A New Method for Recognizing the Input Congestion in Data Envelopment Analysis

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Abstract: One of the most attractive issues in Data Envelopment Analysis (DEA) is the evaluation of congestion of Decision Making Units (DMUs), which occurs if the decreases in certain inputs lead to increases in certain outputs without worsening any other input or output and equivalently, if the increases in certain inputs lead to decreases in certain outputs without improving any other input or output. Most of the existing approaches for detecting the congestion of DMUs in the literature on DEA employ the traditional definition of congestion and consider the situation in which the inputs and outputs can only change proportionally. This study proposes a method for recognizing the congestion of each input of a selected unit under evaluation for the case when the input and output of the respective unit can change disproportionally. The potential application of the proposed method is illustrated based on a dataset related to 16 institutes of basic research of the Chinese Academy of Sciences (CAS) for the year 2010, which was also reported in the literature.

Keywords: Data Envelopment Analysis, Efficiency, Inefficiency, Congestion.

1. Introduction

DEA is a powerful mathematical technique based on linear programming for measuring the performance efficiency of organizational units known as decision-making units (DMUs), which has been widely used in the recent years (Charnes et al., 1962; Banker et al., 1984; Färe et al., 1985). Congestion is often evidenced in the real-world problems when the decreases in some inputs cause the maximum possible increase in some outputs without worsening any other input or output (Cooper et al., 2004). Färe et al. (1985) suggested a model based on the strong and weak disposability assumption related to inputs and used a radial measure for estimating the congestion of DMUs. Many scholars have been attracted to measuring the congestion of units. For example, Cooper et al. (1996) proposed an approach to measure congestion of DMUs.

Wei & Yan (2004) introduced an alternative production possibility set and proposed Data Envelopment Analysis models for determining the effect of the congestion of Decision-Making Units. Jahanshahloo & Khodabakhshi (2004) developed a model based on the relaxed combinations of inputs to measure the amount of congestion in inputs in the textile industry of China. Noura et al. (2010) also presented a method for measuring the congestion of DMUs. Kao (2010) used the model of Wei and Yan (2004) to measure the effect of congestion on Taiwan forests. Khoveyni et al. (2019) proposed an integer-valued slack-based DEA approach for recognizing the right-hand and left-hand congestion of the DMUs with both negative and/or non-negative continuous and integer data. Shadab et al. (2020) developed an algorithm based on the connection between the definition of congestion and the anchor points for recognizing the congestion of units. Ren et al. (2021) reviewed the development of the concept of congestion in the framework of DEA. Salehi et al. (2020) used the non-radial movement in the PPS and presented a new method for identifying the congestion of units based on the definition of congestion.

There are two basic concepts, strong and weak congestion, in the literature on DEA. A unit has strong congestion if decreases in all inputs result in increases in all outputs of DMU. A unit has weak congestion if decreases in some inputs of the DMU result in increases in some outputs of that unit. Yang (2015) pointed out that these definitions have some main drawbacks. For example, in the case of strong or weak congestion, the precise direction, along which the congestion occurs, cannot be recognized. To tackle this drawback, Yang (2015), Khezri et al (2021) considered the definition of the directional congestion along certain input and output directions. This paper concentrates on offering a method to recognize the congestion of units so that the DM may decide about increasing or decreasing the size of a particular DMU. Also, knowing the amount of congestion of each input,

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which is calculated independently of the other inputs, can provide useful information to the DM. However, most of the existing methods in the DEA literature consider the congestion of all inputs, simultaneously, and they cannot determine the congestion of each input independently of the other inputs. To tackle this issue, this study proposes a method for recognizing the congestion of each input of a selected unit under evaluation for the case when the input and output of the respective unit can change disproportionally. The main contribution of this paper lies in the fact that the proposed approach can measure the congestion of each input, separately, by applying one linear programming model and then it can determine the congestion status of each unit without solving any model.

The remainder of this paper is organized as follows. Section 2 reviews the key developments in the literature on DEA to recognize the congestion of units. Section 3 presents the proposed method for the congestion of units. Section 4 illustrates the proposed method by introducing a numerical example. Finally, Section 5 concludes this paper and proposes further possible research directions.

2. Literature Review

Suppose that there are decision making units, $DMU_j, j = 1,..,n$, and each DMU consumes m inputs to generate s different outputs. Suppose that, $x_{ij}, i = 1,..,m$ and $y_{rj}, r = 1,...,s$,denote the i^{th} input and r^{th} output for DMU_j. It is assumed that all input and output values are non-negative and at least one of them is non-zero. Let $DMU_o = (x_o, y_o)$ be the unit under evaluation. Banker et al. (1984) defined the following production possibility set (PPS), T_v :

$$Tv = \left\{ (x, y) \left| \begin{array}{l} \exists \lambda j \geq 0, j = 1,...,n, x_j \geq \sum_{j=1}^{n} \lambda_j x_{ij}, i = 1,...,m, \\ y_r \leq \sum_{j=1}^{n} \lambda_j y_{rj}, r = 1,...,s, \sum_{j=1}^{n} \lambda_j = 1 \end{array} \right. \right\} \quad (1)$$

In equation (1), $\lambda = (\lambda_1,...,\lambda_n)$ is a non-negative vector of variables. They proposed the output-oriented model to assess the efficiency of DMU_o as follows:

$$\begin{aligned} &\zeta^* = \max \rho + \varepsilon(\sum_{i=1}^{m} s_i^- + \sum_{i=1}^{m} s_r^+) \\ &s.t. \\ &\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1,...,m \\ &\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = \rho y_{ro} \quad r = 1,...,s \\ &\sum_{j=1}^{n} \lambda_j = 1 \\ &\lambda_j \geq 0 \quad j = 1,...,n \\ &s_i^- \geq 0 \quad i = 1,...,m \\ &s_r^+ \geq 0 \quad r = 1,...,s \end{aligned} \quad (2)$$

where ε is a non-Archimedean constant and variables s_i^- and s_r^+ are slacks, ρ is a real variable and ζ^* represents the optimal objective values of model (2).

Definition 1. Suppose that $(\rho^*, s^{-*}, s^{+*}, \lambda^*)$ is an optimal solution for model (2). DMU_o is considered a technically efficient unit if $\rho^* = 1$. Moreover, DMU_o is a strongly efficient unit if $\zeta^* = 1$.

The classical definition of congestion was introduced by Cooper et al. (2001) and Brockett et al. (2004) as follows:

Definition 2. $DMU_o = (x_o, y_o)$ has congestion, if the decreases in some inputs result in the increases in some outputs without worsening other inputs or outputs and also, if the increases in some inputs result in decreases in some outputs without improving other inputs or outputs.

There are several methods in the literature on DEA for measuring the congestion of units. In this section, the method of Tone & Sahoo (2004) and the method of Khoveyni et al. (2013) are analyzed.

2.1 The Method of Tone & Sahoo

Tone & Sahoo (2004) introduced a new PPS in which all postulates for building T_v are accepted except the strong disposability assumption. They defined the weak disposal assumption as follows:

Definition 3. The weak disposability assumption is held in the PPS, T, if for each $\widehat{DMU} = (\hat{x}, \hat{y}) \in T$ and $DMU = (x, y) \in T$ where $x = \hat{x}$ and $y \leq \hat{y}$ it is concluded that $(\hat{x}, \hat{y}) \in T$.

Tone & Sahoo (2004) defined PPS, P_{convex} as follows:

$$P_{convex} = \left\{ (x, y) \left| \begin{array}{l} \exists \lambda j \geq 0, j = 1,...,n, x_j = \sum_{j=1}^{n} \lambda_j x_{ij}, i = 1,...,m, \\ y_r \leq \sum_{j=1}^{n} \lambda_j y_{rj}, r = 1,...,s, \sum_{j=1}^{n} \lambda_j = 1 \end{array} \right. \right\} \quad (3)$$

They proposed model (4) to evaluate the efficiency score of DMU_o with respect to P_{convex}:

$$\begin{aligned} & \max \phi + \varepsilon (\sum_{i=1}^{m} s_r^{+}) \\ & s.t. \\ & \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} \quad i = 1,...,m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^{+} = \phi y_{ro} \quad r = 1,...,s \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j \geq 0 \quad j = 1,...,n \\ & s_r^{+} \geq 0 \quad r = 1,...,s \end{aligned} \quad (4)$$

where ϕ is a real variable. The target unit for DMU_o located on the strongly efficient frontier of P_{convex} can be expressed as follows:

$$\begin{aligned} & \widehat{x_{io}} = x_{io} \quad i = 1,...,m \\ & \widehat{y_{ro}} = \phi^* y_{ro} + s_r^{+} \quad r = 1,...,s \end{aligned} \quad (5)$$

Definition 4. Suppose that $DMU_o = (x_o, y_o) \in P_{convex}$. If $\phi^* = 1$ then DMU_o is strongly efficient with respect to P_{convex}.

Definition 5. Suppose that $DMU_o = (x_o, y_o) \in P_{convex}$ is strongly efficient with respect to P_{convex}. If there exists $(\widehat{x}_o, \widehat{y}_o) \in P_{convex}$ such that $\widehat{x}_o = \alpha x_o$ for $0 < \alpha < 1$ (is a factor that allows us to decrease inputs) and $\widehat{y}_o \geq \beta y_o$ (β is a factor that allows us to increase outputs) for $\beta > 1$ then DMU_o has strong congestion.

Definition 6. Suppose that $DMU_o = (x_o, y_o) \in P_{convex}$ is strongly efficient with respect to P_{convex}. If there is an activity in P_{convex} that uses less resources in one or more inputs for making more products in one or more outputs, then DMU_o is weakly congested.

Tone & Sahoo (2004) assumed that DMU_o is a strongly efficient unit with respect to P_{convex}. If not, that unit is projected onto the strongly efficient frontier of P_{convex} and then the following method is applied to determine the strongly and weakly congested units:

Step 1: Solve model (2). Therefore, the following is obtained:

a. If $\rho^* = 1, s^{-*} = 0, s^{+*} = 0$ then $DMU_o = (x_o, y_o)$ is BCC-efficient and not congested.
b. If $\rho^* = 1, s^{-*} \neq 0, s^{+*} = 0$ then $DMU_o = (x_o, y_o)$ is BCC-inefficient.
c. If $\rho^* = 1, s^{+*} \neq 0$ or $\rho^* > 1$ then $DMU_o = (x_o, y_o)$ displays congestion. Go to step 2.

Step 2: Solve model (6):

$$\begin{aligned} & \max \ u_o \\ & s.t. \\ & \sum_{r=1}^{s} u_r y_{ro} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o \leq 0 \quad j = 1,...,n \\ & \sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} + u_o = 0 \\ & u_r \geq 0 \quad r = 1,...,s \\ & v_i \ and \ u_o \ free \quad i = 1,...,m \end{aligned} \quad (6)$$

where u_r and v_i are the weights of the output r and the input i, respectively and u_o is scalar, it can be positive, negative or zero. Suppose that $\bar{u}$ is the optimal value of model (6) and $\psi = 1 + \bar{u}$. If $\psi < 0$ then DMU_o is strongly congested. If $\psi \geq 0$ then DMU_o is weakly congested.

2.2 Method of Khoveyni et al. (2013)

This section reviews a method presented by Khoveyni et al. (2013), to estimate the congestion status of DMUs.

Khoveyni et al. (2013) considered P_{convex} in (3), and they supposed that $DMU_o, o \in \{1,...,n\}$, is on the strongly efficient frontier of the P_{convex}. They defined two non-empty sets, Γ_o^I and Γ_o^O as follows:

$$\Gamma_o^I = \{i \,|x_{io} > 0\} \neq \varnothing,$$
$$\Gamma_o^O = \{i \,|y_{ro} > 0\} \neq \varnothing$$

They presented the following model for recognizing the congestion of units:

$$\begin{aligned}
& z_{1o}^{*} = \max \frac{1}{card(\Gamma_0^{\,o})}\Big(\sum_{r\in\Gamma_0^{\,o}} \frac{t_r^{+}}{y_{ro}}\Big) \\
& s.t. \\
& \sum_{j=1}^{n} \lambda_j x_{ij} + t_i^{-} = x_{io} \quad i=1,\dots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - t_r^{+} = y_{ro} \quad r=1,\dots,s \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \ge 0 \quad j=1,\dots,n \\
& t_r^{+} \ge \varepsilon \quad r=1,\dots,s \\
& t_i^{-} \ge 0 \quad i=1,\dots,m
\end{aligned} \tag{7}$$

In model (7) t_i^{-} and t_r^{+} are variables and card (Γ_o^O) in relation to the symbol in equation (7) represents the number of members of the Γ_o^O set. If model (7) is feasible, then DMU_o is congested and the following model should be solved to determine the congestion status of that unit:

$$\begin{aligned}
& z_{2o}^{*} = \max \frac{1}{card(\Gamma_0^{\,1})}\Big(\sum_{i\in\Gamma_0^{\,1}} \frac{t_i^{-}}{x_{io}}\Big) \\
& s.t. \\
& \sum_{j=1}^{n} \lambda_j x_{ij} + t_i^{-} = x_{io} \quad i=1,\dots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} = y_{ro} + t_r^{+} \quad r=1,\dots,s \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \ge 0 \quad j=1,\dots,n \\
& t_i^{-} \ge \varepsilon \quad i=1,\dots,m
\end{aligned} \tag{8}$$

where z_{2o}^{*} shows the optimality of model (7). By solving model (8), the following cases are obtained:

a. If model (8) is feasible, then DMU_o is strongly congested.
b. If model (8) is infeasible, then DMU_o is weakly congested.
c. If model (7) is infeasible, then DMU_o does not have strong congestion. In this case, model (9) is solved to determine whether DMU_o is weakly congested or not:

$$\begin{aligned}
& z_{o}^{*} = \max \frac{1}{card(\Gamma_0^{\,o})}\Big(\sum_{r\in\Gamma_0^{\,o}} \frac{t_r^{+}}{y_{ro}}\Big) \\
& s.t. \\
& \sum_{j=1}^{n} \lambda_j x_{ij} + t_i^{-} = x_{io} \quad i=1,\dots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - t_r^{+} = y_{ro} \quad r=1,\dots,s \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \ge 0 \quad j=1,\dots,n \\
& t_r^{+} \ge 0 \quad r=1,\dots,s \\
& t_i^{-} \ge 0 \quad i=1,\dots,m
\end{aligned} \tag{9}$$

The following cases can be obtained:

Case 1: If $Z_o^{*} > 0$, then DMU_o has weak congestion.

Case 2: If $Z_o^{*} = 0$ then DMU_o does not have weak congestion.

3. The Proposed Model

This section proposes a new method for estimating the congestion of units which is based on considering the congestion of each input, separately. Most of the existing methods in the literature on DEA for measuring congestion consider all the inputs, simultaneously. In other words, they are not able to obtain the actual congestion of each input. In many real-world problems, the DM needs to find the actual congestion of a special input in order to make a decision on that input. To tackle this drawback, this paper presents an approach which is based on the non-redial evaluation of DMUs and considers the congestion of each input, separately.

Further on, the proposed method is presented in detail. The first step of -the proposed method determines the efficiency status of units with respect to P_{convex}. If a DMU is efficient with respect to P_{convex} then the congestion status of the DMU is considered and if the unit is inefficient, then the target of DMU is considered to the congestion of that unit. In the second step of the proposed method, model (10) is formulated to consider the congestion of the k^{th} input, for all $k = 1,\dots,m$:

$$
\begin{aligned}
& z_k^* = \max z_k \\
& s.t. \\
& \sum_{j=1}^{n} \lambda_j x_{ij} \le (1-\alpha_k) x_{ko} \quad i = 1,\dots,m \\
& \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} \quad i \ne k \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge (1+\beta_r^k) y_{ro} \quad r = 1,\dots,s \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& z_k \le \alpha_k \\
& z_k \le \sum_{r=1}^{s} \beta_r^k \\
& \lambda_j \ge 0 \quad j = 1,\dots,n \\
& \beta_r^k \ge 0 \quad r = 1,\dots,s \\
& 0 \le \alpha_k \le 1
\end{aligned} \tag{10}
$$

In model (10), z_k^* is the optimal solution of the objective function and α_k, β_r^k are variables. Suppose that $(\alpha_k^*, \beta_1^{*k}, \dots, \beta_s^{*k}, z_k^*)$ is an optimal solution for model (10), then $z_k^* = 0$ if and only if $\alpha_k^* = 0$ or $\sum_{r=1}^{s} \beta_r^{*k} = 0$ and $z_k^* > 0$ if and only if $\alpha_k^* > 0$ and $\sum_{r=1}^{s} \beta_r^{*k} > 0$.

Theorem 1. If $z_k^* = 0$ then the k^{th} input of DMU_o has no congestion.

Proof. Suppose that $z_k^* = 0$. This means that $z_k = 0$ for all $(\alpha_k, \beta_1^k, \dots, \beta_s^k, z_k) \in R^{s+2}$. Regarding the constraints of model (10), the following is obtained:

$$
z_k \le \alpha_k
$$

$$
z_k \le \sum_{r=1}^{s} \beta_r
$$

In other words, $z = \min(\alpha_k, \sum_{r=1}^{s} \beta_r)$, and so, at least one of the values $\alpha_k = 0$ or $\sum_{r=1}^{s} \beta_r = 0$.

This means that the k^{th} input of DMU_o cannot decrease or the decreases in the k^{th} input of DMU_o cannot lead to increases in outputs, hence, DMU_o has no congestion in k^{th} input of DMU_o.

It should be noted that model (10) should be solved times for the unit under evaluation and it determines the congestion of each input of that unit. Now, suppose that $(\alpha_k^*, \beta_1^{*k}, \dots, \beta_s^{*k}, z_k^*)$ is an optimal solution for model (10) in the context of evaluating the k^{th} input of DMU_o. In the next step of the proposed approach, the following values are defined:

$$
\begin{aligned}
\alpha_o^L &= \min \alpha_k^* \quad ,1 \le k \le m \\
\alpha_o^U &= \max \alpha_k^* \quad ,1 \le k \le m
\end{aligned} \tag{11}
$$

Also, let us define:

$$
\begin{aligned}
\beta_o^U &= \max\{\max \beta_r^{*k}\} \ 1 \le k \le m \quad ,1 \le r \le s \\
\beta_o^L &= \min\{\min \beta_r^{*k}\} \ 1 \le k \le m \quad ,1 \le r \le s
\end{aligned} \tag{12}
$$

The following theorem determines the congestion status of each unit by applying the relations (11) and (12).

Theorem 2. Suppose that $(\alpha_k^*, \beta_1^{*k}, \dots, \beta_s^{*k}, z_k^*)$ is an optimal solution for model (10), in the context of evaluating the k^{th} input of DMU_o:

i. If $\alpha_o^U = 0$ or $\beta_o^U = 0$, then DMU_o has no congestion.

ii. If $\alpha_o^U > 0$ and $\beta_o^U > 0$, then DMU_o has weak congestion.

iii. If $\alpha_o^L > 0$ and $\beta_o^L > 0$, DMU_o has strong congestion.

Proof. If $\alpha_o^U = 0$ then $\alpha_o^* = 0$, for all $k = 1,\dots,m$,and so, no input can decrease, hence, DMU_o has no congestion. Also, if $\beta_o^U = 0$, then $\beta_r^{*k} = 0$, for all $r = 1,\dots,s$, and $k = 1,\dots,m$,so, no output of DMU_o can increase, hence, DMU_o has no congestion. Therefore, in both cases of i), DMU_o has no congestion.

Now, suppose that $\alpha_o^U > 0$ and $\beta_o^U > 0$ therefore, there is $k \in \{1,\dots,m\}$, such that $\alpha_k^* > 0$ hence, the k^{th} input of DMU_o can decrease and also, there are $k' \in \{1,\dots,m\}$ and $r \in \{1,\dots,s\}$, such that $\beta_r^{*k'} > 0$ hence the r^{th} output of DMU_o can increase. This means that the decrease of the k^{th} input would result in the increase of the r^{th} output of DMU_o, and so, this unit has weak congestion.

Finally, suppose that $\alpha_o^L > 0$ and $\beta_o^L > 0$, hence, for all and for all $r \in \{1,\dots,s\}$, $\beta_r^{*i} = 0$. This means that all inputs of DMU_o can decrease and all outputs of that unit can increase. Hence, DMU_o has strong congestion. Model (10) can have alternative optimal solutions. As it is stated below, the results of the proposed method are still valid even in the presence of alternative optimal solutions. If $z_k^* = 0$ then $\alpha_k^* > 0$ or $\sum_{r=1}^{s} \beta_r^{*k} > 0$. Alternative optimal solutions have no effect on conclusions. Now, suppose that $(\alpha_k^{*}, \beta_1^{*k}, ..., \beta_s^{*k}, z_k^{*})$ is an optimal solution for model (10), if $z_k^{*}=0$ then $\alpha_k^{*}=0$ or $\sum_{r=1}^{s}\beta_r^{*k}=0$:

a. If $\alpha_k^{*}=0$ then $\sum_{r=1}^{s}\beta_r^{*k}$ can be equal to zero or have a positive value. (Alternative optimal solutions)

b. If $\sum_{r=1}^{s}\beta_r^{*k}=0$ then α_k^{*} can be equal to zero or have a positive value (Alternative optimal solutions)

It can be shown that both states of a and b have no effect on the results of the proposed model.

Suppose that $\alpha_o^{U}=0$, in this case, it can be written according to definition (11): α_o^{U}, $\forall k, \alpha_k^{*}=0 \rightarrow z_k^{*}=0$. According to theorem (1), DMU_o has no congestion. Suppose that, $\beta_o^{U}=0$. In this case, it can be written β_o^{U} according to definition (12):

$$\forall k, r, \beta_{ro}^{*}=0 \rightarrow \forall k, \sum_{r=1}^{s}\beta_r^{*k}=0, z_k^{*}=0,$$

According to theorem (1), DMU_o has no congestion.

So, if at least one of the two expressions α_k^{*} and $\sum_{r=1}^{s}\beta_r^{*k}$ is equal to zero, the value of the other expression will not affect this result. And this means alternative optimal solutions have no effect on conclusions.

The next section illustrates the proposed approach by using the dataset presented by Yang (2015).

4. Case Study

This example illustrates the results of applying the proposed approach to a dataset related to 16 institutes of basic research of the Chinese Academy of Sciences (CAS) for the year 2010, which was also reported in Yang (2015).

This dataset has 16 units with two inputs, namely the full-time equivalent of full-time research staff (x_1) and the amount of total income of each institute (x_2) to produce four outputs, namely the number of international papers indexed by Web of Science from Thomson Reuters (y_1) the number of high-quality papers published in top research journals (y_2) the total graduate student enrolment in 2009 (y_3) and the amount of external research funding from research contracts (y_4). The inputs and outputs data are included in Table 1.

Table 1. The inputs, outputs and congestion status for 16 institutes of basic research of CAS

	Input		Output				
	x_1	x_2	y_1	y_2	y_3	y_4	Congestion status
DMU_1	252	117.945	436	133	184	31.558	No
DMU_2	37	29.431	243	127	43	15.3041	No
DMU_3	240	101.425	164	70	89	33.8365	Weak
DMU_4	356	368.483	810	276	247	183.8434	No
DMU_5	310	195.862	200	55	111	12.9342	No
DMU_6	201	188.829	104	49	33	60.7366	No
DMU_7	157	131.301	113	49	45	72.5368	No
DMU_8	236	77.439	8	1	44	23.7015	Weak
DMU_9	805	396.905	371	118	89	216.9885	Strong
DMU_{10}	886	411.539	607	216	168	88.5561	Strong
DMU_{11}	623	221.428	314	49	89	45.3597	Weak
DMU_{12}	560	264.341	261	79	131	41.1156	Strong
DMU_{13}	1344	900.509	627	168	346	31.558	No
DMU_{14}	508	117.945	436	133	184	15.3041	No
DMU_{15}	380	29.431	243	127	43	33.8365	Weak
DMU_{16}	132	101.425	164	70	89	183.8434	No

The last column of Table 1 shows the congestion status of the selected units by applying the method of Tone & Sahoo (2004). As it can be seen, the units 1, 2, 4, 5, 6, 7, 13 and 14 are not congested units and the units 3, 8, 11 and 15 have weak congestion and the units 9, 10, 12 and 16 are strongly congested.

In the first step of the proposed approach, the efficient units were determined with respect to P_{convex} . For this purpose, model (4) was solved, which yielded that the units 1, 2, 4, 8, 9, 10, 11, 13, 14 and 15 were strongly efficient units. If a unit is strongly efficient with respect to P_{convex} , then that unit is considered in order to determine congestion, otherwise, its target, determined by equation (5), should be used in model (10) to measure the congestion of that DMU. So, in the next step of the proposed approach, the selected DMU or its target were employed for measuring the congestion of each input of the unit under evaluation. The results of model (10) for the first input and second input of each DMU are summarized in Table 2 and Table 3, respectively.

Table 2. The results of model (10) for input 1 of each selected DMU

DMU	α_1^*	β_1^{*1}	β_2^{*1}	β_3^{*1}	β_4^{*1}
DMU_1	0	0	0	0	0
DMU_2	0	0	0	0	0
DMU_3	0.4447	0.3603	0.4283	0	0.1257
DMU_4	0	0	0	0	0
DMU_5	0	0	0	0	0
DMU_6	0	0	0	0	0
DMU_7	0	0	0	0	0
DMU_8	0.6518	39.4105	147.0976	0.6338	0.6526
DMU_9	0.4661	1.0324	1.1654	1.7277	0.2142
DMU_{10}	0.5079	0.3100	0.2373	0.5179	1.4978
DMU_{11}	0.6507	0.7964	3.3138	0.7811	1.4415
DMU_{12}	0.5082	0.1379	0.0812	0.0257	0.2635
DMU_{13}	0	0	0	0	0
DMU_{14}	0	0	0	0	0
DMU_{15}	0.4831	0.1817	0.1779	0.1235	0
DMU_{16}	0.1153	0.0338	0.0412	0.0381	0.0022

Table 3. The results of model (10) for input 2 of each selected DMU

DMU	α_2^*	β_1^{*2}	β_2^{*2}	β_3^{*2}	β_4^{*2}
DMU_1	0	0	0	0	0
DMU_2	0	0	0	0	0
DMU_3	0.1785	0.3314	0.6226	0	0
DMU_4	0	0	0	0	0
DMU_5	0	0	0	0	0
DMU_6	0	0	0	0	0
DMU_7	0	0	0	0	0
DMU_8	0.4700	30.0147	126.5464	0.0690	0
DMU_9	0.2223	0.2164	0.2142	0.5916	0.3314
DMU_{10}	0.5459	0.3100	0.4931	0.1250	0.2464
DMU_{11}	0.6425	0.1388	2.8472	0	0
DMU_{12}	0.3019	0.0427	0.3321	0.0257	0.1452
DMU_{13}	0	0	0	0	0
DMU_{14}	0	0	0	0	0
DMU_{15}	0.1160	0.0246	0.1318	0	0
DMU_{16}	0.0199	0.0041	0.0137	0.0019	0.0337

Table 4 shows the value of congestion for each input, which was calculated as follows:

$$C_{1o} = x_{1o} - (1-\alpha_1^*)x_{1o}$$
$$C_{2o} = x_{2o} - (1-\alpha_2^*)x_{2o}$$

Table 4. The value of congestion for each input of the selected units

DMU	Congestion of input 1	Congestion of input 2
DMU_1	0	0
DMU_2	0	0
DMU_3	106.7280	18.1044
DMU_4	0	0
DMU_5	0	0
DMU_6	0	0
DMU_7	0	0
DMU_8	153.8248	36.3963
DMU_9	375.2105	88.2320
DMU_{10}	449.9994	224.6591
DMU_{11}	405.3861	142.2675
DMU_{12}	284.5920	79.8045
DMU_{13}	0	0
DMU_{14}	0	0
DMU_{15}	183.5780	18.7144
DMU_{16}	15.2196	1.6710

Finally, Table 5 shows the minimum and maximum value for the input decreases and output increases and the status of congestion of all DMUs. As it can be seen in Table 1 and Table 5, the status of congestion determined by the proposed method and by the method of Tone & Sahoo (2004) for all DMUs is the same.

Table 5. The minimum and maximum value for the input decreases and output increases and the status of congestion for all the selected DMUs

DMU	α_o^L	α_o^U	β_o^L	β_o^U	Congestion status
DMU_1	0	0	0	0	No
DMU_2	0	0	0	0	No
DMU_3	0.1785	0.4447	0	0.6226	Weak
DMU_4	0	0	0	0	No
DMU_5	0	0	0	0	No
DMU_6	0	0	0	0	No
DMU_7	0	0	0	0	No
DMU_8	0.470	0.651	0	147.09	Weak
DMU_9	0.2223	0.4661	0.2142	1.7277	Strong
DMU_{10}	0.5079	0.5459	0.1250	1.4978	Strong
DMU_{11}	0.6425	0.6507	0	3.3138	Weak
DMU_{12}	0.3019	0.5082	0.0257	0.3321	Strong
DMU_{13}	0	0	0	0	No
DMU_{14}	0	0	0	0	No
DMU_{15}	0.1160	0.4831	0	0.1817	Weak
DMU_{16}	0.0199	0.1153	0.0019	0.0412	Strong

The main advantage of the proposed method is that it can determine the congestion of each input of each DMU, separately. Hence, the decision maker can make a better decision about decreasing or increasing the size of each DMU.

Further on, the results obtained by employing the method of Khoveyni et al. (2013) are compared for this purpose, model (7) is solved for the selected units and the results are summarized in Table 6. Columns 2 and 3 show the optimal value of model (7) and the congestion status of the proposed units, respectively. As it can be seen in this table, the units 3, 8, 9, 10, 11, 12, 15 and 16 have congestion and the type of congestion for them is determined by solving model (8). Also, the congestion status of the units for which model (7) is infeasible, i.e. 1, 2, 4, 5, 6, 7, 13 and 14, is determined by solving model (9). Columns 4 and 5 show the optimal value of model (8) for DMUs 3, 8, 9, 10, 11, 12, 15,16 and the congestion status for these units, respectively. As it can be seen, model (8) for the units 9, 10, 12 and 16 is feasible, hence, these units are strongly congested. Also, model (8) is infeasible for units 3, 8, 11 and 15, hence these units are weakly congested. Finally, model (9) is solved for units 1, 2, 4, 5, 6, 7, 13 and 14 and the results are illustrated in columns 6 and 7. As it can be seen, the optimal value of model (9) for these units is equal to zero, so, the units 1, 2, 4, 5, 6, 7, 13 and 14 are not congested.

Table 6. The results for the method of Khoveyni et al. (2013)

DMU	Z_{1o}^*	Congestion status	Z_{2o}^*	Congestion status	Z_{3o}^*	Congestion status
DMU_1	Infeasible	-	-	-	0	No
DMU_2	Infeasible	-	-	-	0	No
DMU_3	0.8676	Yes	Infeasible	Weak	-	-
DMU_4	Infeasible	-	-	-	0	-
DMU_5	Infeasible	-	-	-	0	-
DMU_6	Infeasible	-	-	-	0	-
DMU_7	Infeasible	-	-	-	0	No
DMU_8	41.8254	Yes	Infeasible	Weak	-	-
DMU_9	3.9842	Yes	1.7170	Strong	-	-
DMU_{10}	2.9676	Yes	3.5135	Strong	-	-
DMU_{11}	0.7475	Yes	Infeasible	Weak	-	-
DMU_{12}	1.5415	Yes	0.8847	Strong	-	-
DMU_{13}	Infeasible	-	-	-	0	No
DMU_{14}	Infeasible	-	-	-	0	No
DMU_{15}	0.1648	Yes	Infeasible	Weak	-	-
DMU_{16}	0.4778	Yes	0.6786	Strong	-	-

According to the argument above, the existing approaches in the literature on DEA cannot determine the congestion of each input and they can only determine the congestion status of the units. But, the proposed method, in addition to determining the type of congestion of each DMU, can also determine the congestion of each input and so it can help the DM make a better decision on increasing or decreasing the size of decision-making units.

Further on, the inputs of DMU_o are decreased by $(1-\alpha_o^L)$ and the new PPS is constructed based on these new points. In other words, the inputs are decreased by the minimum possible amount in order to reduce the congestion of units. After constructing the new PPS, model (10) should be solved for the new DMUs. The results of model (10) for input 1 and input 2 of the selected units are illustrated in Table 7 and Table 8, respectively. Table 9 shows the maximum possible reduction in inputs and maximum possible increment in outputs and the congestion status of the DMUs in the new PPS. As it is obvious in Table 9, the units 9, 10 and 12, which were strongly congested in Table 5, are not congested in the new PPS. This means that the congestion of units can be eliminated by recognizing the congestion of each input of the selected units.

Table 7. The results of model (10) for input 1 of the new DMUs

DMU	α_1^*	β_1^{*1}	β_2^{*1}	β_3^{*1}	β_4^{*1}
DMU_1	0	0	0	0	0
DMU_2	0	0	0	0	0
DMU_3	0.3159	0.0720	0.3301	0	0
DMU_4	0	0	0	0	0
DMU_5	0	0	0	0	0
DMU_6	0	0	0	0	0
DMU_7	0	0	0	0	0
DMU_8	0.5653	30.1033	126.7572	0.0733	0
DMU_9	0	0	0	0	0
DMU_{10}	0	0	0	0	0
DMU_{11}	0.4999	0.1389	2.8472	0	0
DMU_{12}	0	0	0	0	0
DMU_{13}	0	0	0	0	0
DMU_{14}	0	0	0	0	0
DMU_{15}	0.3682	0.0937	0.2191	0.0554	0
DMU_{16}	0.0734	0.0216	0.0341	0.0178	0

Table 8. The results of model (10) for input 2 of the new DMUs

DMU	α_2^*	β_1^{*2}	β_2^{*2}	β_3^{*2}	β_4^{*2}
DMU_1	0	0	0	0	0
DMU_2	0	0	0	0	0
DMU_3	0.0003	0.0720	0.5306	0	0
DMU_4	0	0	0	0	0
DMU_5	0	0	0	0	0
DMU_6	0	0	0	0	0
DMU_7	0	0	0	0	0
DMU_8	0.0563	29.2525	125.5507	0.0273	0
DMU_9	0	0	0	0	0
DMU_{10}	0	0	0	0	0
DMU_{11}	0.0003	0.1392	2.8462	0	0
DMU_{12}	0	0	0	0	0
DMU_{13}	0	0	0	0	0
DMU_{14}	0	0	0	0	0
DMU_{15}	0.0450	0.0053	0	0	0
DMU_{16}	0.0080	0.0011	0.0006	0	0

Table 9. The maximum possible reduction in inputs, the maximum possible increment in outputs and the congestion status of the new DMUs

DMU	α_o^L	α_o^U	β_o^L	β_o^U	Congestion status
DMU_1	0	0	0	0	No
DMU_2	0	0	0	0	No
DMU_3	0.0003	0.3159	0	0.5306	Weak
DMU_4	0	0	0	0	No
DMU_5	0	0	0	0	No
DMU_6	0	0	0	0	No
DMU_7	0	0	0	0	No
DMU_8	0.0563	0.5653	0	126.7572	Weak
DMU_9	0	0	0	0	No
DMU_{10}	0	0	0	0	No
DMU_{11}	0.0003	0.4999	0	2.8472	Weak
DMU_{12}	0	0	0	0	No
DMU_{13}	0	0	0	0	No
DMU_{14}	0	0	0	0	No
DMU_{15}	0.0450	0.3682	0	0.2191	Weak
DMU_{16}	0.0080	0.0734	0	0.0341	Weak

5. Conclusion

The concept of congestion can help the decision maker to decide about increasing or decreasing the size of a particular DMU. Given the importance of the concept of congestion in the literature on DEA, this paper focused on presenting a method for recognizing the congestion of units. Also, knowing the amount of congestion for each input, which is calculated independently of the other inputs, can provide useful information to the DM. Despite the importance of the concept of input congestion, most of the existing methods in the literature on DEA consider the congestion of all inputs, simultaneously, and they cannot determine the congestion of each input independently of the other inputs. To tackle this issue, this study proposes a method for recognizing the congestion of each input of a selected unit under evaluation for the case when the input and output of the respective unit can change disproportionally. Finally, the paper presents a case study, reported in Yang (2015), to show the discriminative power of the proposed method.

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