Observers Design for Discrete-Event Systems Modelled by S-Nets

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Abstract: This paper addresses the design of observers for Discrete-Event Systems modelled by Output Petri nets. The observer is conceived as a copy of the system and a corrective term based on the execution trajectories. The observer performs a tracking of the transition sequence executed by the net. Based on this information, the observer is able to produce approximations of the initial and current state of the system. The focus is a subclass of Petri nets called S-Nets. A Lyapunov criterion is used for testing the stability of the herein proposed scheme. This criterion allows for proving that the observers are asymptotically stable and it supports characterizing the region of stability of the System/Observer pair, as well. An application example is developed through the paper to illustrate the results. Some graphs are provided to show the approximation error of the observer under different initial conditions.

Keywords: Observer Design, Petri Nets, S-Nets, Discrete-Event Systems, Sequence Observer, Lyapunov Stability.

1. Introduction

The observability is an important property of a Discrete-Event System (DES). Different frameworks have been proposed to face the problems related to the study of this property. In the finite automata (FA) framework, automatons are used for modelling a DES, and the analysis of properties is typically done by linguistic approaches. Mainly, the observability notions in the FA framework consist on dividing the automata language into equivalence classes. The control techniques consider that a specification is feasible to be implemented if the classes induced by the observability are finer than those induced by the controllability. This is a class of “static observability” that does not consider the concept of an observer for progressively discover the system’s state, as part of the control scheme [6]-[12].

In Vector Additive Systems (VAS) framework, similar approaches, as those used in FA, are applied. The plant is modelled as a system of vectors, while a set of linear inequalities are the specifications. In a similar way such as in FA, the observability notions require that if two different states of a VAS produce the same output signal, then they must require the same control action. Accordingly, the observability notion in a VAS produces results that are consistent to those of FA. Consequently, a VAS does not consider the notion of an observer for reconstructing the system state [13], [14].

The design of observers in Petri Nets (PN) framework has been addressed in a fewer number of works than those presenting designs of controllers based on this modeling tool. In [16], the problem of discovering the marking of the net is addressed. The proposed scheme produces marking estimations, which are always a lower bound of the current marking of the net. In [17] and [18], the problem of discovering the marking of an Interpreted PN is considered. The sequence invariance and a geometrical approach are used for the analysis and observer design.

Some related works of the authors are reported in the literature. In [1] and [4], the authors state results about the observability analysis focused on the subclass of PN’s known as Free-choice nets. In [5], the authors show that a combination of a PN observer and a supervisor allows for addressing problems that has no solution with the solely use of a supervisor, as in [14] and [15]. In [3] and [2], a framework of practical interest based on Matlab/Simulink for the study of DES, including controllers and observers, is reported.

This work addresses the design of observers for a class of PN known as S-Nets. A Lyapunov criterion is proposed for the stability analysis. The major contributions of this paper are: a) polynomial algorithms for the observer construction; b) the use of a Lyapunov stability criterion for the observer characterization, and c) analysis of the stability region of the pair, system and observer.

The rest of this paper is organized as follows. Section 2 provides some background notions
on PN and on the sequence detection in S-Nets. The major contribution of this work is in sections 3 and 4. Section 3 details the proposed observer scheme and the metric space for the measurement of its error. Section 4 proposes a functional term based on the number of sequences that the observer is tracking at every step. It also shows that the observer satisfies the Lyapunov stability criteria. An example at the end of this section illustrates the developed technique. Section 5 provides the conclusions and finally, are the bibliographical references.

2. Background

This section shows basic PN notions, and briefly reviews the main results on sequence-detectability that are relevant for this work.

Output Petri Nets

An OPN is a tuple \((B, M_0, \varphi)\) where \(B\) is a PN structure \((P, T, I, O)\), \(M_0\) is the initial marking and \(\varphi\) is the output function. For simplicity, \(B\) also stands for the incidence matrix of the PN structure. The pre-set \(\cdot \circ t_j\) and post-set \(\circ t_j \cdot \) of a transition \(t_j \in T\) are as usual. Likewise, are the pre-set \(\cdot \circ p_i\) and post-set \(\circ p_i \cdot\) of a place \(p_i \in P\). The operator \(\circ\) is extended in a natural way for sets. This work deals with well-formed nets, which in summary are strongly connected, conservative and repetitive nets [19]. The compact representation of a marking \(M_k(x_{p_i})\) for \(M_k(p_i) = x\), with \(p_i \in P\) and \(x \in \mathbb{N}^+\), is as in [1]. The state equation of an OPN is:

\[
M_{k+1} = M_k + B\vec{u}_k, \quad y_k = \varphi(M_k) \tag{1}
\]

The notation \(M_k \rightarrow M_{k+1}\) means that from \(M_k\) the transition \(t_j\) is fired reaching \(M_{k+1}\). A net is safe if \(\forall M_k \in R(B, M_0)\), it holds that \(M_k(p_i) \leq 1, \forall p_i \in P\), and non-safe otherwise. The length of a sequence of transitions \(\sigma\) is denoted by \(|\sigma|\). This sequence or trajectory is denoted by \(M_0 \rightarrow M_S\).

As in the automata theory, \(\sigma^*\) denotes the Kleen closure of a sequence \(\sigma\), which extends in a natural way to sets [20]. The output word associated to a sequence \(\sigma\), defined as \(\varphi(\sigma) = \varphi(M_k)\varphi(M_{k+1})...\varphi(M_{k+|\sigma|})\varphi(M_{k+|\sigma|})\), is the information that an external observer is able to detect from an OPN.

Thus, by (1) \(\varphi M_{k+1} = \varphi M_k + \varphi B\vec{u}_k\). The vector \(\varphi B\vec{u}_k\), denoted by \(\varphi_B(\vec{u}_k)\), is the change, or increment, in the system sensors due to the firing of \(u_k\), which the observer tries approximate. Notice that it is possible \(\varphi_B(\vec{u}_i) = 0\), while \(B\vec{u}_{i} \neq 0\). The transition \(t_i\) is known as silent [11]. This class of transitions are out of the scope of this work. For additional notions about PN see [19].

Sequence Detectability in S-Nets

This work is devoted to the design of an observer for tracking the transition sequence executed by an OPN. The Sequence-Detectability (SD) is a useful concept. An efficient solution for the SD is derived in [1] on Output S-System (OSS), as the one shown in Figure 1. A first step in the testing of the SD is the construction of the Event-Detectability (ED) table \(E_B\). Given a safe OSS \((B, M_0, \varphi)\), with \(T = \{t_1, ..., t_n\}\), \(E_B\) is a square arrangement \([n-1 \times n-1]\), where columns represent transitions from \(t_1\) to \(t_{n-1}\) and rows represent transitions from \(t_2\) to \(t_n\). The entries of \(E_B\), for \(2 \leq i \leq n; 1 \leq j \leq n-1; j < i\), are defined as follows:

\[
\text{if } \varphi_B(\vec{t}_i) \neq \varphi_B(\vec{t}_j), \text{ then } E_B(t_i, t_j) = \emptyset \\
\text{otherwise, } E_B(t_i, t_j) = U(t_i, t_j) = \emptyset \\
\forall t_u \in (t_i \circ), \forall t_v \in (t_j \circ) \tag{2}
\]

The Table 1 shows \(E_B\) for the net in Figure 1. A further refinement of \(E_B\) leads to the Sequence-Detectability (SD) table \(E_B^g\). The entries of \(E_B^g\)
are obtained from the entries in $E_B$ by a
repetitive procedure as follows:

$$E_B^\delta(t_i, t_j) = E_B(t_i, t_j) \setminus (t_{ui}, t_{v_i})$$

if $E_B(t_{ui}, t_{v_i}) = \emptyset$ and both,

$$\circ t_i \cap \circ t_j = \emptyset \text{ and } t_i \circ t_j =: \emptyset$$

The definition (3) is recursively applied until
no new empty entries appear. In [1], the authors
shown that the non-empty entries in $E_B^\delta$ lead to
circuits of transitions, denoted by $\Delta E_B^\delta$, that are
closely related to the SD of the net.

**Theorem 1.** A safe OSS $(B, M_0, \varphi)$ is SD if and
only if $\Delta E_B^\delta = \emptyset$.

**Proposition 1.** Let $E_B^\delta$ be the SD table of the
OSS $(B, M_0, \varphi)$. Then, for any nonempty entry
$E_B^\delta(t_g, t_h)$ there exist a pair of circuits, say $\sigma_g$
and $\sigma_h$, such that $\varphi(\sigma_g) = \varphi(\sigma_h)$.

When an OSS is non-safe, the results of the
safe nets have to be extended.

**Theorem 2.** Let $(B, M_0, \varphi)$ be a non-safe OSS.
Then, the net is SD if and only if $E_B$ is empty.

**Observability in S-Nets**

Besides the tracking of the transition sequence
of an OSS, in some cases, the observer is able to
determine the marking of the net. Informally, an
OPN $(B, M_0, \varphi)$ is observable if its initial
and current markings, $M_0$ and $M_2$, respectively,
could be determined in finite time, by using the
information of the output word $\varphi(\sigma)$ and the
structure of $(B, M_0, \varphi)$.

**Marking-Detectability (MD)** are sufficient
conditions for the Observability in a PN.

**Proposition 2.** An OPN $(B, M_0, \varphi)$, which is
FVD and MD, is Observable.

For the case of OSS nets, the ED and the SD
lead to efficient solutions of the FVD and the
MD.

**Proposition 3.** If a well-formed and safe OSS
$(B, M_0, \varphi)$ is SD then, it is also MD.

Thus, the next holds for the observability.

**Corollary 1.** If a well-formed and safe OSS is
SD, then it is observable.

When the OSS is non-safe, the next theorem
shows how $E_B$ allows for testing the SD.

**Theorem 3.** Let $(B, M_0, \varphi)$ be a non-safe OSS.
Then, the net is SD if and only if $E_B$ is empty.

The SD, as in the case of safe nets, does not
directly imply the MD in a non-safe OSS.

**Theorem 4.** A non-safe OSS $(B, M_0, \varphi)$ is MD
if and only if $\ker \varphi = \emptyset$.

**Corollary 2.** Let $(B, M_0, \varphi)$ be a non-safe OSS.
Then $(B, M_0, \varphi)$ is observable if and only if
$\ker \varphi = \emptyset$.

**3. Observer Design**

Consider Figure 2, which depicts the block
diagram of the proposed observer scheme. The
observer is designed as a copy of the system
plus a corrective term, as shown.
Definition 1. Let \((B, M_0, \varphi)\) be the OPN as in (1). The equation of the Observer \((\bar{B}, \bar{M}_0, \varphi)\) is:
\[
\bar{M}_{k+1} = \bar{M}_k + B\bar{u}_k + \ell(\bar{y}_k - y_k)
\]
\[
\bar{y}_k = \varphi\bar{M}_k
\]
(4)

The term \(\ell(\bar{y}_k - y_k)\) is denoted as \(\ell_k\) where no confusion arises. The error \(e_k\) between the system and the observer is denoted by \(e_k = \bar{M}_k - M_k\). Hence, \(e_{k+1} = \bar{M}_{k+1} - M_{k+1}\). From (1) and (4), \(e_{k+1} = \bar{M}_k + B\bar{u}_k + \ell_k(\varphi\bar{M}_k - \varphi M_k) = (M_k + B\bar{u}_k)\). By expanding and rearranging the terms, it leads to \(e_{k+1} = (\bar{M}_k - M_k) = e_k - \ell_k\varphi e_k\). Thus, the equation defining the dynamics of the error system for Figure 2 is as follows:
\[
e_{k+1} = (1 - \ell_k\varphi)e_k
\]
(5)

The initial error in (5) is \(e_0 = \bar{M}_0 - M_0\), where \(\bar{M}_0\) is an initial estimation, or “guess”, of the observer. In order to emphasize the initial error \(e_0\), let \((E, e_0)\) denotes the system (5). Notice that a “trajectory”, say \(e_k \rightarrow e_{k+1}\), occurs in (5) if and only if \(e_k = \bar{M}_k - M_k\) and \(e_{k+1} = \bar{M}_{k+1} - M_{k+1}\), such that \(\bar{M}_k \in \mathcal{L}(B, \bar{M}_0)\), \(e_k \in \mathcal{L}(B, M_0)\), \(\bar{M}_k \in \mathcal{L}(B, \bar{M}_0)\), \(e_k \in \mathcal{L}(B, M_0)\), and \(e_k \in \mathcal{L}(B, M_0)\). Let \(\mathcal{L}(E, e_0)\) be the set of all the trajectories of (5). In this context, we say that \(\rho_k \in \mathcal{L}(E, e_0)\) corresponds to \(e_k\) and \(\bar{M}_k\) in the system and observer, respectively. Thus, a sequence in (5) may be empty, single or infinite, where \(\mathcal{L}(E, e_0)\) is the middle set of \(\mathcal{L}(E, e_0)\), as with an OPN.

Observer Metrics

In order to measure the error of the system (5), the following notions are considered. Let \(m\) be the number of places of the OPN \((B, M_0, \varphi)\). The Manhattan distance \(\rho\) of \(u, v \in \mathbb{N}^m\), here denoted by \(\|u - v\|\), is defined as usual:
\[
\rho(u, v) = \sum_{i=1}^{m} |u(i) - v(i)|
\]
(6)

The distance from an element \(u \in \mathbb{N}^m\) to a set \(V \subseteq \mathbb{N}^m\) is \(\rho(u, V) = \min\{\|u - v\| : v \in V\}\).

Definition 2. The \(r\)-neighbourhood of a subset \(\mathcal{M} \subseteq \mathbb{N}^m\) is \(\mathcal{S}(\mathcal{M}, r) := \{e \in \mathbb{N}^m : 0 < \|u - v\| < r\}\) for some \(r > 0\).

Since \(\rho\) is defined in \(\mathbb{N}^m\), then it is supposed that \(r \in \mathbb{Z}^+\). The subset \(\mathcal{M} \subseteq \mathbb{N}^m\) is said to be invariant w.r.t. \((E, e_0)\), if firstly, \(e_0 \in \mathcal{M}\), where \(e_0 = (\bar{M}_0 - M_0)\). Secondly, \(\forall e_k \in \mathbb{N}^m\) such that \(e_0 \rightarrow e_k\). Then, necessarily \(e_k \in \mathcal{M}\), for every \(k > 0\), whenever exists an infinite sequence \(\gamma\) in (5), that follows from \(\tau_k\), i.e., \(e_0 \rightarrow e_k \rightarrow \ldots\). Notice that \(\gamma\) implies the existence of infinite sequences, say \(\alpha, \beta\), such that \(e_k, \alpha, e_\alpha \in \mathcal{L}(B, M_0)\) and \(\alpha, \beta \in \mathcal{L}(B, \bar{M}_0)\), respectively, where \(e_k, \alpha, \beta\) are related to \(\tau_k\). In other words, a subset \(\mathcal{M} \subseteq \mathbb{N}^m\) is invariant w.r.t. \((E, e_0)\) if \(e_0\) belongs to \(\mathcal{M}\), and, a) for any \(e_k\), “reachable” from \(e_0\) by a finite trajectory \(\tau_k, k \geq 0\), then \(e_k\) also belongs to \(\mathcal{M}\), whenever exist an infinite trajectory \(\gamma\) “following to” \(\tau_k\) in \((E, e_0)\). Notice that if an OSS \((B, M_0, \varphi)\) is well-formed, then for every \(\alpha \in \mathcal{L}(B, M_0)\), \(\gamma < \alpha < \infty\), it always exists another infinite sequence, say \(\alpha, \beta \in \mathcal{L}(B, M_0)\), such that \(\alpha \in \mathcal{L}(B, M_0)\) (by Prop. 8.2 Home markings of live S-Nets [19]). Thus, the later requirement is trivially satisfied.

Suppose that the structure \((B, M_0, \varphi)\) of the OSS is known and the net is SD. Thus, if \(e_0 = 0\), i.e., \(M_0 = \bar{M}_0\), then \(e_{k+1} = 0\) for any \(k \geq 0\). On the contrary, suppose that for some \(k > 0\), where \(e_k \rightarrow e_{k+1}\) for \(\tau_k \in \mathcal{L}(E, e_0)\), with \(e_k = 0\) and \(|\tau_k| = 1\), it holds that \(e_{k+1} > 0\). Then, there must exist \(\bar{e}_k \in \mathcal{L}(B, \bar{M}_0)\), \(\bar{M}_k \rightarrow \bar{M}_{k+1}\), and \(\sigma_k \in \mathcal{L}(B, M_0)\) such that \(|\sigma_k| = 1\) and \(|\sigma_k| = 1\), with \(e_k = \bar{M}_k - \bar{M}_k\) and \(\sigma_k = \bar{M}_{k+1} - M_{k+1}\), corresponding to \(\tau_k\). Let say, without loss of generality, that \(\sigma_k = t_k\) and \(\sigma_k = \ell_k\). Thus, since \(e_{k+1} > 0\), it implies that \(\bar{M}_{k+1} \neq M_{k+1}\). But, the structure of the OSS \((B, M_0, \varphi)\) is known, and since the observer has wrongly computed \(\bar{M}_{k+1}\) when \(e_k = 0\), this directly implies that there must exists in the net at least one transition besides of \(t_k\), say \(t'_k \in T\), such that \(\varphi_B(t'_k) = \varphi_B(t_k)\) for \(|\varphi(t'_k)|\) and \(|\varphi(t_k)|\). However, if the net is safe,
then \( t_k \rightarrow t_k \), which directly contradicts the SD of the net. On the other hand, if the OSS is non-safe, then \( \varphi_B(t_k^\gamma) = \varphi_B(t_k) \) directly contradicts the ED of the net. Thus, \( \{0\} \) is invariant w.r.t. \((E, e_0)\), as stated.

The Lyapunov stability criterion for \((E, e_0)\), valid in the metric space \((\mathbb{N}_0^m, \rho)\), is as follows.

**Definition 3.** A closed invariant set \( \mathcal{M} \subset \mathbb{N}^m \) of the error system (5) is stable in the sense of Lyapunov w.r.t. \( \mathcal{L}(E, e_0) \), if for every \( \epsilon > 0 \), it is possible to find a \( \delta > 0 \), such that whenever \( \rho(e_0, \mathcal{M}) < \delta \), it holds that \( \rho(e_k, \mathcal{M}) < \epsilon \), where \( e_0 \rightarrow e_k \), for all \( k > 0 \). If furthermore, \( \rho(e_k, \mathcal{M}) \rightarrow 0 \) as \( k \rightarrow \infty \), then \( \mathcal{M} \) is asymptotically stable. The region of asymptotic stability of \( \mathcal{M} \) is the subset \( \mathcal{M}_a \subset \mathbb{N}^m \) of all \( e_0 \in \mathbb{N}^m \) for which \( \mathcal{M} \) is asymptotically stable.

The next theorem provides a necessary and sufficient condition to verify asymptotic stability in a neighbourhood of \((E, e_0)\). See [21] for the formal proof.

**Theorem 5.** Let \((E, e_0)\) be the error system in (5) and let \( \mathcal{M} = \{0\} \). Then, \( \mathcal{M} \) is asymptotically stable in the sense of Lyapunov w.r.t. \( \mathcal{L}(E, e_0) \) if and only if in a sufficiently small neighbourhood \( S(\mathcal{M}, r) \), there exist a functional \( V_k \) fulfilling:

i. For all sufficiently small \( c_4 > 0 \) it is possible to find a \( c_2 > 0 \), such that when \( V_k > c_2 \) it holds that \( \rho(e_k, \mathcal{M}) > c_1 \) for \( e_k \in S(\mathcal{M}, r) \);

ii. For any \( c_3 > 0 \) as small as required, it is possible to find a \( c_3 > 0 \) so small such that when \( \rho(e_k, \mathcal{M}) < c_3 \) it holds that \( V_k \leq c_3 \) for \( e_k \in S(\mathcal{M}, r) \);

iii. \( V_k \) is a non-increasing function for all \( k > 0 \), as long as \( e_0 \to e_k \) with \( e_k \in S(\mathcal{M}, r) \) for which exist an infinite trajectory \( \gamma \) such that \( e_0 \to e_k \to \cdots \);

iv. Moreover, \( V_k \to 0 \) as \( k \to \infty \).

The election of the functional \( V_k \) is the key element in the observer design for the stability testing.

This paper proposes a functional that is based in the number of transitions sequences that the term \( t_k \), shown in Figure 2 as part of the observer, is tracking at every \( k \). Before the formal analysis, some intuitive ideas are given in the following example.

**Example 1.** Consider that the initial state of the OSS in Figure 1 is \( M_0(p_2^0) \) and that its respective output is \( y_0 = \{C\} \). The net includes dashed tokens and output signals after a slash that will be used in a later example. With a simple examination, it is easy to see that the markings \( M_0(p_2^1), M_0(p_2^2), M_0(p_2^3), M_0(p_2^4), M_0(p_5^1) \) and \( M_0(p_5^2) \) all produce the same output \( y_0(C) \).

Let \( [\tilde{M}]_0 = M_0(p_2^1) + M_0(p_2^2) + M_0(p_2^3) + M_0(p_2^4) + M_0(p_5^1) + M_0(p_5^2) \) be the observer estimation. Thus, at \( k = 0 \), the distance of the observer estimation and the system state is four, i.e. \( e_0 = 4 \). Now, suppose that \( t_3^2 \) is fired at \( k = 1 \). The output is now \( y_1(D) \). Notice that \( \varphi_B(t_3^2) = \varphi_B(t_3^2) = \varphi_B(t_3^3) = \varphi_B(t_3^5) \).

Indeed, all of these transitions are enabled at \( [\tilde{M}]_0 \). Therefore, at \( k = 1 \), the observer computes the possible executed sequences as \( \sigma_1 = t_3^3, \sigma_2 = t_3^3, \sigma_3 = t_3^3, \sigma_4 = t_3^5 \) and \( \sigma_5 = t_3^5 \). Let’s denote the set of these sequences as \( \{\sigma\}_1 \). Corresponding to \( \{\sigma\}_1 \), the observer computes the possible reached markings as those with one token in the places with subscript three, i.e. from \( M_1(p_3^1) \) to \( M_1(p_3^5) \). Let \( [\tilde{M}]_1 = M_1(p_3^1) + \cdots + M_1(p_3^5) \) be the summation of these markings. At this point, the observer does not improve the state estimations, i.e. the error remains the same. This is an important characteristic of the SD of an OSS, that is, it is possible that the estimations do not improve at every event execution. In this case, \( e_1 = 4 = e_0 \). However, the estimations must improve in a finite number of events until they match to the system. Indeed, suppose that \( t_3^4 \) is fired, at \( k = 2 \) and the output changes to \( y_2 = \{F\} \). Notice that \( \varphi_B(t_3^1) = \varphi_B(t_3^2) = \varphi_B(t_3^3) = \varphi_B(t_3^4) = \varphi_B(t_3^5) \). By using this new information, the observer is able to update the set of possible executed sequences to

\[
\{\sigma\}_2 = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8\}, \quad \text{where} \quad \\
\sigma_1 = t_3^3 t_4^1, \quad \sigma_2 = t_3^3 t_4^2, \quad \sigma_3 = t_3^3 t_4^3, \quad \sigma_4 = t_3^5 t_4^4, \\
\sigma_5 = t_3^5 t_4^5, \quad \sigma_6 = t_3^6 t_4^6, \quad \sigma_7 = t_3^7 t_4^7, \quad \text{and} \quad \sigma_8 = t_3^8 t_4^8. \]

At this point, the observer is able to discard \( \sigma_8 \), since \( \varphi_B(t_3^4) = \varphi_B(t_3^4) \).

Hence, the possible reached marking \( [\tilde{M}]_2 \), is that with a token on places with subscript four, from \( M_2(p_4^1) \) to \( M_2(p_4^8) \). Notice that now \( \|\{\sigma\}_2\| < \|\{\sigma\}_1\| \), and accordingly, \( e_2 \leq e_1 \).
The next section proposes a functional term based on the number of transition sequences tracked by the observer, which satisfies the Lyapunov stability criterion.

4. Observer Stability Analysis

Generalizing the idea in the example of the previous section, let $y_k$ be the $k$-th system output and $\{\hat{\sigma}\}_k$ be the possible sequences executed by the system. Notice that the number of possible transition sequences that the observer may track does not necessarily decrease at every transition firing, as stated. However, this paper shows that for any evolution of the system of length $k$, with a corresponding set of possible fired transition sequences $\{\hat{\sigma}\}_k$, there exist a finite integer $a$ such that

\[|\{\hat{\sigma}\}_k + \tau| < |\{\hat{\sigma}\}_k|\]

To this aim, consider the following definition for the corrective term of the scheme in Figure 2.

**Definition 4.** Let $(B,M_0,\varphi)$ be a SD and well-formed OSS, where $M_0$ is probably unknown. Let $y_0$ be the initial output of the system. Let $M_0 = \varphi(y_0)$ be the initial estimation for the observer by (5). Let $[M]_0 = M_0 = [\hat{M}]_k$ be the estimations of the initial and current markings. Define $\ell_k$ as follows:

(first step) For $k = 1$:

i. Let $\{\hat{\sigma}\}_1 = \cup \sigma_j$ where $\sigma_j = t_j$ for $t_j \in \{\hat{T}\}_1$, such that $[\hat{M}]_0 \sigma_j$;

ii. Let $\ell_1 = -(\varphi' y_0) + \Sigma B^+ \sigma_j: \sigma_j \in \{\hat{T}\}_1$;

iii. Optionally, update the internal variables $[\hat{M}]_0 = [\hat{M}]_0 - \varphi' y_0 + \Sigma B^- \sigma_j: \sigma_j \in \{\hat{T}\}_1$ and $[\hat{M}]_k = [\hat{M}]_k - \varphi' y_0 + \Sigma B^+ \sigma_j: \sigma_j \in \{\hat{T}\}_1$;

(iteration) For $k > 1$:

i. Let $y_k$ be the $k$-th system output and let $\Delta y_k = y_k - y_{k-1}$. Let $\{\hat{T}\}_k = \{t_j \in \hat{T}: \varphi_B(t_j) = \Delta y_k\}$;

ii. If $|\{\hat{T}\}_k| = 1$ then $\{\hat{\sigma}\}_k = \tau t_j$, for $\tau = \sigma t_j \in \{\hat{\sigma}\}_{k-1}$ and $t_j \in \{\hat{T}\}_k$;

iii. Otherwise, if $|\{\hat{T}\}_k| > 1$ let $\{\hat{\sigma}\}_k = \cup \tau t_j$ for $\tau = \sigma t_j \in \{\hat{\sigma}\}_{k-1}$ and $t_j \in \{\hat{T}\}_k$, such that $[\hat{M}]_{k-1} t_j$;

iv. Let $\{\hat{\sigma}\}_k = \{\hat{\sigma}\}_{k-1} \setminus \{\hat{\sigma}\}_k$, i.e., the set of those elements in $\{\hat{\sigma}\}_{k-1}$ not used in $\{\hat{\sigma}\}_k$;

v. Let $\ell_k^+ = \Sigma B^+ \bar{\ell}_i$ and $\ell_k^- = \Sigma B^- \bar{\ell}_i$ for $\sigma t_i \in \{\hat{\sigma}\}_k$;

vi. Let $\ell_k^+ = \Sigma B^- \bar{\ell}_j$ for $t_j \sigma \in \{\hat{\sigma}\}_k$;

vii. Let $\ell_k = \ell_k^+ - (\ell_k^- + \bar{\ell}_k)$ be the $k$-th correction element;

viii. Optionally, update $[\hat{M}]_0 = [\hat{M}]_0 - \ell_k(y_k)$ and $[\hat{M}]_k = [\hat{M}]_k + \ell_k$.

Clearly, the element $\{\hat{\sigma}\}_k$ track the transition sequences that the system has been probably fired at every $k > 0$. Notice that in the
construction of \(\{\sigma\}_k\) some sequences in \(\{\sigma\}_{k-1}\) may be discarded. Thus, if \(\{\sigma\}_k\) is a singleton (point \(ii\) in the iteration stage), then the transitions sequence is completely known. Besides, three parts comprise the corrective term, in such a way that the observer estimations approach to the system state, and at the same time, the initial state could be progressively updated. The last part is not required; however, it is included since its computation is direct from the definition of the corrective term.

Let \(V_k := |\{\sigma\}_k| - 1\) be the functional for the Lyapunov stability analysis. That is, the number of sequences in \(\{\sigma\}_k\) minus one. Thus, if \(|\{\sigma\}_k| = 1\) then \(V_k = 0\). This make sense because under such a condition, the error in the sequence estimation is zero. Intuitively, \(\{\sigma\}_k\) tends to be a singleton as the events occurrence increase. However, it could be the case that the size of \(\{\sigma\}_k\) does not decrease at each transition firing. Moreover, if the net is safe, the knowledge of the transition sequence leads to a zero error in the estimated state, as established in [1].

Additionally, as highlighted, if the system state is known from the beginning, then the tracking sequence is unique. Thus, trivially \(|\{\sigma\}_0| = 1\) and \(V_0 = |\{\sigma\}_0| - 1 = 0\), as expected. Furthermore, \(\rho(\emptyset, \{\sigma\}_0)\) is a closed invariant set of the error system \((E, e_0)\) represented by (5), as previously analysed. Thus, the next theorem shows that \(V_k\) fulfils the Lyapunov stability requirements.

**Theorem 6.** Let \((B, M_0, \varphi)\) be a well-formed OSS, where \(M_0\) is probably unknown. Let \((B, \bar{M}_0, \varphi)\) be the observer defined by (4), where the initial output is \(y_0\), and the corresponding initial estimation is \(\bar{M}_0 = \varphi' y_0\). Let \((E, e_0)\) be the error system in (5), where the initial error is \(e_0 = (\bar{M}_0 - M_0)\). Let \(r = \|e_0\| - 1 = \rho(\bar{M}_0, M_0) - 1\) and let \(\mathcal{M} = \{\emptyset\}\). Then \(V_k\) satisfies the Theorem 5 in the vicinity \(S(\mathcal{M}, r)\).

**Proof:**

i. Without loose of generality, let \(c_1 > 0\) be sufficiently small such that there exists \(e_k \in S(\mathcal{M}, r)\) with \(e_k \in S(\mathcal{M}, r)\) where \(V_k > c_2\) with \(c_2 \geq c_1 + 1\). Then, it must be the case that \(\rho(e_k, \mathcal{M}) > c_1\). On the contrary, suppose that \(\rho(e_k, \mathcal{M}) \leq c_1\). Then, \(c_1 \leq c_2 - 1\). Moreover, as \(\mathcal{M} = \{\emptyset\}\), then \(\rho(e_k, \mathcal{M}) = \|e_k\|\). Thus, \(\|e_k\| \leq c_2 - 1\), or \(\|e_k\| + 1 \leq c_2\). But \(\|e_k\| = \|\bar{M}_k - M_k\|\) for every \(k \geq 0\).

ii. Without loose of generality, let \(c_4 > 0\) be sufficiently small such that there exists \(e_k \in S(\mathcal{M}, r)\) with \(e_k \in S(\mathcal{M}, r)\) where \(0 < c_3 \leq (c_4 - 1)\). Then, it must be the case that \(V_k \leq c_4\). On the contrary, suppose that \(V_k > c_4\). Since \(V_k = |\{\sigma\}_k| - 1\) and \(c_4 \geq c_3 + 1\), then \(|\{\sigma\}_k| > c_3 + 2\). Moreover, since \(\ell_k^+\) is computed from every element in \(\{\sigma\}_k\), then \(\|\ell_k^+\| > c_3 + 2\). Thus, \(c_3 + 2\), since from Definition 4, \(\bar{M}_{k-1} \geq (\ell_k^+ + \bar{e}_k)\) for every \(k \geq 0\). But, \(\ell_k^+ + \bar{M}_{k-1} - \ell_k^+ + \bar{e}_k = M_k\).

In order to show that the functional $V_k$ is a non-increasing function of $k$, notice that by point (ii) in the iteration stage of Definition 4, if $V_k = 0$ for some $k$, i.e. $\|\{\sigma\}_{k-1}\| = 1$, then $V_{k+1} = 0$ for every $l > 0$. Thus, by contradiction, consider without loss of generality that $V_{k+1} > V_k > 0$, for some $k$. Then $\|\{\sigma\}_{k+1}\| > \|\{\sigma\}_k\| \geq 2$. But, by Definition 4, $\{\sigma\}_{k+1} = \bigcup \tau t_j$ for $\tau = \sigma t_k \in \{\sigma\}_k$ and $t_j \in \{\tau\}_{k+1}$, such that $[\hat{M}_j] t_j$. Since, by contradiction, it holds that $\|\{\sigma\}_{k+1}\| \geq 2$, then it must exist at least a pair of transition sequences, say $\beta t_u \beta t_v \in \{\sigma\}_{k+1}$, constructed from the single sequence $\beta = \sigma t_k \in \{\sigma\}_k$, such that $t_u, t_v \in \{\tau\}_{k+1}$, i.e. $\varphi_B(t_u) = \varphi_B(t_v) = \Delta y_k$. On one hand, if the net is non-safe, then $\varphi_B(t_i) = \varphi_B(t_j)$ is a contradiction to the ED of the OSS. On the other hand, if the net is safe, then $\{t_u\}_k \subset \hat{M}_k \cap t_v$. It implies that $\sigma t_u = \sigma t_v$, which is a contradiction to the SD. Thus, necessarily $V_{k+1} \leq V_k$, for every $k > 0$. iv. In order to show that $V_k \to 0$ as $k \to \infty$, let $V_k > 0$ for some $k$. Then $\|\{\sigma\}_k\| > 1$, which implies that there exist at least two sequences in $\{\sigma\}_k$, say $\sigma_1, \sigma_2 \in \{\sigma\}_k$, where $\sigma_1, \sigma_2 \in L(B, M_0)$ and $\varphi(\sigma_1) = \varphi(\sigma_2)$, which on the one hand, it contradicts the ED if the net is non-safe. On the other hand, if the net is safe, then it should exist an integer $l < \infty$, for which $\|\{\sigma\}_{k+l}\| > \|\{\sigma\}_k\|$. On the contrary, suppose that $\|\{\sigma\}_{k+l}\| \geq \|\{\sigma\}_k\|$ for every $l > 0$. Thus, it must exist at least two transition sequences that could be constructed from $\sigma_1$ and $\sigma_2$, say $\sigma_1 = \sigma_1 t_i t_{i+1} t_{i+2} \ldots t_{i+l-2} t_{i+l-1} t_{i+l}$, ..., $\sigma_2 = \sigma_2 t_i t_{i+1} t_{i+2} \ldots t_{i+l-2} t_{i+l-1} t_{i+l}$ such that $\sigma_1 \sigma_2 \in \{\sigma\}_{k+l}$ for every $0 < i \leq l$, where $\varphi(\sigma_1) = \varphi(\sigma_2)$. Now, let $\{t_i, t_i\}, \{t_{i+1}, t_{i+1}\}, \{t_{i+2}, t_{i+2}\}, \{t_{i+l-1}, t_{i+l-1}\}, \{t_{i+l}, t_{i+l}\}$, ..., be the sequence formed by pairing the corresponding transitions in $\sigma_1$ and $\sigma_2$, that follows from $\sigma_1$ and $\sigma_2$, respectively. Then, it holds that $t_i \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \ldots \sigma_1 \sigma_2 \sigma_1 \sigma_2 \ldots$, $t_i \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \ldots$, $t_i \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \ldots$, etc. Since $\varphi(\sigma_1) = \varphi(\sigma_2)$, then in $E_B$, at least, it holds that $\{t_{i+l}, t_{i+l}\} \in E_B(i, i')$, $\{t_{i+l+1}, t_{i+l+1}\} \in E_B(i + 1, i' + 1)$, ..., $\{t_{i+l-1}, t_{i+l-1}\} \in E_B(i + l - 2, i' + l - 2)$, $\{t_{i+l}, t_{i+l}\} \in E_B(i + l - 1, i' + l - 1)$, ..., etc. As the number of transitions in the net is finite, at least one element in the previous list of entries in $E_B$ must appear twice. Let say that $\{t_i, t_i\} \in E_B(i + l, i' + l)$. This conforms a cyclic dependency implying that $\Delta E_{\sigma_1} = 0$, which directly contradicts the SD of the OSS. Thus, $V_k \to 0$ as $k \to \infty$. v. Notice that the requirement $c_2 \geq c_1 + 1$ with $c_1 > 0$ implies that $r = \|e_0\| - 1 \geq 2$. Additionally, the requirement $0 < c_3 \leq (c_4 - 1)$ with $c_4 > 0$ implies that $c_4 \geq 2$. Thus, it must exist at least two transition sequences that the observer has to track. However, this observer scheme can also be useful in the case of a non-safe net, where $\|\{\sigma\}_k\| = 1, \forall k \geq 0$, as highlighted in the following section.
**Illustrative Example**

Consider again the OSS in Figure 1, which shows a system with five processes represented by the loops in the model. The places and transitions include superscripts, which represent the loop to which they belong. For example, $t_3^3$ is the third transition of the first loop. Similarly, $p_1^4$ is the first place of the fourth loop. Some sensors include a slash. The sensor before the slash is used when the net is considered safe, and that one after the slash is used when the net is considered non-safe. Thus, for $p_3^3$, D is used when the net is safe, and $A_1$ when the net is non-safe. Also, some places include grey-filled tokens used as initial markings when the net is non-safe.

The framework in [2] has been used for the simulation of the observer scheme in Figure 2. The corrective term $\ell_k$ was programmed as in Definition 4. The graphs in Figure 3 correspond to different simulation processes with safe markings. In this plot, the error remains at four up to $k = 6$, for the lines of Error 1, Error 4, Error 5 and Error 6. These errors correspond to the initial conditions $M_0 \{p_2^2\}, M_0 \{p_2^2\}, M_0 \{p_2^2\}$ and $M_0 \{p_2^2\}$, respectively. The line of Error 1 corresponds to the observability constant of the net, i.e., it is the longest convergence error. Indeed, this error reaches zero at $k = 9$, which agrees with the sequences $\varphi(\sigma_1) = \varphi(\sigma_3)$ where $\sigma_1 = t_3^1 t_4^1 t_7^1 t_8^1 t_9^1$ and $\sigma_3 = t_3^2 t_4^2 t_7^2 t_8^2 t_9^2$. These sequences are easily constructed by chaining consecutive entries in the non-empty entries in Table 1. That is, $\sigma_1$ and $\sigma_3$, correspond to $E_{G}^{s}(t_1^1, t_2^1), E_{G}^{s}(t_1^2, t_2^2), E_{G}^{s}(t_3^1, t_4^1), E_{G}^{s}(t_3^2, t_4^2), E_{G}^{s}(t_5^1, t_6^1), E_{G}^{s}(t_5^2, t_6^2)$, and $E_{G}^{s}(t_7^1, t_8^1)$, which are non-empty entries in upper left section of Table 1.

Indeed, the table $E_{G}^{s}$ provides information about the loops of a net and their relation to other loops. Thus, for example, the longest sequence of non-empty entries in $E_{G}^{s}$ corresponds to the observability convergence constant of the sequences observer. Moreover, if some sequence is too long that it is unacceptable for a specific application, a detailed examination of $E_{G}^{s}$ may provide suitable information about the best place to add a new sensor. However, the optimal sensor placement for an OSS is out of the scope of this work.

In the same plot, Error 6 abruptly decreases from 4 to 0 at $k = 6$. Notice that this error corresponds to the initial condition $M_0 \{p_2^4\}$. Finally, Errors 4 and 5 decreases from 2 to 0 at $k = 8$, and from 3 to 0 at $k = 7$, respectively. The region of the asymptotic stability of the net is any safe marking, i.e., $M_\alpha = \{M \in \mathbb{N}^4 : 0 \leq \|M\| \leq 1\}$. It is not hard to conclude that the size of $M_\alpha$ is 41.

The graphs in Figure 4 correspond to simulations of the same net, but with non-safe markings. Errors 1 and 2 are from two consecutive simulation processes of the initial marking $M_0 \{p_1^1, p_2^1, p_4^2, p_5^2, p_7^1, p_8^1, p_9^1, p_8^2\}$. The difference shown in the evolution of the errors is due to the firing of the transitions in the system block in Figure 2, when all the transitions are allowed to be fired, i.e. when $u_k = \bar{1}, \forall k \geq 0$. See [2] for details about the firing of the transitions and other configuration options of the simulation framework.

Errors 3 and 4 correspond to two simulation processes with the initial condition $M_0 \{2p_1^1, 6p_2^1, 6p_4^2, 2p_5^2, 2p_7^1, 2p_8^1, 2p_9^1, 2p_8^2\}$. In this case, the initial error is higher than the former due to a greater number of tokens. The error drops as the number of events increases. Notice that, in spite of the transition sequence is known from $k = 1$, the error oscillates, i.e. it increases and decreases randomly. Depending on suitable transition firings in the net, the error may approach to zero. Moreover, it is possible to show that for non-safe markings, the error in (5) is stable. However, a further analysis of this topic is out of the scope of this work.

**5. Conclusions**

This paper presents the design of observers for DES, which are modelled with OPN. The focus is on a subclass of models called S-Nets. An scheme for tracking the transition sequences executed by the system is proposed, where the feedback for the observer is the system output. The observer is provided with a corrective term $\ell_k$ to update its estimations. This corrective element is tracking for every possible transition sequences executed by the system. Thus, the observer error decreases as the number of transition sequences tracked by $\ell_k$ decreases. A Lyapunov stability criterion for characterizing the observer scheme has been used. It shows that the technique developed in this work produces asymptotically stable observers for the case of a safe OSS such that its sequence-detectability property is verified. Thus, in this case, the initial and current state of the system could be perfectly reconstructed with the proposed scheme. When the net is non-safe, the herein introduced scheme produces estimations such that, in some cases, the error may
oscillate. An application example illustrates the advantages of the developed techniques.

REFERENCES