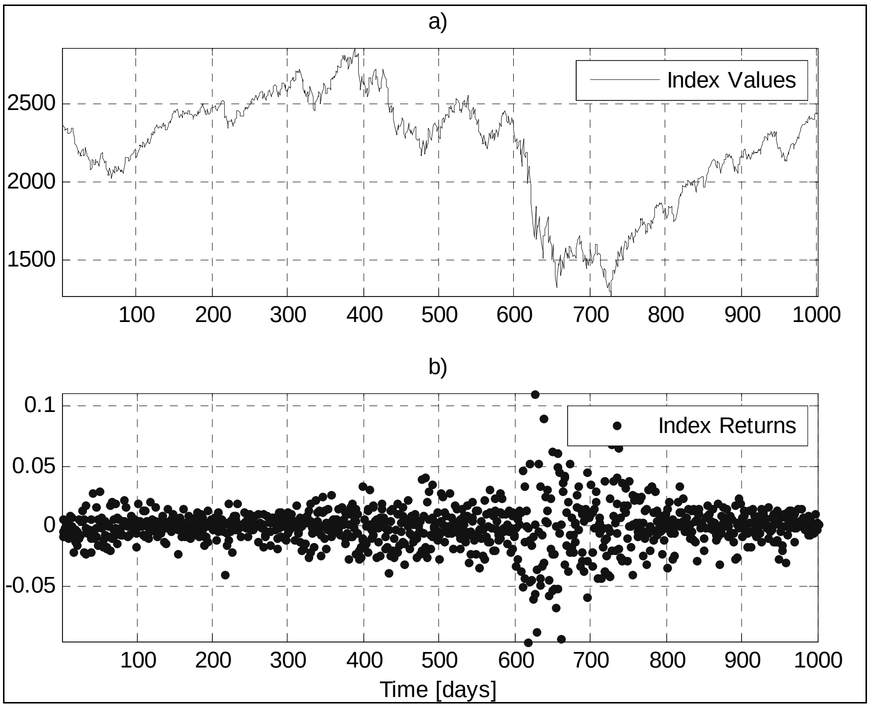

Figure 2. NASDAQ Composite Index and its return series between April 18, 2006 and April 8, 2010.

Table 1. Median and first 4 moments for the NASDAQ Composite index returns between April 18, 2006 and April 8, 2010

Median	Mean	Variance	Skewness	Kurtosis
0.0010	3.366e-05	3.0177e-04	-0.1381	8.9040

4.2.2 Model training

The observations to be considered for NCI's volatility estimation will be the closing value of such index on a daily basis between July 21,

In the following subsection, the uGARCH model in (19), together with the Bootstrap filter, is used to estimate the volatility of the NCI within the mentioned period. These results are compared to a smoothed estimation performed by a GARCH model.

4.2.3 Volatility estimation

Figure 3 presents the index values and its returns series, in addition to the computed estimates. In Subfigure (b) it can be seen how the index fell as a consequence of the start of the recession in t = 50, while (c) shows how the returns present high variance from this point.

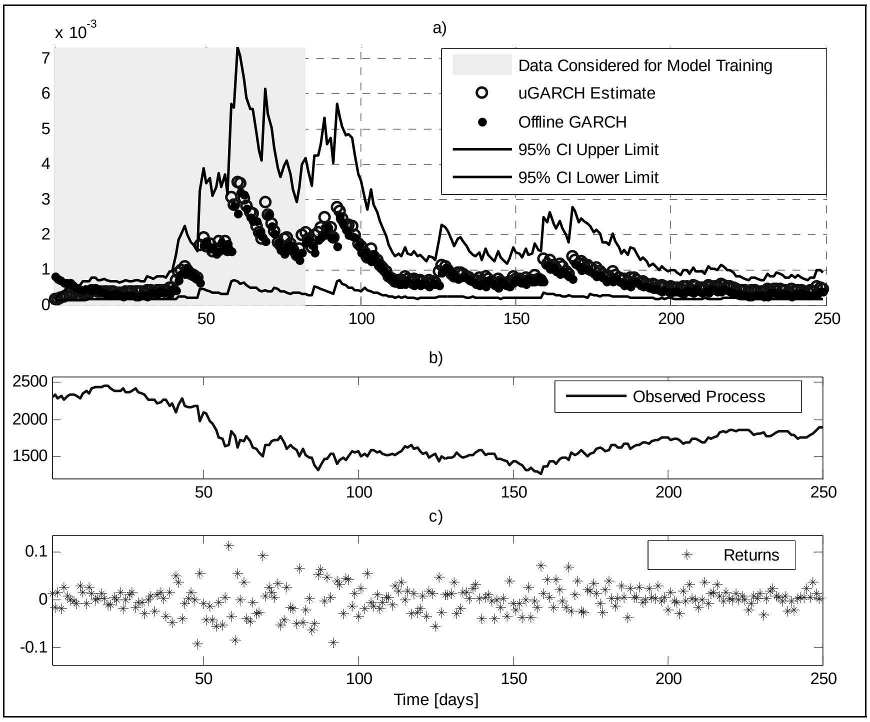

Figure 3. NCI volatility estimates during July 21, 2008 - July 17, 2009 using PF.
a) black dots: volatility smoothed estimate using a GARCH model and ML; blue circles: uGARCH-PF estimate; fine solid lines: uGARCH 95% CI estimate. b) Observed NCI values.
c) Continuously compounded returns.

Regarding the estimates, the uGARCH-PF-based filtering structure realizes the variation of the volatility behavior, since the confidence interval estimate widens in t = 50, and also presents accurate estimates w.r.t the smoothed GARCH estimate.

5. Conclusions

This paper presents, and implements, a novel stochastic volatility model, namely uGARCH, based on the deterministic GARCH model, to estimate the volatility of a given sequence of financial returns using particle filters. Although the proposed structure does not allow straightforward parameter identification, the results show that a set of parameters derived by means of maximum likelihood for the standard GARCH model, can be also used in the uGARCH model.

In the context of estimating the hidden volatility in both simulated and real financial systems, the expected value of the uGARCH estimates is similar to the one of the standard GARCH. Additionally, the uGARCH represents the whole probability density function of the hidden volatility, which is a useful resource in applications such as VaR estimation.

Concerning future research in this line, the authors propose the inclusion of adaptive identification schemes in order to derive a representative set of parameters in case of changing dynamics.

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